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HOMOMORPHISMS OF THE ALGEBRA OF SYMMETRIC ANALYTIC FUNCTIONS ON ℓ_1

The algebra $\mathcal{H}_{bs}(\ell_1)$ of symmetric analytic functions of bounded type is investigated. In particular, we study continuity of some homomorphisms of the algebra of symmetric polynomials on ℓ_p and composition operators of the algebra of symmetric analytic functions. The paper contains several open questions.

Key words and phrases: polynomials and analytic functions on Banach spaces, symmetric polynomials, spectra of algebras.

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INTRODUCTION

Let X be a complex Banach space. By a *symmetric function* on X we mean a function which is invariant with respect to a semigroup of isometric operators on X. In the case $X = \ell_p$ by a symmetric function on ℓ_p we mean a function which is invariant under any reordering of a sequence in ℓ_p.

Let us denote by $\mathcal{P}(\ell_p)$ the algebra of all polynomials on ℓ_p, $1 \le p < \infty$, and by $\mathcal{P}_s(\ell_p)$ the algebra of all symmetric polynomials on ℓ_p. The completion of $\mathcal{P}(\ell_p)$ in the metric of uniform convergence on bounded sets coincides with the algebra of entire analytic functions of bounded type $\mathcal{H}_b(\ell_p)$ on ℓ_p. We use the notations $\mathcal{H}_{bs}(\ell_p)$ for the subalgebra of all symmetric analytic functions in $\mathcal{H}_b(\ell_p)$. Also we use the notation $\mathcal{M}_{bs}(\ell_p)$ for the spectrum (the set of all non-null continuous complex-valued homomorphisms) of the algebra $\mathcal{H}_{bs}(\ell_p)$.

Symmetric polynomials on rearrangement-invariant function spaces were studied in [7, 8]. In [7] it is proved that the polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \qquad k = \lceil p \rceil, \lceil p \rceil + 1, \dots \tag{1}$$

form an algebraic basis in the algebra of all symmetric polynomials on ℓ_p, where $\lceil p \rceil$ is the smallest integer that is greater than or equal to p.

Spectra of algebras of analytic functions were studied in [2, 3, 9, 10]. The spectrum of the algebra $\mathcal{H}_{bs}(\ell_p)$ was investigated in [4–6].

Recall that for any $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ and $f \in \mathcal{H}_{bs}(\ell_p)$, the *symmetric convolution* $\varphi \star \theta$ was defined in [4] as follows

$$(\varphi \star \theta)(f) = \varphi(\theta[T_y^s(f)]),$$

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where $T_y^s(f)(x) = f(x \bullet y) := (x_1, y_1, x_2, y_2, \ldots)$, $x, y \in \ell_p$, $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$.

Let $x, y \in \ell_p$, $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$. In [6] the *multiplicative intertwining* of x and y, $x \diamond y$, was defined as the resulting sequence of ordering the set $\{x_i y_j : \ i, j \in \mathbb{N}\}$ with one single index in some fixed order. It enabled us to define the *multiplicative convolution operator* as a mapping $f \mapsto M_y(f)$, where $M_y(f)(x) = f(x \diamond y)$. And for arbitrary $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ in [6] it was defined their *multiplicative convolution* $\varphi \Diamond \theta$ according to

$$(\varphi \Diamond \theta)(f) = \varphi(\theta[M_x(f)]) \text{ for every } f \in \mathcal{H}_{bs}(\ell_p).$$

Using the symmetric convolution operation and the multiplicative convolution operator in the spectrum of the algebra $\mathcal{H}_{bs}(\ell_1)$, a representation of $\mathcal{M}_{bs}(\ell_1)$ in terms of entire functions of exponential type was obtained.

In this paper we continue to investigate the algebra $\mathcal{H}_{bs}(\ell_1)$ of all symmetric analytic functions on ℓ_1 that are bounded on bounded sets. In particular, we study continuity of some homomorphisms (linear multiplicative operators) of the algebra of symmetric polynomials on ℓ_p and composition operators of the algebra of symmetric analytic functions.

1 CONTINUOUS AND DISCONTINUOUS HOMOMORPHISMS

Let us recall that in [5] it was constructed a family $\{\psi_\lambda : \lambda \in \mathbb{C}\}$ of elements of the set $\mathcal{M}_{bs}(\ell_p)$ such that $\psi_\lambda(F_p) = \lambda$ and $\psi_\lambda(F_k) = 0$ for $k > p$.

Proposition 1. *The homomorphism* $\Gamma : \mathcal{P}_s(\ell_1) \to \mathcal{P}_s(\ell_1)$, *such that* $\Gamma : F_n \mapsto F_{n-1}$, *(in particular,* $\Gamma : F_1 \mapsto 0$,*) is discontinuous.*

Proof. Since $\psi_\lambda \circ F_1 = \lambda$ and $\psi_\lambda \circ F_k = 0$ when $k \neq 1$, we have that $\psi_\lambda \circ \Gamma(F_2) = \lambda$ and $\psi_\lambda \circ \Gamma(F_k) = 0$, $k \neq 2$. It follows that $\psi_\lambda \circ \Gamma$ is discontinuous and we obtain that Γ is discontinuous too. □

Note that Γ acts in the natural way from $\mathcal{P}_s(\ell_2)$ into $\mathcal{P}_s(\ell_1)$.

Question 1. *Does the homomorphism* $\Gamma : \mathcal{P}_s(\ell_2) \longrightarrow \mathcal{P}_s(\ell_1)$ *is discontinuous?*

Proposition 2. *The homomorphism* $\Delta : \mathcal{P}_s(\ell_1) \longrightarrow \mathcal{P}_s(\ell_1)$, $\Delta : F_{n-1} \longmapsto F_n$, *is discontinuous.*

Proof. Let us define

$$m(P(x)) := P(-x) = (-1)^{\deg P} P(x),$$

where P is a homogeneous polynomial. It is easy to see that m is continuous and $m(F_k) = (-1)^k F_k$.

We have $m \circ \Delta \circ m \circ \Delta(F_n) = -F_{n+2}$. Let $x \in \ell_1$, $x \neq 0$. Let us define

$$\Theta_x := \delta_x \circ m \circ \Delta \circ m \circ \Delta.$$

Then $\Theta_x(F_n) = -F_{n+2}(x)$.

Let $x_0 = (-1, 0, 0, \ldots)$. It is easy to see that $\delta_{x_0}(F_n) = \begin{cases} -1, & \text{if} \quad n = 2k-1, \\ 1, & \text{if} \quad n = 2k. \end{cases}$

We have $\Theta_{x_0}(F_n) : (F_1, F_2, \ldots) \longmapsto (0, 0, 1, -1, 1, -1, \ldots)$. According to [5, Theorem 1.6] we have that

$$(\delta_{x_0} \star \Theta_{x_0})(F_1) = \delta_{x_0}(F_1) + \Theta_{x_0}(F_1) = -1 + 0 = -1.$$

Similarly,

$$(\delta_{x_0} \star \Theta_{x_0})(F_2) = 1$$

and

$$(\delta_{x_0} \star \Theta_{x_0})(F_k) = 0 \quad \text{if} \quad k > 2.$$

Hence we obtain that Δ is discontinuous. □

Remark 1. *Propositions 1 and 2 are also true for homomorphisms* $\Gamma : \mathcal{P}_s(\ell_p) \longrightarrow \mathcal{P}_s(\ell_p)$ *and* $\Delta : \mathcal{P}_s(\ell_p) \longrightarrow \mathcal{P}_s(\ell_p)$.

2 COMPOSITION OPERATORS

In this section we consider some homomorphisms which are composition operators, and study their continuity.

1. Let $R : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ be an analytic mapping, $R = (R_1, \ldots, R_m)$. Let us define $T_R : (F_1, \ldots, F_m) \longmapsto (R_1(F_1, \ldots, F_m), \ldots, R_m(F_1, \ldots, F_m))$, that is

$$T_R(F_k) = R_k(F_1, \ldots, F_m).$$

Let P be a symmetric polynomial of degree m on ℓ_1. Then, as it was mentioned above, there exists a polynomial q on $\mathbb{C}^m$ such that $P(x) = q(F_1(x), \ldots, F_m(x))$. Applying T_R we obtain that

$$T_R(P) = q(R_1(F_1, \ldots, F_m), \ldots, R_m(F_1, \ldots, F_m)).$$

Proposition 3. *If* $R : t_n \longmapsto a_n t_n + c_n$, *where* $a_n = \varphi(F_n)$ *for some* $\varphi \in \mathcal{M}_{bs}$ *and* $c_n = \psi(F_n)$ *for some* $\psi \in \mathcal{M}_{bs}$, *then* T_R *is continuous.*

In this case $T_R(f) = (\delta_x \Diamond \varphi) \star \psi(f)$ *for every* $f \in \mathcal{H}_{bs}(\ell_1)$.

Question 2. *For which more* R *the mapping* T_R *is continuous?*

2. Let us consider now an analytic function of one variable $h(t)$ and define

$$T_h(F_k(x)) := \sum_{n=1}^{\infty} (h(x_n))^k.$$

Proposition 4. *The operator* T_h *is continuous.*

Proof. The continuity of T_h can be proved directly. □

3. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of symmetric polynomials such that for every $x \in \ell_1$ the sequence $(P_1(x), \ldots, P_n(x), \ldots) \in \ell_1$.

Let us denote by P a mapping $x \longmapsto (P_1(x), \ldots, P_n(x), \ldots)$. Also for every $f \in \mathcal{H}_{bs}(\ell_1)$ we define

$$C_P(f)(x) := f \circ P(x).$$

Proposition 5. *The composition operator* $C_P(f)$ *is continuous.*

Theorem 1. *Let* $G : \ell_1 \longrightarrow \ell_1$ *be an analytic operator of bounded type. G commutes with permutation operators (in the sense that* $G(\sigma_1 x) = \sigma_2 G(x)$, *where* σ_1, σ_2 *are permutations on the set of positive integers) if and only if the operator* $C_G(f)(x) := f \circ G(x)$, *where* $x \in \ell_1$, $f \in \mathcal{H}_{bs}(\ell_1)$, *is homomorphism.*

Proof. If G commutes with permutation operators, then

$$f(G(\sigma_1 x)) = f(\sigma_2(G(x))) = f(G(x)) \in \mathcal{H}_{bs}(\ell_1).$$

On the contrary: suppose that G does not commute with σ_1, i.e. there exists x such that $G(\sigma_1 x) \neq \sigma_2 G(x)$ for any σ_2. Then there exists G_n such that $G_n(G(\sigma_1 x)) \neq G_n(G(x))$, since $G(\sigma_1 x) \not\sim G(x)$. Hence $G_n \circ G \notin \mathcal{H}_{bs}(\ell_1)$, and we have a contradiction. □

4. Let $P_k \in \mathcal{P}_s(\ell_1)$ and $(P_1(x), P_2(x), \ldots, P_n(x), \ldots) \in \ell_\infty$ for any $x \in \ell_1$. Let us define

$$V_n = \left(\frac{P_1(x)}{n}, \frac{P_2(x)}{n}, \ldots, \frac{P_n(x)}{n}, 0, 0, \ldots\right)$$

and let $\mathcal{U}$ be an arbitrary ultrafilter on $\mathbb{N}$.

Define

$$C_V(f) = \lim_{\mathcal{U}} f(V_n(x)),$$

where f is an arbitrary symmetric analytic function of bounded type on ℓ_1. By constructions of C_V and [1, Example 3.1] it is easy to see that $C_V(F_k) = 0$ if $k > 1$ and $C_V(F_1) \neq 0$ in the generale case.

Proposition 6. *C_V is a continuous operator.*

Theorem 2. *Let $F : \mathcal{H}_{bs}(\ell_1) \longrightarrow \mathcal{H}_{bs}(\ell_1)$ be a homomorphism. Then there exists a mapping $\Lambda : \mathcal{M}_{bs}(\ell_1) \longrightarrow \mathcal{M}_{bs}(\ell_1)$ such that*

$$F(f)(x) = \widehat{f}(\Lambda(\delta_x)), \tag{2}$$

where $f \in \mathcal{H}_{bs}(\ell_1)$ and $\widehat{f}$ is the Gelfand transform of f.

Proof. Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$, then $\psi = \varphi \circ F \in \mathcal{M}_{bs}(\ell_1)$. Let us put $\Lambda(\varphi) = \psi$. Then we have

$$\varphi \circ F(f) = \psi(f) = \Lambda(\varphi)(f).$$

Let $\varphi = \delta_x$ and we obtain

$$\delta_x \circ F(f) = F(f)(x) = \Lambda(\delta_x)(f) = \widehat{f}(\Lambda(\delta_x)).$$

□

It is easy to see that not every mapping $\Lambda : \mathcal{M}_{bs}(\ell_1) \to \mathcal{M}_{bs}(\ell_1)$ generates a continuous homomorphism on $\mathcal{H}_{bs}(\ell_1)$ by the formula (2). We denote by $\mathfrak{M}(\ell_1)$ the class of all mappings which generate continuous homomorphisms.

Question 3. *How can we describe the class $\mathfrak{M}(\ell_1)$?*

From the properties of the operations $\star$ and $\Diamond$ immediately follows the next theorem.

Theorem 3. *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$ and mappings $\Lambda_1, \Lambda_2 : \mathcal{M}_{bs}(\ell_1) \longrightarrow \mathcal{M}_{bs}(\ell_1)$ belong to $\mathfrak{M}(\ell_1)$. Define*

$$\Lambda_\star(\varphi) := \Lambda_1(\varphi) \star \Lambda_2(\varphi),$$

$$\Lambda_\Diamond(\varphi) := \Lambda_1(\varphi) \Diamond \Lambda_2(\varphi).$$

Then $\Lambda_\star$ and $\Lambda_\Diamond$ belong to $\mathfrak{M}(\ell_1)$ as well. In other words, the class $\mathfrak{M}(\ell_1)$ is closed with respect to symmetric operations $\star$ and $\Diamond$.

REFERENCES

[1] Alencar R., Aron R., Galindo P., Zagorodnyuk A. *Algebras of symmetric holomorphic functions on ℓ_p*. Bull. Lond. Math. Soc. 2003, **35** (1), 55–64. doi:10.1112/S0024609302001431

[2] Aron R.M., Cole B.J., Gamelin T.W. *Spectra of algebras of analytic functions on a Banach space*. J. Reine Angew. Math. 1991, **415**, 51–93.

[3] Aron R.M., Galindo P., García D., Maestre M. *Regularity and algebras of analytic funtions in infinite dimensions*. Trans. Amer. Math. Soc. 1996, **348**, 543–559.

[4] Chernega I., Galindo P., Zagorodnyuk A. *Some algebras of symmetric analytic functions and their spectra*. Proc. Edinb. Math. Soc. 2011, **54**, 1–17. doi:10.1017/S0013091509001655

[5] Chernega I., Galindo P., Zagorodnyuk A. *The convolution operation on the spectra of algebras of symmetric analytic functions*. J. Math. Anal. Appl. 2012, **395** (2), 569–577. doi:10.1016/j.jmaa.2012.04.087

[6] Chernega I., Galindo P., Zagorodnyuk A. *A multiplicative convolution on the spectra of algebras of symmetric analytic functions*. Rev. Mat. Complut. 2014, **27** (2), 575–585. doi: 10.1007/s13163-013-0128-0

[7] González M., Gonzalo R., Jaramillo J.A. *Symmetric polynomials on rearrangement-invariant function spaces*. J. London Math. Soc. 1999, **59** (2), 681-697. doi:10.1112/S0024610799007164

[8] Nemirovskii A.S., Semenov S.M. *On polynomial approximation of functions on Hilbert space*. Mat. USSR Sbornik 1973, **21**, 255–277.

[9] Zagorodnyuk A. *Spectra of algebras of entire functions on Banach spaces*. Proc. Amer. Math. Soc. 2006, **134**, 2559–2569.

[10] Zagorodnyuk A. *Spectra of algebras of analytic functions and polynomials on Banach spaces*. Contem. Math. 2007, **435**, 381–394.

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Чернега І. *Гомоморфізми алгебри симетричних аналітичних функцій на просторі ℓ_1* // Карпатські матем. публ. — 2014. — Т.6, №2. — С. 394–398.

Досліджується алгебра $\mathcal{H}_{bs}(\ell_1)$ цілих симетричних аналітичних функцій з ℓ_1 в $\mathbb{C}$, що є обмеженими на обмежених множинах. Зокрема, вивчається неперервність деяких гомоморфізмів алгебри симетричних поліномів на просторі ℓ_p та операторів композиції на алгебрі симетричних аналітичних функцій. В статті поставлено декілька відкритих питань.

Ключові слова і фрази: поліноми та аналітичні функції на банахових просторах, симетричні поліноми, спектри алгебр.

Чернега И. *Гомоморфизмы алгебры симметрических аналитических функций на пространстве ℓ_1* // Карпатские матем. публ. — 2014. — Т.6, №2. — С. 394–398.

В работе исследуется алгебра $\mathcal{H}_{bs}(\ell_1)$ целых симметрических аналитических функций ограниченного типа с ℓ_1 в $\mathbb{C}$. В частности, изучается непрерывность некоторых гомоморфизмов алгебры симметрических полиномов на пространстве ℓ_p и операторов композиции на алгебре симметрических аналитических функций. В статье сформулировано несколько открытых вопросов.

Ключевые слова и фразы: полиномы и аналитические функции на банаховых пространствах, симметрические полиномы, спектры алгебр.