ISSN 2075-9827 http://www.journals.pu.if.ua/index.php/cmp
Carpathian Math. Publ. 2014, **6** (1), 4–7 Карпатські матем. публ. 2014, Т.6, №1, С.4–7
doi:10.15330/cmp.6.1.4-7

ANDRUSYAK I.V.[1], FILEVYCH P.V.[2]

RADIAL BOUNDARY VALUES OF LACUNARY POWER SERIES

We strengthened MacLane's theorem concerning radial boundary values of lacunary power series.

Key words and phrases: analytic function, lacunary power series, radial boundary value, asymptotic value.

[1] Lviv Polytechnic National University, 12 Bandera str., 79013, Lviv, Ukraine
[2] Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: andrusyak.ivanna@gmail.com (Andrusyak I.V.), filevych@mail.ru (Filevych P.V.)

INTRODUCTION

Denote by $\mathcal{H}$ the class of analytic functions on the unite disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. As usual, a value $v \in \overline{\mathbb{C}}$ is called the radial boundary value of a function $f \in \mathcal{H}$ at a point $e^{i\theta} \in \partial\mathbb{D}$ if

$$\lim_{r\uparrow 1} f(re^{i\theta}) = v.$$

By $\mathcal{R}$ we denote the class of functions $f \in \mathcal{H}$ having radial boundary values on a dense set of points $e^{i\theta}$ of $\partial\mathbb{D}$. A value $v \in \overline{\mathbb{C}}$ is called an asymptotic value of a function $f \in \mathcal{H}$ at a point $\omega \in \partial\mathbb{D}$ if there exists a path $\gamma : z = z(t)$, $t \in [0,1]$, such that $z(t) \in \mathbb{D}$ for all $t \in [0,1)$, $z(1) = \omega$ and

$$\lim_{t\uparrow 1} f(z(t)) = v.$$

By $\mathcal{A}$ we denote the MacLane class, i.e. the class of functions $f \in \mathcal{H}$ having asymptotic values on a dense set of points ω of $\partial\mathbb{D}$. Clearly, $\mathcal{R} \subset \mathcal{A}$. It is well known that this inclusion is strict. Recall that, by the classical Fatou theorem, for any bounded function $f \in \mathcal{H}$ we have $f \in \mathcal{R}$ and therefore $f \in \mathcal{A}$.

Let Λ be the class of increasing sequences that consists of nonnegative integers $\lambda = (\lambda_n)$. For any sequence $\lambda \in \Lambda$, let

$$q(\lambda) = \varliminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n}.$$

Denote by $\mathcal{H}(\lambda)$ the class of functions $f \in \mathcal{H}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad z \in \mathbb{D}. \tag{1}$$

G.R. MacLane has proved the following theorems (see [1, Theorem 19]).

УДК 517.53
2010 *Mathematics Subject Classification:* 30B10, 30B30.

Theorem A. *Let* $\lambda \in \Lambda$. *If* $q(\lambda) > 3$, *then* $\mathcal{H}(\lambda) \subset \mathcal{R}$.

Theorem B. *Let* $\lambda \in \Lambda$. *If* $q(\lambda) > 3$, *then for any function* $f \in \mathcal{H}(\lambda)$ *of the form* (1) *such that*

$$\sum_{n=0}^{\infty} |a_n| = +\infty \tag{2}$$

there exists a dense set Θ *in* $[0, 2\pi]$ *such that for any* $\theta \in \Theta$ *the following relation holds*

$$\lim_{r \uparrow 1} \operatorname{Re} f(re^{i\theta}) = +\infty. \tag{3}$$

Note, that if for a function f condition (2) is not satisfied, then this function is bounded in $\mathbb{D}$. Therefore Theorem A is a consequence of the Fatou theorem and Theorem B. It is also clear that in Theorem B the value $\operatorname{Re} f(re^{i\theta})$ can be replaced by one of the values $-\operatorname{Re} f(re^{i\theta})$, $\operatorname{Im} f(re^{i\theta})$ or $-\operatorname{Im} f(re^{i\theta})$ (it is sufficient to apply this theorem to the functions $-f$, $-if$ or if respectively).

If we require only the inclusion $\mathcal{H}(\lambda) \subset \mathcal{A}$, then the condition $q(\lambda) > 3$ can be essentially weakened. This fact follows from the following Murai theorem [2].

Theorem C. *Let* $\lambda \in \Lambda$. *If* $q(\lambda) > 1$, *then* $\mathcal{H}(\lambda) \subset \mathcal{A}$.

In connection with the stated results there is a question: *does there exist* $q \in [1, 3)$ *such that the condition* $q(\lambda) > q$ *is sufficient for the inclusion* $\mathcal{H}(\lambda) \subset \mathcal{R}$?

From our results we can conclude that the condition $q(\lambda) > 3$ in Theorem A is far from being final. Despite this, an answer to the question posed above is not obtained.

Theorem 1. *For any* $q > 1$ *there exists a sequence* $\lambda \in \Lambda$ *such that* $q(\lambda) = q$ *and* $\mathcal{H}(\lambda) \subset \mathcal{R}$.

For a sequence $\lambda \in \Lambda$ let

$$q_1(\lambda) = \min\left\{ \varliminf_{k\to\infty} \frac{\lambda_{2k+1}}{\lambda_{2k}}, \varliminf_{k\to\infty} \frac{\lambda_{2k+2}}{\lambda_{2k+1}} \right\}, \quad q_2(\lambda) = \max\left\{ \varliminf_{k\to\infty} \frac{\lambda_{2k+1}}{\lambda_{2k}}, \varliminf_{k\to\infty} \frac{\lambda_{2k+2}}{\lambda_{2k+1}} \right\}.$$

Theorem 1 is a direct consequence of the Fatou theorem and Theorem 2 stated below, which strengthens Theorem B.

Theorem 2. *Let* $\lambda \in \Lambda$. *If*

$$(q_1(\lambda) - 1) q_2(\lambda) > 6, \tag{4}$$

then for any function $f \in \mathcal{H}(\lambda)$ *of the form* (1) *which satisfies condition* (2) *there exists a dense set* Θ *in* $[0, 2\pi]$ *such that for any* $\theta \in \Theta$ *equality* (3) *holds.*

PROOF OF THEOREM 2

Let for any sequence $\lambda \in \Lambda$ inequality (4) holds. Put

$$p_1 = \varliminf_{k\to\infty} \frac{\lambda_{2k+1}}{\lambda_{2k}}, \quad p_2 = \varliminf_{k\to\infty} \frac{\lambda_{2k+2}}{\lambda_{2k+1}}.$$

Suppose that $p_1 \le p_2$ (in the case $p_1 \ge p_2$ our considerations are similar). Then $q_1(\lambda) = p_1$, $q_2(\lambda) = p_2$, and condition (4) can be written as $(p_1 - 1)p_2 > 6$. It is clear that $p_1 > 1$ and $p_2 > 3$, therefore there exist constants $q_1 \in (1, p_1)$ and $q_2 \in (3, p_2)$ such that $(q_1 - 1)q_2 > 6$, moreover $q_1 < 3$. From the definitions of variables p_1 and p_2 it follows that there exists an integer $k_0 \in \mathbb{N}_0$ such that for all integers $k \ge k_0$ the following inequalities $\lambda_{2k+1} \ge q_1\lambda_{2k}$ and $\lambda_{2k+2} \ge q_2\lambda_{2k+1}$ hold.

In what follows for each segment $I \subset \mathbb{R}$ we denote by $|I|$, $a(I)$, and $b(I)$ its length, the left end and right end respectively.

Consider any segment $I \subset [0, 2\pi]$ and a function $f \in \mathcal{H}(\lambda)$ of the form (1), which satisfies condition (2). Let us prove that there exists a point θ in the segment I such that relation (3) holds. Let Θ be the set of all $\theta \in [0, 2\pi]$, for which (3) holds. Then the set Θ is dense in $[0, 2\pi]$.

Put

$$\varepsilon = \frac{((q_1-1)q_2-6)\pi}{(q_1+1)q_2-2}. \tag{5}$$

It is easy to check that

$$\varepsilon < \frac{(q_1-1)\pi}{q_1+1}. \tag{6}$$

Take $\delta = \cos\frac{\pi-\varepsilon}{2}$. Since $\varepsilon \in (0, \pi)$, we have $\delta > 0$.

Let $n \in \mathbb{N}$, $\alpha_n = \arg a_n$. Then we have $\cos(\lambda_n\theta + \alpha_n) \ge \delta$ on the union of segments

$$\Big[-\frac{\pi-\varepsilon}{2\lambda_n} + \frac{2\pi m-\alpha_n}{\lambda_n}, \frac{\pi-\varepsilon}{2\lambda_n} + \frac{2\pi m-\alpha_n}{\lambda_n}\Big], \qquad m \in \mathbb{Z}, \tag{7}$$

of length $\frac{\pi-\varepsilon}{\lambda_n}$. Obviously, if $n_0 = \min\Big\{n \in \mathbb{N} : |I| \ge \frac{3\pi-\varepsilon}{\lambda_n}\Big\}$, then for every integer $n \ge n_0$ the segment I contains at least one of the segments (7).

Fix an integer $m \ge \max\big\{k_0, \frac{n_0}{2}\big\}$ and let $I_{2m} \subset I$ be a segment of length $\frac{\pi-\varepsilon}{\lambda_{2m}}$ such that $\cos(\lambda_{2m}\theta + \alpha_{2m}) \ge \delta$ for all $\theta \in I_{2m}$. By θ_{2m} we denote the midpoint of the segment I_{2m}. Then $I_{2m} = \Big[\theta_{2m} - \frac{\pi-\varepsilon}{2\lambda_{2m}},\ \theta_{2m} + \frac{\pi-\varepsilon}{2\lambda_{2m}}\Big]$.

Let θ_{2m+1} be a point in the set $\{\theta \in \mathbb{R} : \cos(\lambda_{2m+1}\theta + \alpha_{2m+1}) = -1\}$ that is closest to θ_{2m}. Clearly, $|\theta_{2m+1} - \theta_{2m}| \le \frac{\pi}{\lambda_{2m+1}}$ and $\cos(\lambda_{2m+1}\theta + \alpha_{2m+1}) \ge \delta$ for each segments

$$S_1 = \Big[\theta_{2m+1} - \frac{3\pi-\varepsilon}{2\lambda_{2m+1}},\ \theta_{2m+1} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}}\Big], \qquad S_2 = \Big[\theta_{2m+1} + \frac{\pi+\varepsilon}{2\lambda_{2m+1}},\ \theta_{2m+1} + \frac{3\pi-\varepsilon}{2\lambda_{2m+1}}\Big].$$

Put

$$x = \frac{(q_1-1)\pi - (q_1+1)\varepsilon}{2\lambda_{2m+1}}.$$

Then, according to (6), $x > 0$. Let us show that there exists a segment $I_{2m+1} \subset I_{2m}$ of length x such that $\cos(\lambda_{2m+1}\theta + \alpha_{2m+1}) \ge \delta$ for all $\theta \in I_{2m+1}$.

If $\theta_{2m} - \frac{\pi}{\lambda_{2m+1}} \le \theta_{2m+1} \le \theta_{2m}$, then let $I_{2m+1} = \Big[\theta_{2m+1} + \frac{\pi+\varepsilon}{2\lambda_{2m+1}},\ \theta_{2m+1} + \frac{\pi+\varepsilon}{2\lambda_{2m+1}} + x\Big]$. It is clear that $|I_{2m+1}| = x$ and $a(I_{2m+1}) = a(S_2)$. Since $1 < q_1 < 3$, we have

$$b(I_{2m+1}) = \theta_{2m+1} + \frac{\pi+\varepsilon}{2\lambda_{2m+1}} + x = \theta_{2m+1} + \frac{q_1(\pi-\varepsilon)}{2\lambda_{2m+1}} < \theta_{2m+1} + \frac{3\pi-\varepsilon}{2\lambda_{2m+1}} = b(S_2).$$

Thus $I_{2m+1} \subset S_2$, therefore $\cos(\lambda_{2m+1}\theta + \alpha_{2m+1}) \ge \delta$ for all $\theta \in I_{2m+1}$. In addition, $I_{2m+1} \subset I_{2m}$, because

$$a(I_{2m+1}) = \theta_{2m+1} + \frac{\pi+\varepsilon}{2\lambda_{2m+1}} \ge \theta_{2m} - \frac{\pi}{\lambda_{2m+1}} + \frac{\pi+\varepsilon}{2\lambda_{2m+1}} > \theta_{2m} - \frac{\pi-\varepsilon}{2\lambda_{2m}} = a(I_{2m}),$$

$$b(I_{2m+1}) = \theta_{2m+1} + \frac{\pi+\varepsilon}{2\lambda_{2m+1}} + x \le \theta_{2m} + \frac{q_1(\pi-\varepsilon)}{2\lambda_{2m+1}} \le \theta_{2m} + \frac{\pi-\varepsilon}{2\lambda_{2m}} = b(I_{2m}).$$

If $\theta_{2m} \le \theta_{2m+1} \le \theta_{2m} + \frac{\pi}{\lambda_{2m+1}}$, then let $I_{2m+1} = \left[\theta_{2m+1} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}} - x,\ \theta_{2m+1} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}}\right]$. It is clear that $|I_{2m+1}| = x$ and $b(I_{2m+1}) = b(S_1)$. Since $1 < q_1 < 3$, we obtain

$$a(I_{2m+1}) = \theta_{2m+1} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}} - x = \theta_{2m+1} - \frac{q_1(\pi-\varepsilon)}{2\lambda_{2m+1}} > \theta_{2m+1} - \frac{3\pi-\varepsilon}{2\lambda_{2m+1}} = a(S_1).$$

Thus $I_{2m+1} \subset S_1$, therefore $\cos(\lambda_{2m+1}\theta + \alpha_{2m+1}) \ge \delta$ for all $\theta \in I_{2m+1}$. In addition, $I_{2m+1} \subset I_{2m}$, because

$$b(I_{2m+1}) = \theta_{2m+1} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}} \le \theta_{2m} + \frac{\pi}{\lambda_{2m+1}} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}} < \theta_{2m} + \frac{\pi-\varepsilon}{2\lambda_{2m}} = b(I_{2m}),$$

$$a(I_{2m+1}) = \theta_{2m+1} - \frac{\pi+\varepsilon}{2\lambda_{2m+1}} - x \ge \theta_{2m} - \frac{q_1(\pi-\varepsilon)}{2\lambda_{2m+1}} \ge \theta_{2m} - \frac{\pi-\varepsilon}{2\lambda_{2m}} = a(I_{2m}).$$

From the aforementioned properties it follows the existence of segment I_{2m+1}.

Further, using the inequality $\lambda_{2m+1} \le \lambda_{2m+2}/q_2$ and equality (5) we obtain

$$|I_{2m+1}| = x \ge \frac{(q_1-1)q_2\pi - (q_1+1)q_2\varepsilon}{2\lambda_{2m+2}} = \frac{3\pi-\varepsilon}{\lambda_{2m+2}},$$

whence we see that there exists a segment $I_{2m+2} \subset I_{2m+1}$ of length $\frac{\pi-\varepsilon}{\lambda_{2m+2}}$ such that the inequality $\cos(\lambda_{2m+2}\theta + \alpha_{2m+2}) \ge \delta$ holds for all $\theta \in I_{2m+2}$.

The analysis of our considerations shows that by induction it is possible to construct a system of embedded segments $I_{2m} \supset I_{2m+1} \supset I_{2m+2} \supset I_{2m+3} \supset \dots$ such that for every integer $n \ge 2m$ the inequality $\cos(\lambda_n\theta + \alpha_n) \ge \delta$ holds for all $\theta \in I_n$. Let θ be the common point of all segments I_n, $n \ge 2m$. Then $\theta \in I$ and

$$\operatorname{Re} f(re^{i\theta}) = \sum_{n=0}^{\infty} |a_n| r^n \cos(\lambda_n\theta + \alpha_n) \ge -\sum_{n<2m} |a_n| r^n + \delta \sum_{n\ge 2m} |a_n| r^n,$$

whence, according to (2), we obtain (3). Theorem is proved.

REFERENCES

[1] MacLane G.R. Asumptotic values of holomorphic functions. In: Norbeck E. (Ed.) Monograph in Mathematics, **49** (1). William Marsh Rice University, Houston, 1963.

[2] Murai T. *The boundary behaviour of Hadamard lacunary series*. Nagoya Math. J. 1983, **89**, 65–76.

Received 16.03.2014

Андрусяк І.В., Філевич П.В. *Радіальні граничні значення лакунарних степеневих рядів* // Карпатські матем. публ. — 2014. — Т.6, №1. — С. 4–7.

Підсилено теорему Мак-Лейна про граничні значення лакунарних степеневих рядів.

Ключові слова і фрази: аналітична функція, лакунарний степеневий ряд, радіальне граничне значення, асимптотичне значення.

Андрусяк И.В., Филевич П.В. *Радиальные предельные значения лакунарных степенных рядов* // Карпатские матем. публ. — 2014. — Т.6, №1. — С. 4–7.

Усилена теорема Мак-Лейна о радиальных граничных значениях лакунарных степенных рядов.

Ключевые слова и фразы: аналитическая функция, лакунарный степенной ряд, радиальное граничное значение, асимптотическое значение.