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ON EXTENDED STOCHASTIC INTEGRALS WITH RESPECT TO LÉVY PROCESSES

Let L be a Lévy process on $[0, +\infty)$. In particular cases, when L is a Wiener or Poisson process, any square integrable random variable can be decomposed in a series of repeated stochastic integrals from nonrandom functions with respect to L . This property of L , known as the chaotic representation property (CRP), plays a very important role in the stochastic analysis. Unfortunately, for a general Lévy process the CRP does not hold.

There are different generalizations of the CRP for Lévy processes. In particular, under the Itô's approach one decomposes a Lévy process L in the sum of a Gaussian process and a stochastic integral with respect to a Poisson random measure, and then uses the CRP for both terms in order to obtain a generalized CRP for L . The Nualart–Schoutens's approach consists in decomposition of a square integrable random variable in a series of repeated stochastic integrals from nonrandom functions with respect to so-called orthogonalized centered power jump processes, these processes are constructed with using of a càdlàg version of L . The Lytvynov's approach is based on orthogonalization of continuous polynomials in the space of square integrable random variables.

In this paper we construct the extended stochastic integral with respect to a Lévy process and the Hida stochastic derivative in terms of the Lytvynov's generalization of the CRP; establish some properties of these operators; and, what is most important, show that the extended stochastic integrals, constructed with use of the above-mentioned generalizations of the CRP, coincide.

Key words and phrases: Lévy process, chaotic representation property, extended stochastic integral, Hida stochastic derivative.

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INTRODUCTION

Let $L = (L_t)_{t \in [0, +\infty)}$ be a Lévy process, i.e., a random process on $[0, +\infty)$ with stationary independent increments and such that $L_0 = 0$ (see, e.g., [6, 26, 28] for detailed information about Lévy processes). In particular cases, when L is a Wiener or Poisson process, any square integrable random variable can be decomposed in a series of repeated stochastic integrals from nonrandom functions with respect to L . This property of L is called the *chaotic representation property* (CRP), see, e.g., [23] for more information. The CRP plays a very important role in the stochastic analysis (in particular, it can be used in order to construct extended stochastic integrals, see, e.g., [16, 33, 15]), but, unfortunately, for a general Lévy process this property does not hold (e.g., [31]).

There are different generalizations of the CRP for Lévy processes. The first one was proposed by K. Itô [14] (see also [7]) and consists in the following. By the Lévy–Khintchine formula

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a Lévy process L can be decomposed in the sum of a Gaussian process and a stochastic integral with respect to a Poisson random measure, then one uses chaotic decompositions for both terms in order to obtain a generalized CRP for L .

Another generalization was proposed by D. Nualart and W. Schoutens [24] (see also [29]), now one decomposes a square integrable random variable in a series of repeated stochastic integrals from nonrandom functions with respect to so-called orthogonalized centered power jump processes, these processes are constructed with using of a càdlàg version of the initial Lévy process.

One more generalization (for a Lévy process without Gaussian part) was proposed by E.W. Lytvynov [22], his approach is based on orthogonalization of continuous monomials in the space of square integrable random variables.

The interconnection between above-mentioned generalizations of the CRP is described in, e.g., [22, 2, 30], one more example of a generalized CRP is given in [10, 9].

Let from now L be a Lévy process without Gaussian part and drift (it is comparatively simply to consider such processes from technical point of view). In order to construct an extended stochastic integral with respect to L , one can take any generalization of the CRP described above. Namely, in the case of the "Itô's CRP" the construction of this integral is analogous to the corresponding construction in the Poisson case, cf., e.g., [10] and [15]. In the case of the "Nualart–Schoutens's CRP" one can use term by term integration of a Nualart–Schoutens decomposition for an integrand with respect to a random measure corresponding to L . In the case of the "Lytvynov's CRP" one can construct the extended stochastic integral as in the Meixner case [17] (see also [18]): with use of a "special symmetrization" for kernels from the Lytvynov decomposition, or as the conjugated operator to the Hida stochastic derivative. The reader can find more information about extended stochastic integrals with respect to Lévy processes in, e.g., [3, 21, 10, 8, 11, 25, 9], for a general information about stochastic integration on infinite-dimensional spaces see, e.g., [1].

The main aims of the present paper are to construct the extended stochastic integral with respect to a Lévy process and the Hida stochastic derivative in terms of the Lytvynov's generalization of the CRP; to establish some properties of these operators; and, what is most important, to show that the extended stochastic integrals, constructed with use of three above-mentioned generalizations of the CRP, coincide.

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a convenient for our considerations probability triplet connected with L ; then we consider in details the above-mentioned generalizations of the CRP for L . In particular, we prove a statement about interconnection between the Itô's and Lytvynov's generalizations of the CRP for L . In the second section we introduce extended stochastic integrals in terms of the above-mentioned generalizations of the CRP and prove that these integrals coincide; then we introduce a Hida stochastic derivative in terms of the Lytvynov's generalization of the CRP and establish that this derivative and the extended stochastic integral are conjugated one to another operators.

1 LÉVY PROCESSES AND GENERALIZATIONS OF THE CHAOTIC REPRESENTATION PROPERTY

1.1 Lévy processes

Denote $\mathbb{R}_+ := [0, +\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_t)_{t \in \mathbb{R}_+}$ (a random process on \mathbb{R}_+ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift. By the Lévy–Khintchine formula such a process can be presented in the form (e.g., [10])

$$L_t = \int_0^t \int_{\mathbb{R}} x \tilde{N}(du, dx), \tag{1}$$

where $\tilde{N}(du, dx, \cdot)$ is the compensated Poisson random measure of L ; and the characteristic function of L is

$$\mathbb{E}[e^{iuL_t}] = \exp \left[t \int_{\mathbb{R}} (e^{iux} - 1 - iux) \nu(dx) \right], \tag{2}$$

where ν is the Lévy measure of L , which is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, here and below \mathcal{B} denotes the Borel σ -algebra, \mathbb{E} denotes the expectation. We assume that ν is a Radon measure whose support contains an infinite number of points, $\nu(\{0\}) = 0$, there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} x^2 e^{\varepsilon|x|} \nu(dx) < \infty,$$

and

$$\int_{\mathbb{R}} x^2 \nu(dx) = 1. \tag{3}$$

Let us define a measure of the white noise of L . Let \mathcal{D} denote the set of all real-valued infinite-differentiable functions on \mathbb{R}_+ with compact supports. As is well known, \mathcal{D} can be endowed by the projective limit topology generated by some Sobolev spaces (see, e.g., [5]). Let \mathcal{D}' be the set of linear continuous functionals on \mathcal{D} . For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle \omega, \varphi \rangle$; note that one can understand $\langle \cdot, \cdot \rangle$ as the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on \mathbb{R}_+ . The notation $\langle \cdot, \cdot \rangle$ will be preserved for dual pairings in tensor powers of spaces.

A probability measure μ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$, where \mathcal{C} denotes the cylindrical σ -algebra, with the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} \mu(d\omega) = \exp \left[\int_{\mathbb{R}_+ \times \mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x) d\nu(dx) \right], \quad \varphi \in \mathcal{D}, \tag{4}$$

is called the Lévy white noise measure.

The existence of μ from the Bochner–Minlos theorem (e.g., [13]) follows. Below we will reckon that the σ -algebra $\mathcal{C}(\mathcal{D}')$ is augmented with respect to μ , i.e., $\mathcal{C}(\mathcal{D}')$ contains all subsets of all sets O such that $\mu(O) = 0$.

Denote $(L^2) := L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ the space of (classes of) real-valued square integrable with respect to μ functions on \mathcal{D}' ; let also $\mathcal{H} := L^2(\mathbb{R}_+)$. Substituting in (4) $\varphi = t\psi$, $t \in \mathbb{R}$, $\psi \in \mathcal{D}$, and using the Taylor decomposition by t and (3), one can show that

$$\int_{\mathcal{D}'} \langle \omega, \psi \rangle^2 \mu(d\omega) = \int_{\mathbb{R}_+} (\psi(u))^2 du \tag{5}$$

(this statement follows also from results of [22] and [10]). Let $f \in \mathcal{H}$ and $\mathcal{D} \ni \varphi_k \rightarrow f$ in \mathcal{H} as $k \rightarrow \infty$. It follows from (5) that $\{\langle \circ, \varphi_k \rangle\}_{k \geq 1}$ is a Cauchy sequence in (L^2) , therefore one can define $\langle \circ, f \rangle := \lim_{k \rightarrow \infty} \langle \circ, \varphi_k \rangle \in (L^2)$ (the limit in the topology of (L^2)). It is easy to show (by the method of "mixed sequences") that $\langle \circ, f \rangle$ does not depend on a choice of an approximating sequence for f and therefore is well-definite in (L^2) .

Let us consider $\langle \circ, 1_{[0,t)} \rangle \in (L^2)$, $t \in \mathbb{R}_+$ (here and below 1_A denotes the indicator of a set A). It follows from (2) and (4) that $(\langle \circ, 1_{[0,t)} \rangle)_{t \in \mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$, therefore from now we will identify L_t with $\langle \circ, 1_{[0,t)} \rangle$.

Remark. *In this paper we work in the framework of the so-called "L²-theory of stochastic processes". In particular, it means that it is sufficient for us to understand L_t , $t \in \mathbb{R}_+$, as an element of (L^2) (i.e., as an equivalence class in (L^2)), and, correspondingly, L is a family of elements from (L^2) . But in the probability theory often it is necessary to consider modifications of random processes with some special properties. For example, one can prove that there exists a càdlàg modification of L (i.e., a random process, which is stochastically equivalent to L and has right continuous with finite left limits trajectories), and the random measure \tilde{N} from representation (1) can be constructed with using of such a modification (e.g., [26, 6, 28, 10]).*

1.2 Generalizations of the chaotic representation property for Lévy processes

Let $N = (N_t)_{t \in \mathbb{R}_+}$ be a Poisson random process. Then, as is well known, any square integrable random variable F (square integrability means that $\mathbb{E}|F|^2 < \infty$) can be presented as a series of repeated (Itô) stochastic integrals from nonrandom functions with respect to N (see, e.g., [23] for details). This property of a Poisson process is known as the chaotic representation property (CRP) and plays a very important role in the stochastic analysis. In particular, it is simple to construct an extended (Skorohod) stochastic integral if we use the CRP ([15]).

Unfortunately, for a general L the CRP does not hold (e.g., [31]). Therefore there is a natural question: what can be an appropriate analog of the CRP? There are different answers on this question. The first one was given by K. Itô [14] (see also [7]) and consists in the following. Denote by $\hat{\otimes}$ a symmetric tensor product. For $n \in \mathbb{N}$ and $f_n \in L^2(\lambda \otimes \nu)^{\hat{\otimes} n}$ (here $L^2(\lambda \otimes \nu)$ is the space of square integrable with respect to $\lambda \otimes \nu$ real-valued functions on $\mathbb{R}_+ \times \mathbb{R}$, λ is the Lebesgue measure on \mathbb{R}_+) set

$$I_n(f_n) := \int_{(\mathbb{R}_+ \times \mathbb{R})^n} f_n(u_1, x_1; \dots; u_n, x_n) \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n), \tag{6}$$

where \tilde{N} as in (1), let also $I_0(f_0) := f_0$ for $f_0 \in \mathbb{R}$. Denote $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$; $L^2(\lambda \otimes \nu)^{\hat{\otimes} 0} := \mathbb{R}$.

Theorem. ([14]) *Let $F \in (L^2)$. Then there exists a unique sequence of kernels $f_n \in L^2(\lambda \otimes \nu)^{\hat{\otimes} n}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{7}$$

and

$$\mathbb{E}|F|^2 = \|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2(\lambda \otimes \nu)^{\hat{\otimes} n}}^2. \tag{8}$$

Moreover, for $f_n \in L^2(\lambda \otimes \nu)^{\hat{\otimes} n}$ and $g_m \in L^2(\lambda \otimes \nu)^{\hat{\otimes} m}$, $n, m \in \mathbb{Z}_+$,

$$\mathbb{E}[I_n(f_n)I_m(g_m)] = \delta_{n,m} n! (f_n, g_n)_{L^2(\lambda \otimes \nu)^{\hat{\otimes} n}}. \tag{9}$$

Another approach to a generalization of the CRP was proposed by E.W. Lytvynov [22]. This approach is based on the orthogonalization of continuous polynomials in (L^2) (a suitable procedure of orthogonalization is described in [32]). Note that in the case when a Lévy process is a Poisson one, the repeated stochastic integrals from the chaotic decomposition of an element from (L^2) can be identified with so-called generalized Charlier polynomials that are orthogonal in (L^2) . Let $\mathcal{P} \equiv \mathcal{P}(\mathcal{D}')$ be the set of continuous polynomials on \mathcal{D}' , i.e., elements of \mathcal{P} have a form

$$F(\omega) = \sum_{n=0}^{N_F} \langle \omega^{\otimes n}, f^{(n)} \rangle, \quad \omega \in \mathcal{D}', \quad N_F \in \mathbb{Z}_+, \quad f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}, \quad f^{(N_F)} \neq 0,$$

here N_F is called the *power of a polynomial* F , $\langle \omega^{\otimes 0}, f^{(0)} \rangle := f^{(0)} \in \mathcal{D}^{\widehat{\otimes} 0} := \mathbb{R}$. Since the Lévy white noise measure μ has a holomorphic at zero Laplace transform (this follows from (4) and properties of the measure ν , see also [22]), \mathcal{P} is a dense set in (L^2) ([32]). Denote by \mathcal{P}_n the set of continuous polynomials of power $\leq n$, by $\overline{\mathcal{P}}_n$ the closure of \mathcal{P}_n in (L^2) . Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2) = \bigoplus_{n=0}^{\infty} \mathbf{P}_n.$$

Let $f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\langle \circ^{\otimes n}, f^{(n)} \rangle:$ the orthogonal projection of a monomial $\langle \circ^{\otimes n}, f^{(n)} \rangle$ onto \mathbf{P}_n . Let us define scalar products $\langle \cdot, \cdot \rangle_{ext}$ on $\mathcal{D}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$

$$\langle f^{(n)}, g^{(n)} \rangle_{ext} := \frac{1}{n!} \int_{\mathcal{D}'} : \langle \omega^{\otimes n}, f^{(n)} \rangle :: \langle \omega^{\otimes n}, g^{(n)} \rangle : \mu(d\omega), \quad (10)$$

and let $|\cdot|_{ext}$ be the corresponding norms, i.e., $|f^{(n)}|_{ext} = \sqrt{\langle f^{(n)}, f^{(n)} \rangle_{ext}}$. Denote by $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, the completion of $\mathcal{D}^{\widehat{\otimes} n}$ with respect to the norm $|\cdot|_{ext}$. For $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define $:\langle \circ^{\otimes n}, f^{(n)} \rangle:$ $= (L^2) - \lim_{k \rightarrow \infty} : \langle \circ^{\otimes n}, f_k^{(n)} \rangle :$, where $\mathcal{D}^{\widehat{\otimes} n} \ni f_k^{(n)} \xrightarrow{k \rightarrow \infty} f^{(n)}$ in $\mathcal{H}_{ext}^{(n)}$ (one can easily verify the correctness of this definition). Since, as is easy to see, the sets $\{:\langle \circ^{\otimes n}, f^{(n)} \rangle:; f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}\}$ are dense in \mathbf{P}_n , the following statement is fulfilled.

Theorem. *Let $F \in (L^2)$. Then there exists a unique sequence of kernels $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} : \langle \circ^{\otimes n}, f^{(n)} \rangle : \quad (11)$$

and

$$\mathbb{E}|F|^2 = \|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \sum_{n=0}^{\infty} n! |f^{(n)}|_{ext}^2. \quad (12)$$

Moreover, for $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and $g^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $n, m \in \mathbb{Z}_+$,

$$\begin{aligned} & \mathbb{E}[: \langle \circ^{\otimes n}, f^{(n)} \rangle :: \langle \circ^{\otimes m}, g^{(m)} \rangle :] \\ &= \int_{\mathcal{D}'} : \langle \omega^{\otimes n}, f^{(n)} \rangle :: \langle \omega^{\otimes m}, g^{(m)} \rangle : \mu(d\omega) = \delta_{n,m} n! \langle f^{(n)}, g^{(n)} \rangle_{ext}. \end{aligned} \quad (13)$$

Remark. It was shown in [22] that in the space $(L^2) : \langle \circ^{\otimes 0}, f^{(0)} \rangle := \langle \circ^{\otimes 0}, f^{(0)} \rangle = f^{(0)}$ and $:\langle \circ, f^{(1)} \rangle := \langle \circ, f^{(1)} \rangle$. But for $n > 1 : \langle \circ^{\otimes n}, f^{(n)} \rangle$ is not a continuous polynomial, generally speaking. Moreover, in this case the elements $:\langle \circ^{\otimes n}, f^{(n)} \rangle$ are continuous polynomials (and even generalized Appell polynomials, or Schefer polynomials in another terminology) if and only if our Lévy process L belongs to the so-called Meixner class of random processes, see [22] for details.

Remark. Let $\mathcal{F}_{ext} := \bigoplus_{n=0}^{\infty} \mathcal{H}_{ext}^{(n)} n!$ be the weighted orthogonal sum of the spaces $\mathcal{H}_{ext}^{(n)}$. The space \mathcal{F}_{ext} is called an extended Fock space. This space has important applications in the "Lévy analysis", see, e.g., [22, 4]. The foregoing theorem states that there exists an isometrical isomorphism between \mathcal{F}_{ext} and (L^2) , this isomorphism is described by (11).

One more generalization of the CRP is proposed by D. Nualart and W. Schoutens [24] (see also [29]). Now one decomposes $F \in (L^2)$ in a series of repeated stochastic integrals from nonrandom functions with respect to special random processes generated by a càdlàg version of L . Here we describe a modification of this approach offered by E.W. Lytvynov [22]. Let

$$p_n(x) := x^n + a_{n,n-1}x^{n-1} + \dots + a_{n,1}x, \quad a_{n,j} \in \mathbb{R}, j \in \{1, \dots, n-1\}, n \in \mathbb{N}, \quad (14)$$

be orthogonal in $L^2(\mathbb{R}, \nu)$ polynomials, i.e., for $n, m \in \mathbb{N}, n \neq m, \int_{\mathbb{R}} p_n(x)p_m(x)\nu(dx) = 0$. For $n \in \mathbb{N}$ we define random measures $Y^{(n)}(\Delta)$ on $\mathcal{B}(\mathbb{R}_+)$ by setting

$$Y^{(n)}(\Delta) := \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\Delta}(u)p_n(x)\tilde{N}(du, dx) = I_1(1_{\Delta}p_n). \quad (15)$$

Note that the random processes $Y_t^{(n)}$ ($Y_t^{(1)} = L_t$) from [24] are connected with the measures $Y^{(n)}(\Delta)$ as follows: $Y_t^{(n)} = Y^{(n)}([0, t])$.

Proposition. ([22]) For each $n \in \mathbb{N}$ and $f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}$ we have

$$\begin{aligned} :\langle \circ^{\otimes n}, f^{(n)} \rangle : &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > l_2 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{s_1! \dots s_k! (l_1!)^{s_1} \dots (l_k!)^{s_k}} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} f^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_{k-1}}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \\ &\times Y^{(l_1)}(du_1) \dots Y^{(l_1)}(du_{s_1}) \dots Y^{(l_k)}(du_{s_1 + \dots + s_k}). \end{aligned} \quad (16)$$

One can show that formulas (16) hold true for $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, therefore substituting (16) in (11) one obtains a variant of the Nualart–Schoutens decomposition for $F \in (L^2)$. Moreover, substituting (16) in (10) one can obtain the explicit formulas for the scalar products in $\mathcal{H}_{ext}^{(n)}$. Namely, for $f_n \in L^2(\lambda \otimes \nu)^{\otimes n}, n \in \mathbb{N}$, denote by $[f_n]_{sym}$ the orthogonal projection of f_n onto $L^2(\lambda \otimes \nu)^{\widehat{\otimes} n}$ (i.e., roughly speaking, the symmetrization of $f_n(\cdot_1, \bullet_1; \dots; \cdot_n, \bullet_n)$ by pairs of ar-

guments $(\cdot_1, \bullet_1), (\cdot_2, \bullet_2)$ etc.). Further, denote by $\|\cdot\|_v$ the norm in $L^2(\mathbb{R}, \nu)$. Since (see (15))

$$\begin{aligned} & \int_{\mathbb{R}_+^{s_1+\dots+s_k}} f^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k}) Y^{(l_1)}(du_1) \dots Y^{(l_k)}(du_{s_1+\dots+s_k}) \\ &= \int_{(\mathbb{R}_+ \times \mathbb{R})^{s_1+\dots+s_k}} f^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k}) p_{l_1}(x_1) \dots p_{l_k}(x_{s_1+\dots+s_k}) \\ & \quad \times \tilde{N}(du_1, dx_1) \dots \tilde{N}(du_{s_1+\dots+s_k}, dx_{s_1+\dots+s_k}) \\ &= I_{s_1+\dots+s_k}([f^{(n)}(\underbrace{\cdot_1, \dots, \cdot_1}_{l_1}, \dots, \underbrace{\cdot_{s_1+\dots+s_k}, \dots, \cdot_{s_1+\dots+s_k}}_{l_k}) p_{l_1}(\bullet_1) \dots p_{l_k}(\bullet_{s_1+\dots+s_k})]_{sym}), \end{aligned} \tag{17}$$

the next statement from (9) follows.

Proposition. ([22]) For $f^{(n)}, g^{(n)} \in \mathcal{D}^{\hat{\otimes} n}, n \in \mathbb{N}$, we have

$$\begin{aligned} \langle f^{(n)}, g^{(n)} \rangle_{ext} &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > l_2 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!}\right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!}\right)^{2s_k} \\ & \quad \times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} f^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k}) \\ & \quad \times g^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k}) du_1 \dots du_{s_1+\dots+s_k}. \end{aligned} \tag{18}$$

In particular, for $n = 1$ $\langle f^{(1)}, g^{(1)} \rangle_{ext} = \langle f^{(1)}, g^{(1)} \rangle$; in the case $n = 2$ we have $\langle f^{(2)}, g^{(2)} \rangle_{ext} = \langle f^{(2)}, g^{(2)} \rangle + \frac{\|p_2\|_v^2}{2} \int_{\mathbb{R}_+} f^{(2)}(u, u) g^{(2)}(u, u) du$; in general $\langle f^{(n)}, g^{(n)} \rangle_{ext} = \langle f^{(n)}, g^{(n)} \rangle + \dots$.

As is easy to see, formulas (18) hold true for $f^{(n)}, g^{(n)} \in \mathcal{H}_{ext}^{(n)}$.

It follows from (18) that $\mathcal{H}_{ext}^{(1)} = \mathcal{H} \equiv L^2(\mathbb{R}_+)$: by (14) $p_1(x) = x$ and therefore by (3)

$$\|p_1\|_v = 1; \tag{19}$$

and for $n \in \mathbb{N} \setminus \{1\}$ one can identify $\mathcal{H}^{\hat{\otimes} n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (i.e., $f^{(n)}(u_1, \dots, u_n) = 0$ if there exist $k, j \in \{1, \dots, n\}$ such that $k \neq j$ but $u_k = u_j$). In this sense the space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}^{\hat{\otimes} n}$ (this explains why we used the subindex *ext* in the designations $\mathcal{H}_{ext}^{(n)}, \langle \cdot, \cdot \rangle_{ext}$ and $|\cdot|_{ext}$). (As a consequence, the extended Fock space \mathcal{F}_{ext} is an extension of the Fock space $\bigoplus_{n=0}^{\infty} \mathcal{H}^{\hat{\otimes} n} n!$.)

Remark. A random process L of form (1) is a Poisson one if its Lévy measure $\nu(\Delta) = \delta_1(\Delta)$, i.e., if ν is a point mass at 1. This measure does not satisfy the conditions accepted in this paper, nevertheless, the next statements are fulfilled.

1) Itô decomposition (7) holds true and can be interpreted as the "classical" CRP: now we have $f_n(u_1, x_1; \dots; u_n, x_n) = f^{(n)}(u_1, \dots, u_n) x_1 \dots x_n$ and (see (1))

$$\begin{aligned} I_n(f_n) &= n! \int_0^\infty \int_{\mathbb{R}} \int_0^{u_n} \int_{\mathbb{R}} \dots \int_0^{u_2} \int_{\mathbb{R}} f^{(n)}(u_1, \dots, u_n) x_1 \dots x_n \\ & \quad \times \tilde{N}(du_1, dx_1) \dots \tilde{N}(du_n, dx_n) \\ &= n! \int_0^\infty \int_0^{u_n} \dots \int_0^{u_2} f^{(n)}(u_1, \dots, u_n) dL_{u_1} \dots dL_{u_n}. \end{aligned} \tag{20}$$

2) Decomposition (11) holds true, now : $\langle \circ^{\otimes n}, f^{(n)} \rangle$; $f^{(n)} \in \mathcal{H}_{ext}^{(n)} = \mathcal{H}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, are the generalized Charlier polynomials that can be identified with repeated stochastic integrals (20).

3) The original Nualart–Schoutens decomposition [24] (see also [29, 10, 9]) holds true, now $Y^{(1)} = L$ and $Y^{(l)} = 0$ if $l > 1$.

4) Polynomials (14) are not uniquely defined if $n > 2$; nevertheless, for any "version" of p_n , $n > 1$, we have $\|p_n\|_v = 0$. Therefore one still can define "versions" of the random measures $Y^{(n)}(\Delta)$, but representations (16) takes now the form

$$\begin{aligned} &: \langle \circ^{\otimes n}, f^{(n)} \rangle := \int_{\mathbb{R}_+^n} f^{(n)}(u_1, \dots, u_n) Y^{(1)}(du_1) \cdots Y^{(1)}(du_n) \\ &= n! \int_0^\infty \int_0^{u_n} \cdots \int_0^{u_2} f^{(n)}(u_1, \dots, u_n) Y^{(1)}(du_1) \cdots Y^{(1)}(du_n) \\ &= n! \int_0^\infty \int_0^{u_n} \cdots \int_0^{u_2} f^{(n)}(u_1, \dots, u_n) dL_{u_1} \cdots dL_{u_n} \end{aligned}$$

(all another integrals from (16) are equal to zero in (L^2)).

5) Formula (18) becomes

$$\langle f^{(n)}, g^{(n)} \rangle_{ext} = \int_{\mathbb{R}_+^n} f^{(n)}(u_1, \dots, u_n) g^{(n)}(u_1, \dots, u_n) du_1 \cdots du_n = \langle f^{(n)}, g^{(n)} \rangle,$$

and this is natural because now $\mathcal{H}_{ext}^{(n)} = \mathcal{H}^{\widehat{\otimes} n}$.

Finally we establish the following statement (cf. [2]).

Proposition. The kernels $f_n \in L^2(\lambda \otimes \nu)^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, from Itô decomposition (7) for $F \in (L^2)$, can be presented in the form

$$\begin{aligned} f_n(\cdot_1, \bullet_1; \dots; \cdot_n, \bullet_n) &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ s_1 + \dots + s_k = n}} \frac{(l_1 s_1 + \dots + l_k s_k)!}{s_1! \cdots s_k! (l_1!)^{s_1} \cdots (l_k!)^{s_k}} \\ &\times [f^{(l_1 s_1 + \dots + l_k s_k)}(\underbrace{\cdot_1, \dots, \cdot_1}_{l_1}, \dots, \underbrace{\cdot_{s_1}, \dots, \cdot_{s_1}}_{l_1}, \dots, \underbrace{\cdot_n, \dots, \cdot_n}_{l_k}) \\ &\times p_{l_1}(\bullet_1) \cdots p_{l_1}(\bullet_{s_1}) \cdots p_{l_k}(\bullet_n)]_{sym} \end{aligned} \tag{21}$$

(the equality in $L^2(\lambda \otimes \nu)^{\widehat{\otimes} n}$), where $f^{(k)} \in \mathcal{H}_{ext}^{(k)}$, $k \in \mathbb{N}$, are the kernels from decomposition (11) for F .

Proof. Formally one can obtain (21) by direct calculation with use (7), (6), (11), (16) and (17), but we have to show that the series in the right hand side of (21) converges in $L^2(\lambda \otimes \nu)^{\widehat{\otimes} n}$ and can be integrated term by term by $\tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n)$. Fix $n \in \mathbb{N}$ and for $M \in \mathbb{N}$ set

$$\begin{aligned} S_M(\cdot_1, \bullet_1; \dots; \cdot_n, \bullet_n) &:= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, M \geq l_1 > \dots > l_k, \\ s_1 + \dots + s_k = n}} \frac{(l_1 s_1 + \dots + l_k s_k)!}{s_1! \cdots s_k! (l_1!)^{s_1} \cdots (l_k!)^{s_k}} \\ &\times [f^{(l_1 s_1 + \dots + l_k s_k)}(\underbrace{\cdot_1, \dots, \cdot_1}_{l_1}, \dots, \underbrace{\cdot_{s_1}, \dots, \cdot_{s_1}}_{l_1}, \dots, \underbrace{\cdot_n, \dots, \cdot_n}_{l_k}) \\ &\times p_{l_1}(\bullet_1) \cdots p_{l_1}(\bullet_{s_1}) \cdots p_{l_k}(\bullet_n)]_{sym} \in L^2(\lambda \otimes \nu)^{\widehat{\otimes} n}. \end{aligned}$$

It follows from the Nualart–Schoutens decomposition for F and (15) that

$$\begin{aligned} & \exists (L^2) - \lim_{M \rightarrow \infty} \int_{(\mathbb{R}_+ \times \mathbb{R})^n} S_M(u_1, x_1; \dots; u_n, x_n) \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n) \\ &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ s_1 + \dots + s_k = n}} \frac{(l_1 s_1 + \dots + l_k s_k)!}{s_1! \cdots s_k! (l_1!)^{s_1} \cdots (l_k!)^{s_k}} \\ & \times \int_{(\mathbb{R}_+ \times \mathbb{R})^n} [f^{(l_1 s_1 + \dots + l_k s_k)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_n, \dots, u_n}_{l_k}) \\ & \times p_{l_1}(x_1) \cdots p_{l_1}(x_{s_1}) \cdots p_{l_k}(x_n)]_{sym} \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n). \end{aligned}$$

Further, since $\left(\int_{(\mathbb{R}_+ \times \mathbb{R})^n} S_M(u_1, x_1; \dots; u_n, x_n) \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n) \right)_{M \in \mathbb{N}}$ is a Cauchy sequence in (L^2) , by (6) and (9) $(S_M)_{M \in \mathbb{N}}$ is a Cauchy sequence in the space $L^2(\lambda \otimes \nu)^{\hat{\otimes} n}$, therefore

$$\begin{aligned} & \exists L^2(\lambda \otimes \nu)^{\hat{\otimes} n} - \lim_{M \rightarrow \infty} S_M(\cdot_1, \bullet_1; \dots; \cdot_n, \bullet_n) = S_\infty(\cdot_1, \bullet_1; \dots; \cdot_n, \bullet_n) \\ &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ s_1 + \dots + s_k = n}} \frac{(l_1 s_1 + \dots + l_k s_k)!}{s_1! \cdots s_k! (l_1!)^{s_1} \cdots (l_k!)^{s_k}} \\ & \times [f^{(l_1 s_1 + \dots + l_k s_k)}(\underbrace{\cdot_1, \dots, \cdot_1}_{l_1}, \dots, \underbrace{\cdot_{s_1}, \dots, \cdot_{s_1}}_{l_1}, \dots, \underbrace{\cdot_n, \dots, \cdot_n}_{l_k}) \\ & \times p_{l_1}(\bullet_1) \cdots p_{l_1}(\bullet_{s_1}) \cdots p_{l_k}(\bullet_n)]_{sym}. \end{aligned}$$

Again by (6) and (9)

$$\begin{aligned} & \left\| \int_{(\mathbb{R}_+ \times \mathbb{R})^n} (S_\infty(u_1, x_1; \dots; u_n, x_n) - S_M(u_1, x_1; \dots; u_n, x_n)) \right. \\ & \times \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n) \left. \right\|_{(L^2)}^2 = n! \|S_\infty - S_M\|_{L^2(\lambda \otimes \nu)^{\hat{\otimes} n}}^2 \xrightarrow{M \rightarrow \infty} 0, \end{aligned}$$

therefore $\int_{(\mathbb{R}_+ \times \mathbb{R})^n} S_\infty(u_1, x_1; \dots; u_n, x_n) \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n)$ can be calculated term by term, thus the statement of the proposition is proved. \square

More information about described above generalizations of the CRP and about the interconnection between them is given in [22, 2, 30]. Of course, another generalizations of the CRP are also possible, see, e.g., [10, 9] for a corresponding example.

2 EXTENDED STOCHASTIC INTEGRALS

2.1 Constructions and some properties of extended stochastic integrals

Let \mathcal{N} be a family of all sets $O \in \mathcal{C}(\mathcal{D}')$ such that $\mu(O) = 0$ (we recall that the σ -algebra $\mathcal{C}(\mathcal{D}')$ is augmented with respect to μ); $\tilde{\mathcal{F}}_t = \sigma(L_u : u \leq t)$ be the σ -algebra generated by the random process L up to a moment of time t ; $\mathcal{F}_t := \bigcap_{u>t} \tilde{\mathcal{F}}_u \cup \mathcal{N}$. Then $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a flow of σ -algebras. It follows from the definition of L , its representation in the form $L_t = \langle \circ, 1_{[0,t]} \rangle$, and (13) that L is a locally square integrable random process with orthogonal independent increments.

Therefore L is a martingale with respect to the flow $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ with a Doob-Meyer decomposition $L_t^2 = m_t + A_t$, $t \in \mathbb{R}_+$, where m is an \mathcal{F}_t -martingale and A is an increasing nonrandom function [12]. One can easily show that now $A_t = t$, thus, L is a locally square integrable normal \mathcal{F}_t -martingale. Therefore one can consider the Itô stochastic integral with respect to L $\int_{\mathbb{R}_+} \circ(u)dL_u : (L^2) \otimes \mathcal{H} \rightarrow (L^2)$ with the domain

$$\text{dom}\left(\int_{\mathbb{R}_+} \circ(u)dL_u\right) = \{F \in (L^2) \otimes \mathcal{H} : F \text{ is adapted with respect to } (\mathcal{F}_t)_{t \in \mathbb{R}_+}\}. \tag{22}$$

But since the class of \mathcal{F}_t -adapted functions is a comparatively narrow subset of $(L^2) \otimes \mathcal{H}$, it is natural to try to extend the notion of a stochastic integral to a more wide class of elements from $(L^2) \otimes \mathcal{H}$. An idea of such an extension can be the following. Let $F \in (L^2) \otimes \mathcal{H}$. Then by (7) F can be presented in the form

$$F(\cdot) = \sum_{n=0}^{\infty} I_n(f_{n,\cdot}), \quad f_{n,\cdot} \in L^2(\lambda \otimes \nu)^{\widehat{\otimes} n} \otimes \mathcal{H}. \tag{23}$$

Since the integration by the random measure $Y^{(1)}$ (see (15)) is an extension of the integration in the Itô sense by the Lévy process L ($L_t = Y^{(1)}([0, t])$), and since by (15), (6)

$$\begin{aligned} \int_{\mathbb{R}_+} I_n(f_{n,u})Y^{(1)}(du) &= \int_{\mathbb{R}_+ \times \mathbb{R}} I_n(f_{n,u})x\tilde{N}(du, dx) \\ &= \int_{(\mathbb{R}_+ \times \mathbb{R})^{n+1}} f_{n,u}(u_1, x_1; \dots; u_n, x_n)x\tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n)\tilde{N}(du, dx) \\ &= \int_{(\mathbb{R}_+ \times \mathbb{R})^{n+1}} [f_{n,u}(u_1, x_1; \dots; u_n, x_n)x]_{\text{sym}}\tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n)\tilde{N}(du, dx) \\ &= I_{n+1}(\widehat{f}_n), \end{aligned}$$

where $\widehat{f}_n := [f_{n,\cdot}(\cdot_1, \bullet_1; \dots; \cdot_n, \bullet_n)\bullet]_{\text{sym}} \in L^2(\lambda \otimes \nu)^{\widehat{\otimes} n+1}$, it is natural to define an extended stochastic integral $\int_{\mathbb{R}_+} F(u)\tilde{d}L_u \in (L^2)$ by setting (cf. [10])

$$\int_{\mathbb{R}_+} F(u)\tilde{d}L_u := \sum_{n=0}^{\infty} I_{n+1}(\widehat{f}_n). \tag{24}$$

The domain of this integral, i.e., of the operator $\int_{\mathbb{R}_+} \circ(u)\tilde{d}L_u : (L^2) \otimes \mathcal{H} \rightarrow (L^2)$, consists of $F \in (L^2) \otimes \mathcal{H}$ such that (see (8))

$$\left\| \int_{\mathbb{R}_+} F(u)\tilde{d}L_u \right\|_{(L^2)}^2 = \sum_{n=0}^{\infty} (n+1)! \|\widehat{f}_n\|_{L^2(\lambda \otimes \nu)^{\widehat{\otimes} n+1}}^2 < \infty. \tag{25}$$

Let $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$. We define an extended stochastic integral $\int_{t_1}^{t_2} \circ(u)\tilde{d}L_u : (L^2) \otimes \mathcal{H} \rightarrow (L^2)$ by setting

$$\int_{t_1}^{t_2} \circ(u)\tilde{d}L_u := \int_{\mathbb{R}_+} \circ(u)1_{[t_1, t_2)}(u)\tilde{d}L_u, \tag{26}$$

i.e., instead of $F \in (L^2) \otimes \mathcal{H}$ we integrate the element $F1_{[t_1, t_2)} \in (L^2) \otimes \mathcal{H}$. The domain of integral (26) depends on t_1 and t_2 : now the kernels in estimate (25) depend on t_1 and t_2 . Note that by analogy with (26) one can define an extended stochastic integral $\int_{\Delta} \circ(u)\tilde{d}L_u : (L^2) \otimes \mathcal{H} \rightarrow (L^2)$ for any Borel set $\Delta \subseteq \mathbb{R}_+$: it is necessary to use 1_{Δ} instead of $1_{[t_1, t_2)}$.

The next statement follows directly from results of [10] (see also [9]).

Theorem. Let $F \in (L^2) \otimes \mathcal{H}$ be integrable by Itô (i.e., F satisfies (22)). Then for any $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, F is integrable in the extended sense and

$$\int_{t_1}^{t_2} F(u) \tilde{d}L_u = \int_{t_1}^{t_2} F(u) dL_u. \tag{27}$$

Remark. For convenience of a reader we describe an idea of a proof of this very important theorem. In the first place, by the Nualart–Schoutens decomposition one can show that for an integrable by Itô function F the kernels $f_{n,\cdot}$, $n \in \mathbb{N}$, from decomposition (23) satisfy equalities

$$f_{n,\cdot}(\cdot_1, \bullet_1; \dots; \cdot_n, \bullet_n) = f_{n,\cdot}(\cdot_1, \bullet_1; \dots; \cdot_n, \bullet_n) 1_{[0,\cdot]^n}(\cdot_1, \dots, \cdot_n).$$

Since under this conditions $\int_{\mathbb{R}_+} I_n(f_{n,u}) Y^{(1)}(du) = \int_{\mathbb{R}_+} I_n(f_{n,u}) dL_u$, $n \in \mathbb{Z}_+$, equality (27) follows directly from the construction of the extended stochastic integral. In the second place, since L is a normal martingale, for F satisfying (22) we have

$$\left\| \int_{\mathbb{R}_+} F(u) 1_{[t_1, t_2)}(u) dL_u \right\|_{(L^2)} = \|F 1_{[t_1, t_2)}\|_{(L^2) \otimes \mathcal{H}}$$

therefore condition (25) for $F 1_{[t_1, t_2)}$ is fulfilled.

Another idea of an extension of the Itô stochastic integral is based on term by term integration by $Y^{(1)}(du)$ of the Nualart–Schoutens decomposition for $F \in (L^2) \otimes \mathcal{H}$. Namely, by (11) and (16) we have

$$\begin{aligned} F(\cdot) &= f^{(0)} + \sum_{n=1}^{\infty} \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{s_1! \dots s_k! (l_1!)^{s_1} \dots (l_k!)^{s_k}} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} f^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \\ &\times Y^{(l_1)}(du_1) \dots Y^{(l_k)}(du_{s_1 + \dots + s_k}), \quad f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}, \end{aligned} \tag{28}$$

therefore it is natural to define an extended stochastic integral $\int_{\mathbb{R}_+} F(u) \widehat{d}L_u \in (L^2)$ by setting

$$\begin{aligned} \int_{\mathbb{R}_+} F(u) \widehat{d}L_u &= \int_{\mathbb{R}_+} f_u^{(0)} dL_u + \sum_{n=1}^{\infty} \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{s_1! \dots s_k! (l_1!)^{s_1} \dots (l_k!)^{s_k}} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k + 1}} f_u^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \\ &\times Y^{(l_1)}(du_1) \dots Y^{(l_k)}(du_{s_1 + \dots + s_k}) Y^{(1)}(du). \end{aligned} \tag{29}$$

In order to describe the domain of this integral, denote

$$\begin{aligned} &\widehat{f}_{l,s}^{(n)}(\underbrace{\cdot_1, \dots, \cdot_1}_{l_1}, \dots, \underbrace{\cdot_{s_1 + \dots + s_k}, \dots, \cdot_{s_1 + \dots + s_k}}_{l_k}) \\ &:= \begin{cases} f^{(n)}(\underbrace{\cdot_1, \dots, \cdot_1}_{l_1}, \dots, \underbrace{\cdot_{s_1 + \dots + s_k}, \dots, \cdot_{s_1 + \dots + s_k}}_{l_k}), & \text{if } l_k > 1 \\ \sqrt{s_k + 1} f^{(n)}(\underbrace{\cdot_1, \dots, \cdot_1}_{l_1}, \dots, \cdot_{s_1 + \dots + s_{k-1} + 1}, \dots, \cdot_{s_1 + \dots + s_k}), & \text{if } l_k = 1 \end{cases}. \end{aligned}$$

Then the domain of $\int_{\mathbb{R}_+} \circ(u) \widehat{dL}_u : (L^2) \otimes \mathcal{H} \rightarrow (L^2) \otimes \mathcal{H}$ consists of $F \in (L^2) \otimes \mathcal{H}$ such that

$$\begin{aligned}
 & \left\| \int_{\mathbb{R}_+} F(u) \widehat{dL}_u \right\|_{(L^2)}^2 = \int_{\mathbb{R}_+} |f_u^{(0)}|^2 du \\
 & + \sum_{n=1}^{\infty} \sum_{\substack{k, l, j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{s_1! \cdots s_k!} \left(\frac{\|p_{l_1}\|_{\nu}}{l_1!} \right)^{2s_1} \cdots \left(\frac{\|p_{l_k}\|_{\nu}}{l_k!} \right)^{2s_k} \\
 & \times \int_{\mathbb{R}_+^{s_1 + \dots + s_k + 1}} |\widehat{f}_{l, s}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}, u)|^2 \\
 & \times du_1 \cdots du_{s_1 + \dots + s_k} du < \infty,
 \end{aligned} \tag{30}$$

the representation for $\left\| \int_{\mathbb{R}_+} F(u) \widehat{dL}_u \right\|_{(L^2)}$ can be obtained by direct calculation with use (15), (6), (9), (19) and the orthogonality of polynomials (14) in $L^2(\mathbb{R}, \nu)$, see also Lemma 4.1 in [22].

Theorem. *The extended stochastic integrals $\int_{\mathbb{R}_+} \circ(u) \widetilde{dL}_u$ and $\int_{\mathbb{R}_+} \circ(u) \widehat{dL}_u$, given by (24) and (29) respectively, coincide.*

Proof. By definition, for $F \in (L^2) \otimes \mathcal{H}$, $\int_{\mathbb{R}_+} F(u) \widetilde{dL}_u$ is the result of term by term integration of Itô decomposition (23) for F by $Y^{(1)}(du)$; and $\int_{\mathbb{R}_+} F(u) \widehat{dL}_u$ is the result of term by term integration of Nualart–Schoutens decomposition (28) for F by $Y^{(1)}(du)$, if the results of such integration belong to (L^2) . Therefore it is sufficient to show that for each $n \in \mathbb{N}$ $\int_{\mathbb{R}_+} I_n(f_{n, \cdot}) Y^{(1)}(du)$ is the result of term by term integration of decomposition (28) for $I_n(f_{n, \cdot})$ by $Y^{(1)}(du)$.

Fix $n \in \mathbb{N}$. By (the proof of) (21), (6) and (15) decomposition (28) for $I_n(f_{n, \cdot})$ can be presented in the form

$$\begin{aligned}
 I_n(f_{n, \cdot}) &= \sum_{\substack{k, l, j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ s_1 + \dots + s_k = n}} \frac{(l_1 s_1 + \dots + l_k s_k)!}{s_1! \cdots s_k! (l_1!)^{s_1} \cdots (l_k!)^{s_k}} \\
 & \times \int_{(\mathbb{R}_+ \times \mathbb{R})^n} [f^{(l_1 s_1 + \dots + l_k s_k)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_n, \dots, u_n}_{l_k}) \\
 & \times p_{l_1}(x_1) \cdots p_{l_1}(x_{s_1}) \cdots p_{l_k}(x_n)]_{sym} \widetilde{N}(du_1, dx_1) \cdots \widetilde{N}(du_n, dx_n).
 \end{aligned} \tag{31}$$

For $M \in \mathbb{N}$ set

$$\begin{aligned}
 S_M(\bullet_1, \bullet_1; \dots; \bullet_n, \bullet_n; \bullet) &:= \sum_{\substack{k, l, j, s_j \in \mathbb{N}: j=1, \dots, k, M \geq l_1 > \dots > l_k, \\ s_1 + \dots + s_k = n}} \frac{(l_1 s_1 + \dots + l_k s_k)!}{s_1! \cdots s_k! (l_1!)^{s_1} \cdots (l_k!)^{s_k}} \\
 & \times [f^{(l_1 s_1 + \dots + l_k s_k)}(\underbrace{\bullet_1, \dots, \bullet_1}_{l_1}, \dots, \underbrace{\bullet_{s_1}, \dots, \bullet_{s_1}}_{l_1}, \dots, \underbrace{\bullet_n, \dots, \bullet_n}_{l_k}) \\
 & \times p_{l_1}(\bullet_1) \cdots p_{l_1}(\bullet_{s_1}) \cdots p_{l_k}(\bullet_n)]_{sym} \in L^2(\lambda \otimes \nu)^{\widehat{\otimes} n} \otimes \mathcal{H}.
 \end{aligned}$$

Then by (6), (9) and (3)

$$\begin{aligned}
 & \left\| \int_{(\mathbb{R}_+ \times \mathbb{R})^{n+1}} (f_{n, u}(u_1, x_1; \dots; u_n, x_n) - S_M(u_1, x_1; \dots; u_n, x_n; u)) \right. \\
 & \quad \left. \times \widetilde{N}(du_1, dx_1) \cdots \widetilde{N}(du_n, dx_n) x \widetilde{N}(du, dx) \right\|_{(L^2)}^2 \\
 & = (n+1)! \|\widehat{f}_n - [S_M \bullet]_{sym}\|_{L^2(\lambda \otimes \nu)^{\widehat{\otimes} n+1}}^2 \leq (n+1)! \|f_{n, \cdot} - S_M\|_{L^2(\lambda \otimes \nu)^{\widehat{\otimes} n} \otimes \mathcal{H}}^2 \xrightarrow{M \rightarrow \infty} 0,
 \end{aligned}$$

therefore

$$\begin{aligned} & \int_{(\mathbb{R}_+ \times \mathbb{R})^{n+1}} S_M(u_1, x_1; \dots; u_n, x_n; u) \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n) x \tilde{N}(du, dx) \\ & \xrightarrow{M \rightarrow \infty} \int_{(\mathbb{R}_+ \times \mathbb{R})^{n+1}} f_{n,u}(u_1, x_1; \dots; u_n, x_n) \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n) x \tilde{N}(du, dx) \\ & = \int_{\mathbb{R}_+} I_n(f_{n,u}) Y^{(1)}(du) \end{aligned}$$

in (L^2) (see (6), (15)). But by construction

$$\int_{(\mathbb{R}_+ \times \mathbb{R})^{n+1}} S_M(u_1, x_1; \dots; u_n, x_n; u) \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n) x \tilde{N}(du, dx)$$

tends in (L^2) to the result of term by term integration of the right hand side of (31) by $Y^{(1)}(du)$ as $M \rightarrow \infty$, thus the necessary statement is obtained.

Finally, the domains of integrals (24) and (29) coincide because both these domains are given by the condition: the result of integration is an element of (L^2) , see (25) and (30). \square

By analogy with (26) for $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, set

$$\int_{t_1}^{t_2} \circ(u) \widehat{dL}_u := \int_{\mathbb{R}_+} \circ(u) 1_{[t_1, t_2]}(u) \widehat{dL}_u, \tag{32}$$

then $\int_{t_1}^{t_2} \circ(u) \widehat{dL}_u = \int_{t_1}^{t_2} \circ(u) \widetilde{dL}_u$. In what follows, we denote integrals (24), (29) by $\int_{\mathbb{R}_+} \circ(u) \widehat{dL}_u$, integrals (26) and (32) by $\int_{t_1}^{t_2} \circ(u) \widehat{dL}_u$.

Remark. Let $F \in (L^2) \otimes \mathcal{H}$ be such that the kernels from decomposition (28) $f^{(n)} \in \mathcal{H}^{\otimes n} \otimes \mathcal{H} \subset \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ (the inclusion in the generalized sense described above), $n \in \mathbb{N}$, i.e., all $f^{(n)}$ "vanish on diagonals". In this case (28) has a form

$$\begin{aligned} F(\cdot) &= f^{(0)} + \sum_{n=1}^{\infty} \int_{\mathbb{R}_+^n} f^{(n)}(u_1, \dots, u_n) Y^{(1)}(du_1) \cdots Y^{(1)}(du_n) \\ &= f^{(0)} + \sum_{n=1}^{\infty} n! \int_0^{\infty} \int_0^{u_n} \cdots \int_0^{u_2} f^{(n)}(u_1, \dots, u_n) dL_{u_1} \cdots dL_{u_n}; \end{aligned} \tag{33}$$

(23) reduces by (31) to

$$\begin{aligned} F(\cdot) &= f_{0,\cdot} + \sum_{n=1}^{\infty} \int_{(\mathbb{R}_+ \times \mathbb{R})^n} f^{(n)}(u_1, \dots, u_n) x_1 \cdots x_n \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_n, dx_n) \\ &= f_{0,\cdot} + \sum_{n=1}^{\infty} \int_{\mathbb{R}_+^n} f^{(n)}(u_1, \dots, u_n) Y^{(1)}(du_1) \cdots Y^{(1)}(du_n) \end{aligned}$$

(see (14) and (15)); and the extended stochastic integral can be constructed as in the Poisson analysis: by (24)

$$\begin{aligned} & \int_{\mathbb{R}_+} F(u) \widehat{dL}_u \\ &= \sum_{n=0}^{\infty} \int_{(\mathbb{R}_+ \times \mathbb{R})^{n+1}} \widehat{f}^{(n)}(u_1, \dots, u_{n+1}) x_1 \cdots x_{n+1} \tilde{N}(du_1, dx_1) \cdots \tilde{N}(du_{n+1}, dx_{n+1}) \\ &= \sum_{n=0}^{\infty} (n+1)! \int_0^{\infty} \int_0^{u_{n+1}} \cdots \int_0^{u_2} \widehat{f}^{(n)}(u_1, \dots, u_{n+1}) dL_{u_1} \cdots dL_{u_{n+1}}, \end{aligned} \tag{34}$$

where $\widehat{f}^{(n)} \in \mathcal{H}^{\widehat{\otimes}n+1} \subset \mathcal{H}_{ext}^{(n+1)}$, $n \in \mathbb{Z}_+$, are the symmetrizations of $f^{(n)}$ by all arguments (more exactly, the projections of $f^{(n)} \in \mathcal{H}^{\widehat{\otimes}n} \otimes \mathcal{H}$ onto $\mathcal{H}^{\widehat{\otimes}n+1}$).

Finally, by (11) any $F \in (L^2) \otimes \mathcal{H}$ can be uniquely presented in the form

$$F(\cdot) = \sum_{n=0}^{\infty} : \langle \circ^{\otimes n}, f^{(n)} \rangle :, \quad f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}, \tag{35}$$

therefore it is natural to construct an extended stochastic integral that is based on this decomposition and correlated with the structure of the spaces $\mathcal{H}_{ext}^{(n)}$. In the case when L is a process of Meixner type (e.g., [22]), such an integral is constructed and studied in [17]. The idea of its construction is the following. Let at first the kernels from (35) $f^{(n)} \in \mathcal{H}^{\widehat{\otimes}n} \otimes \mathcal{H} \subset \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{Z}_+$. Then by (16) decomposition (35) reduces to (33) and the extended stochastic integral can be defined by (34) that now can be written in the form

$$\int_{\mathbb{R}_+} F(u) \widehat{d}L_u = \sum_{n=0}^{\infty} : \langle \circ^{\otimes n+1}, \widehat{f}^{(n)} \rangle :, \quad \widehat{f}^{(n)} \in \mathcal{H}^{\widehat{\otimes}n+1}. \tag{36}$$

Of course, general elements of $\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ can not be projected onto $\mathcal{H}_{ext}^{(n+1)}$; nevertheless, by $f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ one can construct kernels $\widehat{f}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ that can be used in order to define the extended stochastic integral by (36). For a Lévy process L that we consider in this paper, the situation is quite analogous. Namely, let $f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{N}$. We select a representative (a function) $\dot{f}^{(n)} \in f^{(n)}$ such that

$$\dot{f}_u^{(n)}(u_1, \dots, u_n) = 0 \text{ if for some } k \in \{1, \dots, n\} u = u_k. \tag{37}$$

Let $\widetilde{f}^{(n)}$ be the symmetrization of $\dot{f}^{(n)}$ by $n + 1$ variables. Define $\widehat{f}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ as the equivalence class in $\mathcal{H}_{ext}^{(n+1)}$ generated by $\widetilde{f}^{(n)}$.

Lemma. For each $f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{N}$, the element $\widehat{f}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ is well-definite (in particular, $\widehat{f}^{(n)}$ does not depend on a choice of a representative $\dot{f}^{(n)} \in f^{(n)}$ satisfying (37)) and

$$|\widehat{f}^{(n)}|_{ext} \leq |f^{(n)}|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}}. \tag{38}$$

Proof. Let $f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{N}$, and $\dot{f}^{(n)} \in f^{(n)}$ be a representative of $f^{(n)}$ satisfying (37). Without loss of generality we can assume that $\dot{f}^{(n)}(\cdot_1, \dots, \cdot_n)$ is a symmetric function by the arguments \cdot_1, \dots, \cdot_n , therefore

$$\begin{aligned} \widetilde{f}^{(n)}(u_1, \dots, u_n, u) &= \frac{1}{n+1} \left[\dot{f}_u^{(n)}(u_1, \dots, u_n) \right. \\ &\left. + \dot{f}_{u_n}^{(n)}(u, u_1, \dots, u_{n-1}) + \dots + \dot{f}_{u_1}^{(n)}(u_2, \dots, u_n, u) \right]. \end{aligned} \tag{39}$$

Denote $\check{f}^{(n)}(u_1, \dots, u_n, u) := \dot{f}_u^{(n)}(u_1, \dots, u_n)$. Using (18) and the well-known inequality

$|\sum_{l=1}^p a_l|^2 \leq p \sum_{l=1}^p |a_l|^2$ we obtain

$$\begin{aligned} |\tilde{f}^{(n)}|_{ext}^2 &= \sum_{\substack{k,l,j,s_j \in \mathbb{N}: j=1,\dots,k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n+1}} \frac{(n+1)!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!}\right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!}\right)^{2s_k} \\ &\times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} |\tilde{f}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k})|^2 du_1 \dots du_{s_1+\dots+s_k} \\ &\leq \sum_{\substack{k,l,j,s_j \in \mathbb{N}: j=1,\dots,k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n+1}} \frac{(n+1)!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!}\right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!}\right)^{2s_k} \frac{n+1}{(n+1)^2} \\ &\times \left[\int_{\mathbb{R}_+^{s_1+\dots+s_k}} |\check{f}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k})|^2 du_1 \dots du_{s_1+\dots+s_k} \right. \\ &+ \int_{\mathbb{R}_+^{s_1+\dots+s_k}} |\check{f}^{(n)}(u_{s_1+\dots+s_k}, \underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k-1})|^2 du_1 \dots du_{s_1+\dots+s_k} \\ &\left. + \dots + \int_{\mathbb{R}_+^{s_1+\dots+s_k}} |\check{f}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1-1}, \dots, \underbrace{u_{s_1+\dots+s_k}, \dots, u_{s_1+\dots+s_k}}_{l_k}, u_1)^2 du_1 \dots du_{s_1+\dots+s_k} \right] \end{aligned}$$

(arguments over $\underbrace{\quad}_0$ are absent). It follows from (37) that if $l_k > 1$ then all terms in square brackets $[\dots]$ are equal to zero; and if $l_k = 1$ then for a fixed collection k, l, s . all nonzero terms in square brackets $[\dots]$ coincide and the quantity of such terms is equal to s_k . Therefore we can continue our calculation as follows:

$$\begin{aligned} |\tilde{f}^{(n)}|_{ext}^2 &\leq \sum_{\substack{k,l,j,s_j \in \mathbb{N}: j=1,\dots,k, l_1 > \dots > 1, \\ l_1 s_1 + \dots + (s_k-1) = n}} \frac{n!}{s_1! \dots s_{k-1}! (s_k-1)!} \left(\frac{\|p_{l_1}\|_v}{l_1!}\right)^{2s_1} \dots \left(\frac{\|p_{l_{k-1}}\|_v}{l_{k-1}!}\right)^{2s_{k-1}} \\ &\times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} |\check{f}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, u_{s_1+\dots+s_{k-1}+1}, \dots, u_{s_1+\dots+s_k})|^2 du_1 \dots du_{s_1+\dots+s_k} \\ &= \sum_{\substack{k',l',j,s'_j \in \mathbb{N}: j=1,\dots,k', l'_1 > \dots > l'_{k'}, \\ l'_1 s'_1 + \dots + l'_{k'} s'_{k'} = n}} \frac{n!}{s'_1! \dots s'_{k'}!} \left(\frac{\|p_{l'_1}\|_v}{l'_1!}\right)^{2s'_1} \dots \left(\frac{\|p_{l'_{k'}}\|_v}{l'_{k'}!}\right)^{2s'_{k'}} \\ &\times \int_{\mathbb{R}_+^{s'_1+\dots+s'_{k'}+1}} |\check{f}_u^{(n)}(\underbrace{u_1, \dots, u_1}_{l'_1}, \dots, \underbrace{u_{s'_1+\dots+s'_{k'}}, \dots, u_{s'_1+\dots+s'_{k'}}}_{l'_{k'}})|^2 du_1 \dots du_{s'_1+\dots+s'_{k'}} du \\ &= |\check{f}^{(n)}|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}}^2 \end{aligned}$$

(here we used (19)), hence $\tilde{f}^{(n)}$ generates an equivalence class $\hat{f}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ and estimate (38) is fulfilled.

Let $\dot{g}^{(n)} \in f^{(n)}$ be another representative of $f^{(n)}$ with property (37), $\hat{g}^{(n)}$ be the corresponding element of $\mathcal{H}_{ext}^{(n+1)}$. Then, obviously, $\dot{h}^{(n)} := \dot{f}^{(n)} - \dot{g}^{(n)} \in 0 \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ satisfies (37) and the corresponding to $\dot{h}^{(n)}$ element of $\mathcal{H}_{ext}^{(n+1)}$ $\hat{h}^{(n)} = \hat{f}^{(n)} - \hat{g}^{(n)} = 0$ by (38). So, $\hat{f}^{(n)}$ does not depend on a choice of a representative $\dot{f}^{(n)} \in f^{(n)}$. \square

For $F \in (L^2) \otimes \mathcal{H}$ we define an extended stochastic integral $\int_{\mathbb{R}_+} F(u) \delta L_u \in (L^2)$ by setting

$$\int_{\mathbb{R}_+} F(u) \delta L_u := \sum_{n=0}^{\infty} : \langle \circ^{\otimes n+1}, \widehat{f}^{(n)} \rangle : , \quad \widehat{f}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}, \quad (40)$$

where $\widehat{f}^{(0)} := f^{(0)} \in \mathcal{H} = \mathcal{H}_{ext}^{(1)}$, and $\widehat{f}^{(n)}$, $n \in \mathbb{N}$, are constructed by the kernels $f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ from decomposition (35) for F . The domain of this integral, i.e., of the operator $\int_{\mathbb{R}_+} \circ(u) \delta L_u : (L^2) \otimes \mathcal{H} \rightarrow (L^2)$, consists of $F \in (L^2) \otimes \mathcal{H}$ such that (see (12))

$$\left\| \int_{\mathbb{R}_+} F(u) \delta L_u \right\|_{(L^2)}^2 = \sum_{n=0}^{\infty} (n+1)! |\widehat{f}^{(n)}|_{ext}^2 < \infty. \quad (41)$$

Theorem. *The extended stochastic integrals $\int_{\mathbb{R}_+} \circ(u) \widehat{d}L_u$ and $\int_{\mathbb{R}_+} \circ(u) \delta L_u$, given by (29) and (40) respectively, coincide.*

Proof. Let at first $F(\cdot) = : \langle \circ^{\otimes n}, f^{(n)} \rangle : \in (L^2) \otimes \mathcal{H}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{N}$. Using (40), (16), the construction of the kernel $\widehat{f}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ (in particular, (39)) and (29) we obtain

$$\begin{aligned} \int_{\mathbb{R}_+} F(u) \delta L_u &= : \langle \circ^{\otimes n+1}, \widehat{f}^{(n)} \rangle : = \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n+1}} \frac{(n+1)!}{s_1! \dots s_k! (l_1!)^{s_1} \dots (l_k!)^{s_k}} \\ &\quad \times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} \widehat{f}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \\ &\quad \times Y^{(l_1)}(du_1) \dots Y^{(l_k)}(du_{s_1 + \dots + s_k}) \\ &= \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_{k-1}, \\ l_1 s_1 + \dots + l_{k-1} s_{k-1} + (s_k - 1) = n}} \frac{n!(n+1)s_k}{s_1! \dots s_{k-1}! (s_k - 1)! s_k (l_1!)^{s_1} \dots (l_{k-1}!)^{s_{k-1}} (n+1)} \\ &\quad \times \int_{\mathbb{R}_+^{s_1 + \dots + (s_k - 1) + 1}} \dot{f}_u^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, u_{s_1 + \dots + s_{k-1} + 1}, \dots, u_{s_1 + \dots + (s_k - 1)}) \\ &\quad \times Y^{(l_1)}(du_1) \dots Y^{(1)}(du_{s_1 + \dots + (s_k - 1)}) Y^{(1)}(du) \\ &= \int_{\mathbb{R}_+} : \langle \circ^{\otimes n}, f_u^{(n)} \rangle : Y^{(1)}(du) = \int_{\mathbb{R}_+} F(u) \widehat{d}L_u. \end{aligned}$$

In a general case the result follows from the obtained equality, (40) and (29); the domains of integrals (29) and (40) coincide because both these domains are given by the condition: the result of integration is an element of (L^2) , see (30) and (41). \square

By analogy with (26), (32) for $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, set

$$\int_{t_1}^{t_2} F(u) \delta L_u := \int_{\mathbb{R}_+} F(u) 1_{[t_1, t_2)}(u) \delta L_u, \quad (42)$$

then $\int_{t_1}^{t_2} F(u) \delta L_u = \int_{t_1}^{t_2} F(u) \widehat{d}L_u$. In what follows, we denote integrals (40) and (42) by $\int_{\mathbb{R}_+} F(u) \delta L_u$ and $\int_{t_1}^{t_2} F(u) \widehat{d}L_u$ respectively. Of course, the domain of integral (42) depends on t_1 and t_2 and can be described by (41), where the kernels $\widehat{f}^{(n)}$, $n \in \mathbb{Z}_+$, are replaced by the kernels $\widehat{f}_{[t_1, t_2)}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ constructed with use of $F(\cdot) 1_{[t_1, t_2)}(\cdot)$ instead of $F(\cdot)$.

Remark. Since integrals (32) and (42) coincide with the extended stochastic integral (26), these integrals are extensions of the Itô stochastic integral. Note that for integral (42) this fact can be proved by analogy with the corresponding proof in the "Meixner analysis" [17] (see also [19]) with using results of [4].

2.2 A Hida stochastic derivative and its interconnection with the extended stochastic integral

As is well known, in the "Poisson analysis" the extended stochastic integral is the conjugate operator of the Hida stochastic derivative. In the "Meixner analysis" the situation is analogous [17]. Now we will show that this result holds true in the "Lévy analysis".

In order to define a stochastic derivative on (L^2) we need some preparation. Let $g^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, $\dot{g}^{(n)} \in g^{(n)}$ be a representative of $g^{(n)}$. We consider $\dot{g}^{(n)}(\cdot)$, i.e., separate one argument of $g^{(n)}$, and define $g^{(n)}(\cdot) \in \mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}$ as the equivalence class in $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}$ generated by $\dot{g}^{(n)}(\cdot)$.

Lemma. For each $g^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, the element $g^{(n)}(\cdot) \in \mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}$ is well-definite (in particular, $g^{(n)}(\cdot)$ does not depend on a choice of a representative $\dot{g}^{(n)} \in g^{(n)}$) and

$$|g^{(n)}(\cdot)|_{\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}} \leq |g^{(n)}|_{ext}. \tag{43}$$

Proof. Without loss of generality we can assume that $\dot{g}^{(n)}$ is a symmetric function, therefore one can separate the last argument. Using (18) and (19) one can write

$$\begin{aligned} |g^{(n)}|_{ext}^2 &= |\dot{g}^{(n)}|_{ext}^2 = \sum_{\substack{k,l,j,s_j \in \mathbb{N}: j=1,\dots,k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!}\right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!}\right)^{2s_k} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} |\dot{g}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k})|^2 du_1 \dots du_{s_1 + \dots + s_k} \\ &= \sum_{\substack{k,l,j,s_j \in \mathbb{N}: j=1,\dots,k, l_1 > \dots > l_k > 1, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!}\right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!}\right)^{2s_k} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} |\dot{g}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k})|^2 du_1 \dots du_{s_1 + \dots + s_k} \\ &+ \sum_{\substack{k,l,j,s_j \in \mathbb{N}: j=1,\dots,k, l_1 > \dots > l_{k-1} = 1, \\ l_1 s_1 + \dots + l_{k-1} s_{k-1} + (s_k - 1) = n - 1}} \frac{(n-1)!n}{s_1! \dots s_{k-1}!(s_k-1)!s_k} \left(\frac{\|p_{l_1}\|_v}{l_1!}\right)^{2s_1} \dots \left(\frac{\|p_{l_{k-1}}\|_v}{l_{k-1}!}\right)^{2s_{k-1}} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + (s_k-1) + 1}} |\dot{g}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, u_{s_1 + \dots + s_{k-1} + 1}, \dots, u_{s_1 + \dots + (s_k-1)}, u)|^2 \\ &\times du_1 \dots du_{s_1 + \dots + (s_k-1)} du \geq |\dot{g}^{(n)}(\cdot)|_{\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}}^2 = |g^{(n)}(\cdot)|_{\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}}^2 \end{aligned}$$

because $n \geq s_k$, hence $\dot{g}^{(n)}(\cdot)$ generates an equivalence class $g^{(n)}(\cdot) \in \mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}$ and estimate (43) is fulfilled.

Let $\dot{f}^{(n)} \in g^{(n)}$ be another representative of $g^{(n)}$, $f^{(n)}(\cdot)$ be the corresponding element of $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}$. Then $\dot{h}^{(n)} := \dot{g}^{(n)} - \dot{f}^{(n)} \in 0 \in \mathcal{H}_{ext}^{(n)}$ and the corresponding to $\dot{h}^{(n)}$ element of

$\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H} h^{(n)}(\cdot) = g^{(n)}(\cdot) - f^{(n)}(\cdot) = 0$ by (43). So, $g^{(n)}(\cdot)$ does not depend on a choice of a representative $\dot{g}^{(n)} \in g^{(n)}$. \square

Remark. Note that in spite of estimate (43) the space $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N} \setminus \{1\}$, can not be considered as a subspace of $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}$ because different elements of $\mathcal{H}_{ext}^{(n)}$ can coincide as elements of $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}$.

Let $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$. We define a Hida stochastic derivative $1_{[t_1, t_2]}(\cdot)\partial.G \in (L^2) \otimes \mathcal{H}$ for $G \in (L^2)$ by setting

$$1_{[t_1, t_2]}(\cdot)\partial.G := \sum_{n=0}^{\infty} (n+1) : \langle \circ^{\otimes n}, g^{(n+1)}(\cdot) 1_{[t_1, t_2]}(\cdot) \rangle :, \tag{44}$$

where $g^{(n+1)} \in \mathcal{H}_{ext}^{(n+1)}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (11) for G , in point as elements of $\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$. The domain of this derivative, i.e., of the operator $1_{[t_1, t_2]}(\cdot)\partial. : (L^2) \rightarrow (L^2) \otimes \mathcal{H}$, consists of $G \in (L^2)$ such that

$$\|1_{[t_1, t_2]}(\cdot)\partial.G\|_{(L^2) \otimes \mathcal{H}}^2 = \sum_{n=0}^{\infty} (n+1)!(n+1) |g^{(n+1)}(\cdot) 1_{[t_1, t_2]}(\cdot)|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}}^2 < \infty. \tag{45}$$

Theorem. For arbitrary $t_1, t_2 \in [0, +\infty]$, $t_1 < t_2$, the extended stochastic integral $\int_{t_1}^{t_2} \circ(u) \widehat{dL}_u : (L^2) \otimes \mathcal{H} \rightarrow (L^2)$ and the Hida stochastic derivative $1_{[t_1, t_2]}(\cdot)\partial. : (L^2) \rightarrow (L^2) \otimes \mathcal{H}$ are conjugated one to another:

$$\int_{t_1}^{t_2} \circ(u) \widehat{dL}_u = (1_{[t_1, t_2]}(\cdot)\partial.)^* \circ, \quad 1_{[t_1, t_2]}(\cdot)\partial. = \left(\int_{t_1}^{t_2} \circ \widehat{dL} \right)^*. \tag{46}$$

In particular, $\int_{t_1}^{t_2} \circ(u) \widehat{dL}_u$ and $1_{[t_1, t_2]}(\cdot)\partial.$ are closed operators.

Proof. First we note that the operators $(1_{[t_1, t_2]}(\cdot)\partial.)^*$ and $(\int_{t_1}^{t_2} \circ \widehat{dL})^*$ are well-definite because the domains of $1_{[t_1, t_2]}(\cdot)\partial.$ and $\int_{t_1}^{t_2} \circ(u) \widehat{dL}_u$ are dense sets in the corresponding spaces. Further, let us show that for $F \in \text{dom}(\int_{t_1}^{t_2} \circ(u) \widehat{dL}_u)$ and $G \in \text{dom}(1_{[t_1, t_2]}(\cdot)\partial.)$

$$\left(\int_{t_1}^{t_2} F(u) \widehat{dL}_u, G \right)_{(L^2)} = (F(\cdot), 1_{[t_1, t_2]}(\cdot)\partial.G)_{(L^2) \otimes \mathcal{H}}. \tag{47}$$

By (42), (40), (11) and (13)

$$\left(\int_{t_1}^{t_2} F(u) \widehat{dL}_u, G \right)_{(L^2)} = \sum_{n=0}^{\infty} (n+1)! \langle \widehat{f}_{[t_1, t_2]}^{(n)}, g^{(n+1)} \rangle_{ext},$$

where $\widehat{f}_{[t_1, t_2]}^{(n)}, g^{(n+1)} \in \mathcal{H}_{ext}^{(n+1)}$, $n \in \mathbb{Z}_+$, are the kernels from decompositions (40) and (11) for $\int_{t_1}^{t_2} F(u) \widehat{dL}_u$ and G correspondingly. On the other hand, it follows from (35), (44) and (13) that

$$(F(\cdot), 1_{[t_1, t_2]}(\cdot)\partial.G)_{(L^2) \otimes \mathcal{H}} = \sum_{n=0}^{\infty} (n+1)! \langle f^{(n)}, g^{(n+1)}(\cdot) 1_{[t_1, t_2]}(\cdot) \rangle_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}'}$$

where $f^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (35) for F , $g^{(n+1)} \in \mathcal{H}_{ext}^{(n+1)}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (11) for G , in point as elements of $\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$. Therefore in order to prove (47) it is sufficient to show that for each $n \in \mathbb{Z}_+$

$$\langle \widehat{f}_{[t_1, t_2]}^{(n)}, g^{(n+1)} \rangle_{ext} = (f^{(n)}, g^{(n+1)}(\cdot)1_{[t_1, t_2]}(\cdot))_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}}.$$

The case $n = 0$ is trivial, so we consider the case $n \in \mathbb{N}$. Let $f^{(n)}(\cdot_1, \dots, \cdot_n) \in f^{(n)}$ be a representative of $f^{(n)}$ satisfying (37) and symmetric by the arguments \cdot_1, \dots, \cdot_n , $\dot{g}^{(n+1)} \in g^{(n+1)}$ be a symmetric representative of $g^{(n+1)}$. Denote $\check{f}_{[t_1, t_2]}^{(n)}(u_1, \dots, u_n, u) := \check{f}_u^{(n)}(u_1, \dots, u_n)1_{[t_1, t_2]}(u)$, and let $\widetilde{f}_{[t_1, t_2]}^{(n)}$ be the symmetrization of $\check{f}_{[t_1, t_2]}^{(n)}$ by all arguments. Then, obviously, $\widetilde{f}_{[t_1, t_2]}^{(n)} \in \widetilde{f}_{[t_1, t_2]}^{(n)}$. Using (18) and (19) we obtain

$$\begin{aligned} & \langle \widehat{f}_{[t_1, t_2]}^{(n)}, g^{(n+1)} \rangle_{ext} = \langle \widetilde{f}_{[t_1, t_2]}^{(n)}, \dot{g}^{(n+1)} \rangle_{ext} \\ &= \sum_{\substack{k, l, j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n+1}} \frac{(n+1)!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!} \right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!} \right)^{2s_k} \\ & \times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} \widetilde{f}_{[t_1, t_2]}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \\ & \times \dot{g}^{(n+1)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) du_1 \dots du_{s_1 + \dots + s_k} \\ &= \sum_{\substack{k, l, j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n+1}} \frac{n!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!} \right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!} \right)^{2s_k} \\ & \times \left[\int_{\mathbb{R}_+^{s_1 + \dots + s_k}} \check{f}_{[t_1, t_2]}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \right. \\ & \times \dot{g}^{(n+1)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) du_1 \dots du_{s_1 + \dots + s_k} \\ & + \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} \check{f}_{[t_1, t_2]}^{(n)}(u_{s_1 + \dots + s_k} \underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_{k-1}}) \\ & \times \dot{g}^{(n+1)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) du_1 \dots du_{s_1 + \dots + s_k} \\ & + \dots + \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} \check{f}_{[t_1, t_2]}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1 - 1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}, u_1) \\ & \left. \times \dot{g}^{(n+1)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) du_1 \dots du_{s_1 + \dots + s_k} \right] \\ &= \sum_{\substack{k, l, j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > \dots > l_{k-1}, \\ l_1 s_1 + \dots + l_{k-1} s_{k-1} + (s_k - 1) = n}} \frac{n!}{s_1! \dots s_{k-1}! (s_k - 1)!} \left(\frac{\|p_{l_1}\|_v}{l_1!} \right)^{2s_1} \dots \left(\frac{\|p_{l_{k-1}}\|_v}{l_{k-1}!} \right)^{2s_{k-1}} \\ & \times \int_{\mathbb{R}_+^{s_1 + \dots + (s_k - 1) + 1}} \check{f}_{[t_1, t_2]}^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, u_{s_1 + \dots + s_{k-1} + 1}, \dots, u_{s_1 + \dots + (s_k - 1)}, u) \end{aligned}$$

$$\begin{aligned} & \times \dot{g}^{(n+1)}(\underbrace{u_1, \dots, u_1}_{I_1}, \dots, u_{s_1+\dots+s_{k-1}+1}, \dots, u_{s_1+\dots+(s_k-1)}, u) du_1 \cdots du_{s_1+\dots+(s_k-1)} du \\ & = (\dot{f}^{(n)}, \dot{g}^{(n+1)}(\cdot) 1_{[t_1, t_2]}(\cdot))_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}} = (f^{(n)}, g^{(n+1)}(\cdot) 1_{[t_1, t_2]}(\cdot))_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}}. \end{aligned}$$

Now in order to prove equalities (46) it remains to show that the domains of the corresponding operators coincide.

1) Let us show that $\text{dom}((1_{[t_1, t_2]}(\cdot)\partial.)^*) = \text{dom}(\int_{t_1}^{t_2} \circ(u)\widehat{dL}_u)$. By definition, F belongs to $\text{dom}((1_{[t_1, t_2]}(\cdot)\partial.)^*)$ if and only if $F \in (L^2) \otimes \mathcal{H}$ and

$$(L^2) \supset \text{dom}(1_{[t_1, t_2]}(\cdot)\partial.) \ni G \mapsto (F(\cdot), 1_{[t_1, t_2]}(\cdot)\partial.G)_{(L^2) \otimes \mathcal{H}}$$

is a linear continuous functional. By the Riesz's theorem this means that

$$(F(\cdot), 1_{[t_1, t_2]}(\cdot)\partial.G)_{(L^2) \otimes \mathcal{H}} = (H, G)_{(L^2)}$$

with some $H \in (L^2)$. But it follows from uniqueness of representation (11) for elements of (L^2) and (the proof of) (47) that $H = \int_{t_1}^{t_2} F(u)\widehat{dL}_u$, i.e.,

$$F \in \text{dom}((1_{[t_1, t_2]}(\cdot)\partial.)^*) \Leftrightarrow H \in (L^2) \Leftrightarrow \int_{t_1}^{t_2} F(u)\widehat{dL}_u \in (L^2) \Leftrightarrow F \in \text{dom}\left(\int_{t_1}^{t_2} \circ(u)\widehat{dL}_u\right)$$

(see (41)).

2) Let us show that $\text{dom}\left(\left(\int_{t_1}^{t_2} \circ\widehat{dL}\right)^*\right) = \text{dom}(1_{[t_1, t_2]}(\cdot)\partial.)$. By definition, G belongs to $\text{dom}\left(\left(\int_{t_1}^{t_2} \circ\widehat{dL}\right)^*\right)$ if and only if $G \in (L^2)$ and

$$(L^2) \otimes \mathcal{H} \supset \text{dom}\left(\int_{t_1}^{t_2} \circ(u)\widehat{dL}_u\right) \ni F \mapsto \left(\int_{t_1}^{t_2} F(u)\widehat{dL}_u, G\right)_{(L^2)}$$

is a linear continuous functional. By the Riesz's theorem this is possible if and only if $\left(\int_{t_1}^{t_2} F(u)\widehat{dL}_u, G\right)_{(L^2)} = (F, H)_{(L^2) \otimes \mathcal{H}}$ with some $H \in (L^2) \otimes \mathcal{H}$. But it follows from uniqueness of representation (35) for elements of $(L^2) \otimes \mathcal{H}$ and (the proof of) (47) that $H = 1_{[t_1, t_2]}(\cdot)\partial.G$, i.e.,

$$G \in \text{dom}\left(\left(\int_{t_1}^{t_2} \circ\widehat{dL}\right)^*\right) \Leftrightarrow H \in (L^2) \otimes \mathcal{H} \Leftrightarrow 1_{[t_1, t_2]}(\cdot)\partial.G \in (L^2) \otimes \mathcal{H} \Leftrightarrow G \in \text{dom}(1_{[t_1, t_2]}(\cdot)\partial.)$$

(see (45)). □

Note that equalities (46) can be used as alternative definitions of the extended stochastic integral and the Hida stochastic derivative.

Remark. Equality (47) can be written in the form

$$\left(\int_{t_1}^{t_2} F(u)\widehat{dL}_u, G\right)_{(L^2)} = \int_{t_1}^{t_2} (F(u), \partial_u G)_{(L^2)} du \equiv \int_{t_1}^{t_2} (\partial_u^\dagger F(u), G)_{(L^2)} du,$$

therefore it is natural to write the operator $\int_{t_1}^{t_2} \circ(u)\widehat{dL}_u$ formally as

$$\int_{t_1}^{t_2} \circ(u)\widehat{dL}_u = \int_{t_1}^{t_2} \partial_u^\dagger \circ(u) du$$

(cf. [17]), here ∂_u^\dagger is the formal operator conjugated to the Hida stochastic derivative at the point u (cf. [27, 20]). Strongly speaking, now for fixed $u \in \mathbb{R}_+$ the operators ∂_u and ∂_u^\dagger are not well-defined (e.g., for $G \in (L^2)$ $\partial_u G$ is not uniquely defined), but such operators can be defined on suitable spaces of test and generalized functions respectively; a detailed presentation will be given in another paper.

Let us say several words about possible simple generalizations of the results of the present paper. In the first place, instead of the Lebesgue measure on \mathbb{R}_+ one can use a non-atomic measure σ that satisfies some additional assumptions (cf. [22, 17]). In the second place, one can consider a complex-valued Lévy process, define (L^2) as the space of (classes of) complex-valued functions, and obtain "complex versions" of results presented above (cf. [17]). In the third place, one can consider the operators $\int_\Delta \circ(u) dL_u$ and $1_\Delta(\cdot)\partial$. for any measurable $\Delta \subseteq \mathbb{R}_+$ using 1_Δ instead of $1_{[t_1, t_2]}$ in the corresponding places. Finally, it is possible to construct and to study the extended stochastic integral and the Hida stochastic derivative in the case when one uses instead of \mathbb{R}_+ a much more general space (cf. [22]).

Remark. Since the extended stochastic integral and the Hida stochastic derivative are not continuous operators, it can be some problems with their applications. For example, the Itô stochastic integral has the following property: for any $t_1, t_2, t_3 \in [0, +\infty]$, $t_1 < t_2 < t_3$,

$$\int_{t_1}^{t_2} \circ(u) dL_u + \int_{t_2}^{t_3} \circ(u) dL_u = \int_{t_1}^{t_3} \circ(u) dL_u. \quad (48)$$

This property, in particular, plays an important role in the theory of stochastic differential and integral equations. Formally the extended stochastic integral also satisfies (48), but since the domain of this integral depends on the interval of integration, the application of (48) in some situations can be impossible. In the forthcoming paper we will consider the extended stochastic integral of form (40), (42) and the Hida stochastic derivative of form (44) as linear continuous operators on suitable riggings of (L^2) .

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Качановський М.О. *Про розширені стохастичні інтеграли за процесами Леві* // Карпатські математичні публікації. — 2013. — Т.5, №2. — С. 256–278.

Позначимо через L процес Леві на $[0, +\infty)$. У частинних випадках, коли L — вінерівський чи пуассонівський процес, будь-яку квадратично інтегровну випадкову величину можна розкласти у ряд з повторних стохастичних інтегралів за L від невідповідних функцій. Ця властивість L , відома як властивість хаотичного розкладу (ВХР), відіграє дуже важливу роль у стохастичному аналізі. На жаль, взагалі кажучи, процес Леві не володіє ВХР.

Існують різноманітні узагальнення ВХР для процесів Леві. Зокрема, при підході Іто процес Леві L розкладають у суму гауссівського процесу та стохастичного інтеграла за пуассонівською випадковою мірою, після цього використовують ВХР для обох доданків з метою отримання узагальненої ВХР для L . Підхід Нуаларта та Скоутенса полягає у розкладі квадратично інтегровної випадкової величини у ряд з повторних стохастичних інтегралів від невідповідних функцій за так званими ортогоналізованими центрованими процесами степенів стрибків, ці процеси побудовані з використанням càdlàg версії L . Підхід Литвинова заснований на ортогоналізації неперервних поліномів у просторі квадратично інтегровних випадкових величин.

У цій статті ми будемо розширений стохастичний інтеграл за процесом Леві та стохастичну похідну Хіди у термінах узагальненої ВХР, запропонованої Литвиновим; встановлюємо деякі властивості цих операторів; та, що є найбільш важливим, показуємо, що розширені стохастичні інтеграли, побудовані із застосуванням вищезгаданих узагальнень ВХР, співпадають.

Ключові слова і фрази: процес Леві, властивість хаотичного розкладу, розширений стохастичний інтеграл, стохастична похідна Хіди.

Качановский Н.А. *О расширенных стохастических интегралах по процессам Леви* // Карпатские математические публикации. — 2013. — Т.5, №2. — С. 256–278.

Обозначим через L процесс Леви на $[0, +\infty)$. В частных случаях, когда L — винеровский или пуассоновский процесс, любую квадратично интегрируемую случайную величину можно разложить в ряд из повторных стохастических интегралов по L от неслучайных функций. Это свойство L , известное как свойство хаотического разложения (СХР), играет очень важную роль в стохастическом анализе. К сожалению, вообще говоря, процесс Леви не имеет СХР.

Существуют различные обобщения СХР для процессов Леви. В частности, при подходе Ито процесс Леви L раскладывают в сумму гауссовского процесса и стохастического интеграла по пуассоновской случайной мере, затем используют СХР для обоих слагаемых с целью получения обобщенного СХР для L . Подход Нуаларта и Скоутенса состоит в разложении квадратично интегрируемой случайной величины в ряд из повторных стохастических интегралов от неслучайных функций по так называемым ортогонализированным центрированным процессам степеней скачков, эти процессы сконструированы с использованием càdlàg версии L . Подход Литвинова основан на ортогонализации непрерывных полиномов в пространстве квадратично интегрируемых случайных величин.

В этой статье мы конструируем расширенный стохастический интеграл по процессу Леви и стохастическую производную Хиды в терминах обобщенного СХР, предложенного Литвиновым; устанавливаем некоторые свойства этих операторов; и, что наиболее важно, показываем, что расширенные стохастические интегралы, построенные с использованием вышеупомянутых обобщений СХР, совпадают.

Ключевые слова и фразы: процесс Леви, свойство хаотического разложения, расширенный стохастический интеграл, стохастическая производная Хиды.