ISSN 2075-9827 `http://www.journals.pu.if.ua/index.php/cmp`
Carpathian Math. Publ. 2014, **6** (1), 161–165 Карпатськi матем. публ. 2014, Т.6, №1, С.161–165
doi:10.15330/cmp.6.1.161-165

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(δ, γ)-DUNKL LIPSCHITZ FUNCTIONS IN THE SPACE $\mathrm{L}^2(\mathbb{R}, |x|^{2\alpha+1}dx)$

Using a generalized Dunkl translation, we obtain an analog of Theorem 5.2 in Younis' paper [2] for the Dunkl transform for functions satisfying the (δ, γ)-Dunkl Lipschitz condition in the space $\mathrm{L}^2(\mathbb{R}, |x|^{2\alpha+1}dx)$.

Key words and phrases: Dunkl operator, Dunkl transform, generalized Dunkl translation.

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INTRODUCTION AND PRELIMINARIES

Younis Theorem 5.2 [2] characterizes the set of functions in $\mathrm{L}^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms; namely, we have the following statement.

Theorem 1 ([2]). *Let $f \in \mathrm{L}^2(\mathbb{R})$. Then the following are equivalent:*

1) $\|f(x+h) - f(x)\|_2 = O\left(\frac{h^\alpha}{(\log \frac{1}{h})^\beta}\right)$ *as* $h \to 0$, $0 < \alpha < 1, \beta > 0$,

2) $\int_{|x|\geq r} |\mathcal{F}(f)(x)|^2 dx = O\left(\frac{r^{-2\alpha}}{(\log r)^{2\beta}}\right)$ *as* $r \to +\infty$,

where $\mathcal{F}$ stands for the Fourier transform of f.

In this paper we obtain an analog of Theorem 1 for the Dunkl transform. For this purpose we use a generalized Dunkl translation.

Assume that $\mathrm{L}_{2,\alpha} = \mathrm{L}^2(\mathbb{R}, |x|^{2\alpha+1}dx)$, $\alpha > -\frac{1}{2}$, is the Hilbert space of measurable functions $f(t)$ on $\mathbb{R}$ with the norm

$$\|f\|_{2,\alpha} = \left(\int_{\mathbb{R}} |f(t)|^2 |t|^{2\alpha+1} dt\right)^{1/2}.$$

The Dunkl operator is a differential-difference operator D which satisfies the condition

$$\mathrm{D}f(x) = \frac{df}{dx}(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x}.$$

Let $j_\alpha(x)$ be a normalized Bessel function of the first kind, i.e.,

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{x}{2}\right)^{2n}.$$

УДК 517.44
2010 *Mathematics Subject Classification:* 44A35.

The Dunkl kernel is defined by

$$e_\alpha(x) = j_\alpha(x) + ic_\alpha x j_{\alpha+1}(x),$$

where $c_\alpha = (2\alpha+2)^{-1}$. The function $y = e_\alpha(x)$ satisfies the equation $\mathrm{D}y = iy$ with the initial condition $y(0) = 1$. In the limit case with $\alpha = -\frac{1}{2}$ the Dunkl kernel coincides with the usual exponential function e^{ix}.

Lemma 1 ([1]). *For $x \in \mathbb{R}$ the following inequalities are fulfilled*

(i) $|e_\alpha(x)| \le 1$,

(ii) $|1 - e_\alpha(x)| \le 2|x|$,

(iii) $|1 - e_\alpha(x)| \ge c$ *with* $|x| \ge 1$, *where* $c > 0$ *is a certain constant which depends only on* α.

The Dunkl transform is the integral transform

$$\widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e_\alpha(\lambda x)|x|^{2\alpha+1}dx.$$

The inverse Dunkl transform is defined by the formula

$$f(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-2}\int_{-\infty}^{\infty}\widehat{f}(\lambda)e_\alpha(-\lambda x)|\lambda|^{2\alpha+1}d\lambda.$$

The Dunkl transform satisfies the Parseval's equality ($f \in \mathrm{L}_{2,\alpha}$)

$$\|f\|_{2,\alpha} = (2^{\alpha+1}\Gamma(\alpha+1))^{-1}\|\widehat{f}\|_{2,\alpha}.$$

Consider the generalized Dunkl translation T_h in $\mathrm{L}_{2,\alpha}$, defined by

$$\mathrm{T}_h f(x) = C\left(\int_0^\pi f_e(G(x,h,\varphi))h^e(x,h,\varphi)\sin^{2\alpha}\varphi\, d\varphi + \int_0^\pi f_0(G(x,h,\varphi))h^0(x,h,\varphi)\sin^{2\alpha}\varphi\, d\varphi\right),$$

where

$$C = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}, \quad G(x,h,\varphi) = \sqrt{x^2+h^2-2|xh|\cos\varphi}, \quad h^e(x,h,\varphi) = 1 - \mathrm{sgn}(xh)\cos\varphi\,,$$

and

$$h^0(x,h,\varphi) = \frac{(x+h)h^e(x,h,\varphi)}{G(x,h,\varphi)} \quad \text{for } (x,h) \neq (0,0), \qquad h^0(x,h,\varphi) = 0 \quad \text{for } (x,h) = (0,0),$$

$$f_e(x) = \frac{1}{2}(f(x)+f(-x)), \qquad f_0(x) = \frac{1}{2}(f(x)-f(-x)).$$

From [1] we have: if $f \in \mathrm{L}_{2,\alpha}$, then

$$\widehat{(\mathrm{T}_h f)}(\lambda) = e_\alpha(\lambda h)\widehat{f}(\lambda). \tag{1}$$

MAIN RESULT

In this section we give the main result of this paper. We need first to define (δ,γ)-Dunkl Lipschitz class.

Definition. *Let $0<\delta<1$ and $\gamma>0$. A function $f\in L_{2,\alpha}$ is said to be in the (δ,γ)-Dunkl Lipschitz class, denoted by $Lip(\delta,\gamma,2)$, if*

$$\|T_h f(t)-f(t)\|_{2,\alpha}=O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right)\quad \text{as } h\to 0.$$

Theorem 2. *Let $f\in L_{2,\alpha}$. Then the following conditions are equivalent*

1. $f\in Lip(\delta,\gamma,2)$,
2. $\int_{|\lambda|\geq r}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda=O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right)$ *as* $r\to+\infty$.

Proof. 1) $\Longrightarrow$ 2) Assume that $f\in Lip(\delta,\gamma,2)$. Then we have

$$\|T_h f(t)-f(t)\|_{2,\alpha}=O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right)\quad \text{as } h\to 0.$$

Formula (1) and Parseval's equality give

$$\|T_h f(t)-f(t)\|_{2,\alpha}^2=\frac{1}{(2^{\alpha+1}\Gamma(\alpha+1))^2}\int_{-\infty}^{+\infty}|1-e_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda.$$

If $|\lambda|\in[\frac{1}{h},\frac{2}{h}]$, then $|\lambda h|\geq 1$ and (iii) of Lemma 1 implies that $1\leq\frac{1}{c^2}|1-e_\alpha(\lambda h)|^2$. Then

$$\begin{aligned}
\int_{\frac{1}{h}\leq|\lambda|\leq\frac{2}{h}}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &\leq \frac{1}{c^2}\int_{\frac{1}{h}\leq|\lambda|\leq\frac{2}{h}}|1-e_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\
&\leq \frac{1}{c^2}\int_{-\infty}^{+\infty}|1-e_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\
&\leq \frac{1}{c^2}(2^{\alpha+1}\Gamma(\alpha+1))^2\|T_h f(t)-f(t)\|_{2,\alpha}^2\\
&= O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right).
\end{aligned}$$

We obtain

$$\int_{r\leq|\lambda|\leq 2r}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\leq C\frac{r^{-2\delta}}{(\log r)^{2\gamma}},$$

where C is a positive constant. Now,

$$\begin{aligned}
\int_{|\lambda|\geq r}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= \left[\int_{r\leq|\lambda|\leq 2r}+\int_{2r\leq|\lambda|\leq 4r}+\int_{4r\leq|\lambda|\leq 8r}+\dots\right]|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\
&\leq C\frac{r^{-2\delta}}{(\log r)^{2\gamma}}+C\frac{(2r)^{-2\delta}}{(\log 2r)^{2\gamma}}+C\frac{(4r)^{-2\delta}}{(\log 4r)^{2\gamma}}+\dots\\
&\leq C\frac{r^{-2\delta}}{(\log r)^{2\gamma}}(1+2^{-2\delta}+(2^{-2\delta})^2+(2^{-2\delta})^3+\dots)\leq CC_\delta\frac{r^{-2\delta}}{(\log r)^{2\gamma}},
\end{aligned}$$

where $C_\delta = (1-2^{-2\delta})^{-1}$ since $2^{-2\delta} < 1$.

Consequently

$$\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right) \text{ as } r \to +\infty.$$

2) $\Longrightarrow$ 1) Suppose now that

$$\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right) \text{ as } r \to +\infty.$$

We write

$$\int_{-\infty}^{+\infty} |1-e_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = \mathrm{I}_1 + \mathrm{I}_2,$$

where

$$\mathrm{I}_1 = \int_{|\lambda|<\frac{1}{h}} |1-e_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda, \qquad \mathrm{I}_2 = \int_{|\lambda|\geq\frac{1}{h}} |1-e_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda.$$

Firstly, we use the formulas $|e_\alpha(\lambda h)| \leq 1$ and

$$\mathrm{I}_2 \leq 4\int_{|\lambda|\geq\frac{1}{h}} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda = O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right).$$

Set

$$\psi(x) = \int_x^{+\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda.$$

Integrating by parts we obtain

$$\begin{aligned}\int_0^x \lambda^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= \int_0^x -\lambda^2\psi'(x)dx = -x^2\psi(x) + 2\int_0^x \lambda\psi(\lambda)d\lambda \\ &\leq C_1\int_0^x \lambda\lambda^{-2\delta}(\log\lambda)^{-2\gamma}d\lambda = O(x^{2-2\delta}(\log x)^{-2\gamma}),\end{aligned}$$

where C_1 is a positive constant.

We use the formula (ii) of lemma 1:

$$\begin{aligned}\int_{-\infty}^{+\infty} |1-e_\alpha(\lambda h)|^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= O(h^2\int_{|\lambda|<\frac{1}{h}} \lambda^2|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda) + O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right) \\ &= O\left(h^2h^{2\delta-2}(\log\frac{1}{h})^{-2\gamma}\right) + O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right) \\ &= O\left(\frac{h^{2\delta}}{(\log\frac{1}{h})^{2\gamma}}\right)\end{aligned}$$

and this ends the proof. □

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Received 17.07.2013

Ель Хамма М.Е., Лахлалі Х., Дахер Р. (δ,γ)-*Данкл-Ліпшицеві функції в просторі* $L^2(\mathbb{R},|x|^{2\alpha+1}dx)$ // Карпатські матем. публ. — 2014. — Т.6, №1. — С. 161–165.

За допомогою узагальненого зсуву Данкла отримано аналог теореми 5.2 зі статті Юніса [2] для перетворення Данкла для функцій, що задовольняють умову (δ,γ)-Данкла-Ліпшиця в просторі $L^2(\mathbb{R},|x|^{2\alpha+1}dx)$.

Ключові слова і фрази: оператор Данкла, перетворення Данкла, узагальнений зсув Данкла.

Эль Хамма М., Лахлали Х., Дахер Р. (δ,γ)-*Данкл-Липшицевы функции в пространстве* $L^2(\mathbb{R},|x|^{2\alpha+1}dx)$ // Карпатские матем. публ. — 2014. — Т.6, №1. — С. 161–165.

С помощью обобщенного сдвига Данкла получен аналог теоремы 5.2 из статьи Юниса [2] для преобразования Данкла для функций, удовлетворяющих (δ,γ)-Данкл-Липшицево условие в пространстве $L^2(\mathbb{R},|x|^{2\alpha+1}dx)$.

Ключевые слова и фразы: оператор Данкла, преобразование Данкла, обобщенный сдвиг Данкла.