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ALMOST KENMOTSU f-MANIFOLDS

In this paper we consider a generalization of almost Kenmotsu f-manifolds. We get basic Riemannian curvature, sectional curvatures and scalar curvature properties of such type manifolds. Finally, we give two examples.

Key words and phrases: f-structure, almost Kenmotsu f-manifolds.

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1 INTRODUCTION

Let M be a real $(2n+s)$-dimensional smooth manifold. M admits f-structure [8] if there exists a non-null smooth $(1,1)$ tensor field φ, tangent bundle TM, satisfying $\varphi^3+\varphi=0$, $rank\,\varphi=2n$. An f-structure is a generalization of almost complex $(s=0)$ and almost contact $(s=1)$ structure. In the latter case M is orientable [9]. Corresponding to two complementary projection operators P and Q applied to TM, defined by $P=-\varphi^2$ and $Q=\varphi^2+I$, where I identity operator, there exist two complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that $\dim(\mathcal{D})=2n$ and $\dim\left(\mathcal{D}^\perp\right)=s$. The following relations hold [6]

$$\varphi P=P\varphi=\varphi,\quad \varphi Q=Q\varphi=0,\quad \varphi^2P=-P,\quad \varphi^2Q=0.$$

Thus, we have an almost complex distribution $\left(\mathcal{D},\ J=\varphi_{|\mathcal{D}},\ J^2=-I\right)$ and φ acts on $\mathcal{D}^\perp$ as a null operator. It follows that

$$TM=\mathcal{D}\oplus\mathcal{D}^\perp,\quad \mathcal{D}\cap\mathcal{D}^\perp=\{0\}.$$

Assume that $\mathcal{D}_p^\perp$ is spanned by s globally defined orthonormal vector $\{\xi_i\}$ at each point $p\in M$, $1\le i\le s$, with its dual set $\left\{\eta^i\right\}$. Then one obtains

$$\varphi^2=-I+\sum_{i=1}^{s}\eta^i\otimes\xi_i.$$

In the above case, M is called a globally framed manifold (or simply an f-manifold) ([1], [5] and [4]) and we denote its frame structure by $M(\varphi,\xi_i)$. From the above conditions one has

$$\varphi\xi_i=0,\quad \eta^i\circ\varphi=0,\quad \eta^i\left(\xi_j\right)=\delta_i^j.$$

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Now we consider Riemannian metric g on M that is compatible with an f-structure such that

$$g(\varphi X, Y) + g(X, \varphi Y) = 0,\ g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y),\ g(X, \xi_i) = \eta^i(X).$$

In the above case, we say that M is a metric f-manifold and its associated structure will be denoted by $M(\varphi, \xi_i, \eta^i, g)$.

A framed structure $M(\varphi, \xi_i)$ is normal [5] if the torsion tensor N_φ of φ is zero i.e., if

$$N_\varphi = N + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0,$$

where N denotes the Nijenhuis tensor field of φ.

Define a 2-form Φ on M by $\Phi(X, Y) = g(\varphi X, Y)$, for any $X, Y \in \Gamma(TM)$. The Levi-Civita connection ∇ of a metric f-manifold satisfies the following formula [1]:

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) &= 3d(X, \varphi Y, \varphi Z) - 3d(X, Y, Z) \\ &\quad + g(N(Y, Z), \varphi X) + N_j^2(Y, Z)\eta^j(X) \\ &\quad + 2d\eta^j(\varphi Y, X)\eta^j(Z) - 2d\eta^j(\varphi Z, X)\eta^j(Y), \end{aligned}$$

where the tensor field N_j^2 is defined by

$$N_j^2(X, Y) = \left(L_{\varphi X}\eta^j\right)Y - \left(L_{\varphi Y}\eta^j\right)X = 2d\eta^j(\varphi X, Y) - 2d\eta^j(\varphi Y, X),$$

for each $j \in \{1, \ldots, s\}$. Following the terminology introduced by Blair [1], we say that a normal metric f-manifold is a K-manifold if its 2-form Φ closed (i.e., $d\Phi = 0$). Since $\eta^1 \wedge \cdots \wedge \eta^s \wedge \Phi^n \neq 0$, a K-manifold is orientable. Furthermore, we say that a K-manifold is a C-manifold if each η^i is closed, an S-manifold if $d\eta^1 = d\eta^2 \cdots = d\eta^s = \Phi$.

Note that, if $s = 1$, namely if M is an almost contact metric manifold, the condition $d\Phi = 0$ means that M is quasi-Sasakian. M is said a K-contact manifold if $d\eta = \Phi$ and ξ is Killing.

Falcitelli and Pastore introduced and studied a class of manifolds which is called almost Kenmotsu f-manifold [3]. Such manifolds admit an f-structure with s-dimensional parallelizable kernel. A metric $f.pk$-manifold of dimension $(2n + s)$, $s \geq 1$, with $f.pk$-structure $(\varphi, \xi_i, \eta^i, g)$, is said to be a almost Kenmotsu $f.pk$-manifold if the 1-forms η^i's are closed and $d\Phi = 2\eta^1 \wedge \Phi$. Several foliations canonically associated with an almost Kenmotsu $f.pk$-manifold are studied and locally conformal almost Kenmotsu $f.pk$-manifolds are characterized by Falcitelli and Pastore. Öztürk et al. studied almost α-cosymplectic f-manifolds [6].

In this paper we consider a generalization of almost Kenmotsu f-manifolds. We get some curvature properties.

Throughout this paper we use the notations $\overline{\eta} = \eta^1 + \eta^2 + \cdots + \eta^s$, $\overline{\xi} = \xi_1 + \xi_2 + \cdots + \xi_s$ and $\overline{\delta}_i^j = \delta_i^1 + \delta_i^2 + \cdots + \delta_i^s$.

2 ALMOST KENMOTSU f-MANIFOLDS

Almost Kenmotsu f-manifolds firstly defined and studied by Aktan et al. as mentioned below [6].

Definition 2.1 ([6]). *Let $M\left(\varphi,\xi_i,\eta^i,g\right)$ be $(2n+s)$-dimensional metric f-manifold. For each η^i, $1\leq i\leq s$, 1-forms and each Φ 2-forms, if $d\eta^i=0$ and $d\Phi=2\overline{\eta}\wedge\Phi$ satisfy, then M is called almost Kenmotsu f-manifold.*

Let M be an almost Kenmotsu f-manifold. Since the distribution $\mathcal{D}$ is integrable, we have $L_{\xi_i}\eta^j=0$, $\left[\xi_i,\xi_j\right]\in\mathcal{D}$ and $\left[X,\xi_j\right]\in\mathcal{D}$ for any $X\in\Gamma\left(\mathcal{D}\right)$. Then the Levi-Civita connection is given by

$$2g\left(\left(\nabla_X\varphi\right)Y,Z\right)=2\left(\sum_{j=1}^{s}\left(g\left(\varphi X,Y\right)\xi_j-\eta^j\left(Y\right)\varphi X\right),Z\right)+g\left(N\left(Y,Z\right),\varphi X\right),\qquad(1)$$

for any $X,Y,Z\in\Gamma\left(TM\right)$. Putting $X=\xi_i$ we obtain $\nabla_{\xi_i}\varphi=0$ which implies $\nabla_{\xi_i}\xi_j\in\mathcal{D}^{\perp}$ and then $\nabla_{\xi_i}\xi_j=\nabla_{\xi_j}\xi_i$, since $\left[\xi_i,\xi_j\right]=0$.

We put $A_iX=-\nabla_X\xi_i$ and $h_i=\dfrac{1}{2}\left(L_{\xi_i}\varphi\right)$, where L denotes the Lie derivative operator.

Proposition 2.1 ([6]). *For any $i\in\{1,\ldots,s\}$ the tensor field A_i is a symmetric operator such that*

1) $A_i\left(\xi_j\right)=0$, *for any* $j\in\{1,\ldots,s\}$,
2) $A_i\circ\varphi+\varphi\circ A_i=-2\varphi$,
3) $tr(A_i)=-2n$.

Proof. Equality $d\eta^i=0$ implies that A_i is symmetric.

1) For any $i,j\in\{1,\ldots,s\}$ deriving $g\left(\xi_i,\xi_j\right)=\delta_i^j$ with respect to ξ_k, using $\nabla_{\xi_i}\xi_j=\nabla_{\xi_j}\xi_i$, we get $2g\left(\xi_k,A_i\left(\xi_j\right)\right)=0$. Since $\nabla_{\xi_i}\xi_j\in\mathcal{D}^{\perp}$, we conclude $A_i\left(\xi_j\right)=0$.

2) For any $Z\in\Gamma\left(TM\right)$, we have $\varphi\left(N\left(\xi_i,Z\right)\right)=\left(L_{\xi_i}\varphi\right)Z$ and, on the other hand, since $\nabla_{\xi_i}\varphi=0$,

$$L_{\xi_i}\varphi=A_i\circ\varphi-\varphi\circ A_i.\qquad(2)$$

One can easily obtain from (2)

$$-A_iX=-\varphi^2X-\varphi h_iX.\qquad(3)$$

Applying (1) with $Y=\xi_i$, we have

$$2g\left(\varphi A_iX,Z\right)=-2g\left(\varphi X,Z\right)-g\left(\varphi N\left(\xi_i,Z\right),X\right),$$

which implies the desired result.

3) Considering local adapted orthonormal frame $\{X_1,\ldots,X_n,\varphi X_1,\ldots,\varphi X_n,\xi_1,\ldots\xi_s\}$, by 1) and 2), one has

$$trA_i=\sum_{j=1}^{n}\left(g\left(A_iX_j,X_j\right)+g\left(A_i\varphi X_j,\varphi X_j\right)\right)=-2\sum_{j=1}^{n}g\left(\varphi X_j,\varphi X_j\right)=-2n.$$

□

Proposition 2.2 ([1]). *For any $i\in\{1,\ldots,s\}$ the tensor field h_i is a symmetric operator and satisfies*

i) $h_i\xi_j=0$, *for any* $j\in\{1,\ldots,s\}$,
ii) $h_i\circ\varphi+\varphi\circ h_i=0$,
iii) $trh_i=0$,
iv) $tr\varphi h_i=0$.

Proposition 2.3. *∇_φ satisfies the following relation [6]:*

$$\left(\nabla_{X\varphi}\right) Y + \left(\nabla_{\varphi X}\varphi\right) \varphi Y = \sum_{i=1}^{s} \left[-\left(\eta^i (Y) \varphi X + 2g (X, \varphi Y) \xi_i \right) - \eta^i (Y) h_i X \right]. \tag{4}$$

Proof. By direct computations, we get

$$\varphi N (X, Y) + N (\varphi X, Y) = 2 \sum_{i=1}^{s} \eta^i (X) h_i Y,$$

and

$$\eta^i (N (\varphi X, Y)) = 0.$$

From (1) and the equations above, the proof is completed. □

3 ALMOST KENMOTSU f-MANIFOLDS WITH ξ BELONGING TO THE (κ, μ, ν)-NULLITY DISTRIBUTION

Definition 3.1. *Let M be an almost Kenmotsu f-manifold, κ, μ and ν are real constants. We say that M verifies the (κ, μ, ν)-nullity condition if and only if for each $i \in \{1, \ldots, s\}$, $X, Y \in \Gamma(TM)$ the following identity holds*

$$\begin{aligned} R (X, Y) \xi_i = {} & \kappa \left(\overline{\eta} (X) \varphi^2 Y - \overline{\eta} (Y) \varphi^2 X \right) + \mu \left(\overline{\eta} (Y) h_i X - \overline{\eta} (X) h_i Y \right) \\ & + \nu \left(\overline{\eta}(Y) \varphi h_i X - \overline{\eta}(X) \varphi h_i Y \right). \end{aligned} \tag{5}$$

Lemma 3.1. *Let M be an almost Kenmotsu f-manifold verifiying (κ, μ, ν)-nullity condition. Then we have:*

(i) $h_i \circ h_j = h_j \circ h_i$ for each $i, j \in \{1, 2, \ldots, s\}$,
(ii) $\kappa \leq -1$,
(iii) if $\kappa < -1$ then, for each $i \in \{1, 2, \ldots, s\}$, h_i has eigenvalues $0, \pm\sqrt{-(\kappa+1)}$.

Proof. From (5) it follows that for each $X \in \Gamma(TM)$, $i, j \in \{1, 2, \ldots, s\}$

$$R(\xi_j, X)\xi_i - \varphi R(\xi_j, \varphi X)\xi_i = 2\kappa\varphi^2 X.$$

Using

$$R(\xi_j, X)\xi_i - \varphi R(\xi_j, \varphi X)\xi_i = 2 \left[-\varphi^2 X + \left(h_i \circ h_j \right) X \right]$$

we obtain

$$\left(h_i \circ h_j \right) X = (\kappa + 1) \varphi^2 X = \left(h_j \circ h_i \right) X \tag{6}$$

and then (i) is verified. Next from (6) we get

$$h_i^2 X = (\kappa + 1) \varphi^2 X, \tag{7}$$

$$h_i^2 X = -(\kappa + 1) X, \quad X \in \Gamma(\mathcal{D}). \tag{8}$$

Then, using Proposition 2 and (8) we obtain that the eigenvalues of h_i^2 are 0 and $-(\kappa+1)$. Moreover h_i is symmetric: $\|h_i X\|^2 = -(\kappa+1) \|X\|^2$. Hence $\kappa \leq -1$. Finally let t be a real eigenvalue of h_i and X be an eigenvector corresponding to t. Then $t^2 \|X\|^2 = -(\kappa+1) \|X\|^2$ and $t = \pm\sqrt{-(\kappa+1)}$. Taking Proposition 2 into account we get (iii). □

Proposition 3.1. *Let M be an almost Kenmotsu f-manifold verifying (κ,μ,ν)-nullity condition. Then*

$$h_1 = \cdots = h_s. \tag{9}$$

Proof. If $\kappa = -1$ then from (7) and the symmetry of each h_i we have $h_1 = \cdots = h_s = 0$. Let now $\kappa < -1$. We fix $x \in M$ and $i \in \{1,2,\ldots,s\}$. Since h_i is symmetric then we have $\mathcal{D}_x = (\mathcal{D}_+)_x \oplus (\mathcal{D}_-)_x$, where $(\mathcal{D}_+)_x$ is the eigenspace of h_i corresponding to the eigenvalue $\lambda = \sqrt{-(\kappa+1)}$ and $(D_-)_x$ is the eigenspace of h_i corresponding to the eigenvalue $-\lambda$. If $X \in \mathcal{D}_x$ then we can write $X = X_+ + X_-$, where $X_+ \in (\mathcal{D}_+)_x$, $X_- \in (\mathcal{D}_-)_x$ so that $h_i X = \lambda(X_+ + X_-)$. We fix $j \in \{1,2,\ldots,s\}$, $j \neq i$. Then from (6) we get $h_j X = h_j(X_+ + X_-) = h_j(\frac{1}{\lambda}h_i X_+ - \frac{1}{\lambda}h_i X_-) = \frac{1}{\lambda}\left(h_j \circ h_i\right)(X_+ + X_-) = \lambda(X_+ + X_-) = h_i X$. Taking into account Proposition 2 we obtain (9). □

Remark 3.1. *Throughout the paper whenever* (5) *holds we put $h := h_1 = \cdots = h_s$. Then* (5) *becomes*

$$\begin{aligned} R(X,Y)\xi_i = \kappa\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X\right) + \mu\left(\overline{\eta}(Y)hX - \overline{\eta}(X)hY\right) \\ + \nu\left(\overline{\eta}(Y)\varphi hX - \overline{\eta}(X)\varphi hY\right). \end{aligned} \tag{10}$$

Furthermore, using (10)*, the symmetry properties of the curvature tensor and the symmetry of φ^2 and h, we get*

$$\begin{aligned} R(\xi_i,X)Y = \kappa\left(\overline{\eta}(Y)\varphi^2 X - g\left(X,\varphi^2 Y\right)\overline{\xi}\right) + \mu\left(g(X,hY)\overline{\xi} - \overline{\eta}(Y)hX\right) \\ + \nu\left(g(\varphi hX,Y)\overline{\xi} - \overline{\eta}(Y)\varphi hX\right). \end{aligned} \tag{11}$$

Remark 3.2. *Let M be an almost Kenmotsu f-manifold verifying (κ,μ,ν)-nullity condition, with $\kappa \neq -1$. We denote by $\mathcal{D}_+$ and $\mathcal{D}_-$ the n-dimensional distributions of the eigenspaces of $\lambda = \sqrt{-(\kappa+1)}$ and $-\lambda$, respectively. We have that $\mathcal{D}_+$ and $\mathcal{D}_-$ are mutually orthogonal. Moreover, since φ anticommutes with h, we have $\varphi(\mathcal{D}_+) = \mathcal{D}_-$ and $\varphi(\mathcal{D}_-) = \mathcal{D}_+$. In other words, $\mathcal{D}_+$ is a Legendrian distribution and $\mathcal{D}_-$ is the conjugate Legendrian distribution of $\mathcal{D}_+$.*

Proposition 3.2. *Let M be an almost Kenmotsu f-manifold verifying (κ,μ,ν)-nullity condition. Then M is a Kenmotsu f-manifold if and only if $\kappa = -1$.*

Proof. We observed in the proof of Proposition 3.1 that if $\kappa = -1$ then $h = 0$. It follows that (10) reduces to $R(X,Y)\xi_i = \overline{\eta}(Y)\varphi^2 X - \overline{\eta}(X)\varphi^2 Y$. From [2, Proposition 3.4, Theorem 4.3] we get the claim. □

4 PROPERTIES OF THE CURVATURE

Let $M\left(\varphi,\xi_i,\eta^i,g\right)$ be a $(2n+s)$-dimensional almost Kenmotsu f-manifold. We consider the $(1,1)$-tensor fields defined by

$$l_{ij}(.) = R_{.\xi_i}\xi_j$$

for each $i,j \in \{1,\ldots,s\}$ and put $l_i = l_{ii}$.

Lemma 4.1. *For each $i, j, k \in \{1, \ldots, s\}$ the following identities hold:*

$$\varphi \circ l_{ji} \circ \varphi - l_{ji} = 2\left[h_i \circ h_j - \varphi^2\right], \tag{12}$$

$$\eta_k \circ l_{ji} = 0, \tag{13}$$

$$l_{ji}(\xi_k) = 0, \tag{14}$$

$$\nabla_{\xi_j} h_i = -\varphi \circ l_{ji} - \varphi - (h_j + h_i) - \varphi \circ h_i \circ h_j, \tag{15}$$

$$\nabla_{\xi_i} h_i = -\varphi \circ l_{ji} - \varphi - 2h_i - \varphi h_i^2. \tag{16}$$

Proof. Identity (12) is a rewriting of [7, (3.4)]. Formulas (13) and (14) are an immediate consequence of (12). Next from (3) and $\eta_l \circ (\nabla_{\xi_i} h_k) = 0$ we get

$$l_{ij} = \left(\varphi\left(\nabla_{\xi_j} h_i\right) + \varphi^2 + \varphi \circ h_i + \varphi \circ h_j - h_j \circ h_i\right).$$

Applying φ to both sides we get

$$\left(\nabla_{\xi_j} h_i\right) = \left(-\varphi \circ l_{ij} - \varphi - h_i - h_j - \varphi \circ h_j \circ h_i\right),$$

from which it follows (15). Finally, identity (16) is (15) when $i = j$. □

Remark 4.1. *Let M be an almost Kenmotsu f-manifold verifying (κ, μ, ν)-nullity condition. Then for each $i, j \in \{1, \ldots, s\}$ we have*

$$l_{ji} = -\kappa\varphi^2 + \mu h + \nu\varphi h. \tag{17}$$

It follows that all the l_{ji}'s coincide. We put $l = l_{ji}$.

Lemma 4.2. *Let M be an almost Kenmotsu f-manifold verifying (κ, μ, ν)-nullity condition. Then for each $i \in \{1, \ldots, s\}$, the following identities hold:*

$$\nabla_{\xi_i} h = -\mu\varphi h + \nu h - 2h, \tag{18}$$

$$l\varphi - \varphi l = 2\mu h\varphi + 2\nu h, \tag{19}$$

$$l\varphi + \varphi l = 2\kappa\varphi, \tag{20}$$

$$Q\xi_i = 2n\kappa\overline{\xi}. \tag{21}$$

Proof. From (16), using (17), we obtain (18). Identities (19) and (20) follow directly from (17) using $h \circ \varphi = -\varphi \circ h$. For the proof of (21) we fix $x \in M$ and $\{E_1, \ldots, E_{2n+s}\}$ a local φ-basis around x with $E_{2n+1} = \xi_1, \ldots, E_{2n+s} = \xi_s$. Then using (11) and $trace\,(h) = 0$ we get $Q\xi_i = \sum_{j=1}^{2n} R_{\xi_i E_j} E_j = \sum_{j=1}^{2n} \kappa g\left(\varphi^2 E_j, E_j\right)\overline{\xi} = \kappa \sum_{j=1}^{2n} \delta_{jj}\overline{\xi}$. □

Lemma 4.3. *Let $(M, \varphi, \xi_i, \eta_j, g)$ be a $(2n+s)$-dimensional almost Kenmotsu f-manifold. Then the curvature tensor satisfies the identities*

$$\begin{aligned} g\left(R_{\xi_i X} Y, Z\right) &= \sum_{j=1}^{s} \eta_j(Z)\, g\left(\varphi^2 Y, X\right) - \sum_{j=1}^{s} \eta_j(Y)\, g\left(\varphi^2 Z, X\right) \\ &+ \sum_{j=1}^{s} \eta_j(Z)\, g\left(\varphi h_j Y, X\right) - \sum_{j=1}^{s} \eta_j(Y)\, g\left(\varphi h_j Z, X\right) + g\left(\left(\nabla_Z \varphi h_i\right) Y - \left(\nabla_Y \varphi h_i\right) Z, X\right) \end{aligned} \tag{22}$$

and

$$\begin{aligned} &g\left(R_{\xi_i X} Y, Z\right) - g\left(R_{\xi_i X} \varphi Y, \varphi Z\right) + g\left(R_{\xi_i \varphi X} Y, \varphi Z\right) + g\left(R_{\xi_i \varphi X} \varphi Y, Z\right) \\ &= 2g\left(\left(\nabla_{h_i X} \varphi\right) Y, Z\right) + 2\overline{\eta}(Z)\, g\left(h_i X - \varphi X, \varphi Y\right) - 2\overline{\eta}(Y)\, g\left(h_i X - \varphi X, \varphi Z\right) \end{aligned} \tag{23}$$

for each $i = 1, \ldots, s$ and $X, Y, Z \in \Gamma(TM)$.

Proof. Using the Riemannian curvature tensor and (8), we obtain (22).

We introduce the operators A and B_i, $i \in \{1, \ldots, s\}$, defined by

$$A(X,Y,Z) := 2\overline{\eta}(Y)\, g(\varphi X, \varphi Z) - 2\overline{\eta}(Z)\, g(\varphi X, \varphi Y)$$

and

$$\begin{aligned} B_i(X,Y,Z) := & -g\left(\varphi X, \left(\nabla_Y (\varphi \circ h_i)\right)(\varphi Z)\right) - g\left(\varphi X, \left(\nabla_Y (\varphi \circ h_i)\right) Z\right) \\ & - g\left(X, \left(\nabla_Y (\varphi \circ h_i)\right) Z\right) + g\left(X, \left(\nabla_{\varphi Y} (\varphi \circ h_i)\right)(\varphi Z)\right) \end{aligned}$$

for each $X, Y, Z \in \Gamma(TM)$. By a direct computation and using (22) we get that the left hand side of (23) equals $A(X,Y,Z) + B_i(X,Y,Z) - B_i(X,Z,Y)$. Since

$$\eta_j\left(\left(\nabla_{\varphi Y} h_i\right) Z\right) = \eta_j\left(\nabla_{\varphi Y}(h_i Z)\right),$$

we can write

$$\begin{aligned} B_i(X,Y,Z) = & -g\left(X, \left(\nabla_Y (\varphi \circ h_i) Z\right)\right) + g\left(X, (\varphi \circ h_i)(\nabla_Y Z)\right) \\ & + g\left(X, \left(\nabla_{\varphi Y} (\varphi \circ h_i \circ \varphi) Z\right)\right) + g\left(X, (\varphi \circ h_i)\left(\nabla_{\varphi Y} \varphi Z\right)\right) \\ & - g\left(\varphi X, \left(\nabla_Y (\varphi \circ h_i \circ \varphi) Z\right)\right) + g\left(\varphi X, (\varphi \circ h_i)\left(\nabla_Y (\varphi Z)\right)\right) \\ & - g\left(\varphi X, \left(\nabla_{\varphi Y} (\varphi \circ h_i) Z\right)\right) + g\left(\varphi X, (\varphi \circ h_i)\left(\nabla_{\varphi Y} (h_i Z)\right)\right) \\ = & -g\left(X, (\nabla_Y \varphi)(h_i Z)\right) + g\left(X, h_i\left((\nabla_Y \varphi) Z\right)\right) \\ & + g\left(X, (h_i \circ \varphi)\left((\nabla_{\varphi Y} \varphi) Z\right)\right) + g\left(X, \varphi\left((\nabla_{\varphi Y} \varphi)(h_i Z)\right)\right) \\ & + \sum_{j=1}^{s} \eta_j\left(\left(\nabla_{\varphi Y} h_i\right) Z\right) \eta_j(X). \end{aligned} \tag{24}$$

Moreover, from (3), (4) and Proposition 1 it follows that

$$\begin{aligned} \left(\varphi \circ (\nabla_{\varphi X} \varphi)\right) Y = & \left(\nabla_{\varphi X} \varphi^2\right) Y - \left(\nabla_{\varphi X} \varphi\right)(\varphi Y) \\ & = \sum_{j=1}^{s} \left(\left(\nabla_{\varphi X} \eta_j\right) Y \xi_j\right) + \sum_{j=1}^{s} \left(\eta_j(Y) \nabla_{\varphi X} \xi_j\right) \\ & - \left(\nabla_{\varphi X} \varphi\right)(\varphi Y) = \sum_{j=1}^{s} \left(\left(\nabla_{\varphi X}\right)\left(g\left(\xi_j, Y\right)\right) \xi\right. \\ & \left. - g\left(\nabla_{\varphi X} Y, \xi_j\right) \xi_j\right) + \sum_{j=1}^{s} \eta_j(Y)\left(\varphi X - h_j X\right) \\ & + \sum_{j=1}^{s} \eta_j(Y) h_j X + \overline{\eta}(Y) \varphi X + 2g(X, \varphi Y)\, \overline{\xi} + (\nabla_X \varphi) Y. \end{aligned}$$

Hence

$$\begin{aligned} \left(\varphi \circ (\nabla_{\varphi X} \varphi)\right) Y \;\; = & \;\; \sum_{j=1}^{s} g(X, \varphi Y)\, \xi_j - \sum_{j=1}^{s} g\left(Y, h_j X\right) \xi_j \\ + & \;\; 2\sum_{j=1}^{s} \eta_j(Y)\, \varphi X + (\nabla_X \varphi) Y. \end{aligned}$$

Furthermore, from (4), for each $j \in \{1, \dots, s\}$ we have

$$\begin{aligned} \eta_i\left(\left(\nabla_{\varphi Y} h_j\right) Z\right) &= \eta_i\left(\nabla_{\varphi Y}\left(h_j Z\right)\right) = \left(\nabla_{\varphi Y} \eta_i\right)\left(h_j Z\right) \\ &= -g\left(h_j Z, \nabla_{\varphi Y} \xi_i\right) = g\left(h_j Z, h_i Y - \varphi Y\right). \end{aligned} \tag{25}$$

Then, using (24) and (25), we get

$$\begin{aligned} B_i(X,Y,Z) \;=\; & -g\left(X, \left(\nabla_Y \varphi\right)\left(h_i Z\right)\right) + g\left(X, h_i\left(\left(\nabla_Y \varphi\right) Z\right)\right) + 2\overline{\eta}\left(Z\right) g\left(h_i X, \varphi Y\right) \\ & + g\left(h_i X, \left(\nabla_Y \varphi\right) Z\right) + \overline{\eta}\left(X\right) g\left(Y, \varphi h_i Z\right) - \sum_{j=1}^{s} \eta_j\left(X\right) g\left(h_i Z, h_j Y\right) \\ & + \sum_{j=1}^{s} \eta_j\left(X\right) g\left(h_i Z, h_j Y\right) + g\left(X, \left(\nabla_Y \varphi\right)\left(h_i Z\right)\right) - \overline{\eta}\left(X\right) g\left(\varphi Y, h_i Z\right) \\ & = 2\left(g\left(h_i X, \left(\nabla_Y \varphi\right) Z\right) + \overline{\eta}\left(Z\right) g\left(h_i X, \varphi Y\right) - \overline{\eta}\left(X\right) g\left(\varphi Y, h_i Z\right)\right). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} & A(X,Y,Z) + B_i(X,Y,Z) - B_i(X,Z,Y) \\ & = 2\left(\nabla_Y \Phi\right)\left(h_i X, Z\right) - 2\left(\nabla_Z \Phi\right)\left(h_i X, Y\right) \\ & + 2\overline{\eta}\left(Z\right) g\left(h_i X - \varphi X, \varphi Y\right) - 2\overline{\eta}\left(Y\right) g\left(h_i X - \varphi X, \varphi Z\right) \end{aligned}$$

and hence (23) follows. □

Remark 4.2. *Let M be an almost Kenmotsu f-manifold. Then from (23) using $\left(\nabla_{h_i X} \Phi\right)(Y,Z) = -g\left(\left(\nabla_{h_i X} \varphi\right) Y, Z\right)$, for each $X, Y, Z \in \Gamma(TM)$, we get*

$$\begin{aligned} \left(\nabla_{h_i X} \varphi\right) Y &= \frac{1}{2}\left(\varphi R_{\xi_i \varphi X} Y - R_{\xi_i \varphi X} \varphi Y - \varphi R_{\xi_i X} \varphi Y - R_{\xi_i X} Y\right) \\ &\quad + g\left(h_i X - \varphi X, \varphi Y\right) \overline{\xi} + \overline{\eta}\left(Y\right)\left(\varphi h_i X - \varphi^2 X\right). \end{aligned} \tag{26}$$

Lemma 4.4. *Let M be an almost Kenmotsu f-manifold verifying (κ, μ, ν)-nullity condition. Then the following identities hold:*

$$(\nabla_X \varphi) Y = g(\varphi X + hX, Y)\xi - \eta(Y)(\varphi X + hX), \tag{27}$$

$$\begin{aligned} (\nabla_X h) Y - (\nabla_Y h) X &= (\kappa + 1)(\eta(Y)\varphi X - \eta(X)\varphi Y + 2g(\varphi X, Y)\xi) \\ &\quad + \mu(\eta(Y)\varphi h X - \eta(X)\varphi h Y) + (1 - \nu)(\eta(Y) h X - \eta(X) h Y). \end{aligned} \tag{28}$$

Proof. From (26) we obtain

$$\left(\nabla_{hX} \varphi\right) Y = -\left(\kappa + 1\right) g\left(X, Y\right) \overline{\xi} + \left(\kappa + 1\right) \overline{\eta}\left(Y\right) X + \overline{\eta}\left(Y\right) \varphi h X + g\left(hX, \varphi Y\right) \overline{\xi}.$$

Here we replace X with hX and by a direct computation, taking into account (3), (7), we get (27). From (27), since h and φ^2 are self-adjoint, we have

$$\left(\nabla_X\left(\varphi \circ h\right)\right) Y - \left(\nabla_Y\left(\varphi \circ h\right)\right) X = \varphi\left(\left(\nabla_X h\right) Y - \left(\nabla_Y h\right) X\right).$$

It follows that for each $Z \in \Gamma(TM)$

$$\begin{aligned} g\left(R_{XY} \xi_i, Z\right) &= \overline{\eta}\left(Y\right) g\left(\varphi^2 X + \varphi h X, Z\right) - \overline{\eta}\left(X\right) g\left(\varphi^2 Y + \varphi h Y, Z\right) \\ &\quad + g\left(\varphi\left(\left(\nabla_Y h\right) X - \left(\nabla_X h\right) Y\right), Z\right), \end{aligned} \tag{29}$$

where we use (5) of [6] and (27). From (29) and the symmetry of h and φ^2 it follows that

$$\varphi\left(\left(\nabla_Y h\right)X-\left(\nabla_X h\right)Y\right)=R_{XY}\xi_i-\overline{\eta}\left(Y\right)\left(\varphi^2X+\varphi hX\right)+\overline{\eta}\left(X\right)\left(\varphi^2Y+\varphi hY\right).$$

Then, applying φ to both sides of the last identity, using (10) and

$$\eta_l\left(\left(\nabla_Y h\right)X-\left(\nabla_X h\right)Y\right)=-2\left(\kappa+1\right)g\left(\varphi X,Y\right),\quad l\in\{1,\dots,s\},$$

we get (28). □

Theorem 1. *Let* $\boldsymbol{Z}=\left(M,\varphi,\xi_i,\eta_j,g\right)$ *be a* $(2n+s)$*-dimensional almost Kenmotsu* f*-manifold and* $\left(\widetilde{\varphi},\widetilde{\xi}_i,\widetilde{\eta}_j,\widetilde{g}\right)$ *be an almost* f*-structure on* M *obtained by a* $\mathcal{D}$*-homothetic transformation of constant* α*. If* $\boldsymbol{Z}$ *verifies the* (κ,μ,ν)*-nullity condition for certain real constants* (κ,μ,ν) *then* $\left(M,\widetilde{\varphi},\widetilde{\xi}_i,\widetilde{\eta}_j,\widetilde{g}\right)$ *verifies the* $(\widetilde{\kappa},\widetilde{\mu},\widetilde{\nu})$*-nullity condition, where*

$$\widetilde{\kappa}=\frac{\kappa}{\alpha},\quad\widetilde{\mu}=\frac{\mu}{\alpha},\quad\widetilde{\nu}=\frac{\nu}{\alpha}.$$

Proof. From (18) and (9) it follows that $\widetilde{h}_1=\cdots=\widetilde{h}_s$. Then, using (27), by a direct calculation we get the claim. □

Lemma 4.5. *Let* M *be an almost Kenmotsu* f*-manifold verifying the* (κ,μ,ν)*-nullity condition. Then*

$$X,Y\in\Gamma\left(\mathcal{D}_+\right)\Rightarrow\nabla_XY\in\Gamma\left(\mathcal{D}_+\right),\tag{30}$$

$$X,Y\in\Gamma\left(\mathcal{D}_-\right)\Rightarrow\nabla_XY\in\Gamma\left(\mathcal{D}_-\right),\tag{31}$$

$$X\in\Gamma\left(\mathcal{D}_+\right),\ Y\in\Gamma\left(\mathcal{D}_-\right)\Rightarrow\nabla_XY\in\Gamma\left(\mathcal{D}_-\oplus\ker\left(\varphi\right)\right),\tag{32}$$

$$X\in\Gamma\left(\mathcal{D}_-\right),\ Y\in\Gamma\left(\mathcal{D}_+\right)\Rightarrow\nabla_XY\in\Gamma\left(\mathcal{D}_+\oplus\ker\left(\varphi\right)\right).\tag{33}$$

Proof. From (28) we get $g\left(\left(\nabla_X h\right)\varphi Z-\left(\nabla_{\varphi Z}h\right)X,Y\right)=0$, for each $X,Y,Z\in\Gamma\left(\mathcal{D}_+\right)$. On the other hand, since h is symmetric, from Remark 2 we have

$$g\left(\left(\nabla_X h\right)\varphi Z-\left(\nabla_{\varphi Z}h\right)X,Y\right)=-2\lambda g\left(\nabla_X\left(\varphi Z\right),Y\right).$$

Then

$$g\left(\varphi Z,\nabla_XY\right)=-g\left(\nabla_X\left(\varphi Z\right),Y\right),$$

i.e. ∇_XY is normal to $\mathcal{D}_-$. Moreover from (3) and Remark 2 it follows that, for each $i\in\{1,\dots,s\}$, $g\left(\nabla_XY,\xi_i\right)=-g\left(Y,\nabla_X\xi_i\right)=0$. Then we have (30). The proof of (31) is analogous. If $X\in\Gamma\left(\mathcal{D}_+\right)$, $Y\in\Gamma\left(\mathcal{D}_-\right)$ then from (30) and Remark 2 we get that for each $Z\in\Gamma\left(\mathcal{D}_+\right)$ $g\left(\nabla_XY,Z\right)=-g\left(Y,\nabla_XZ\right)=0$ and then we have (32). Analogously we prove (33). □

Remark 4.3. *It follows from (30) and (31) that* $\mathcal{D}_\pm$ *define two orthogonal totally geodesic Legendrian foliations* $F_\pm$ *on* M*.*

Lemma 4.6. *Let* M *be an almost Kenmotsu* f*-manifold verifying the* (κ,μ,ν)*-nullity condition. Then for each* $X,Y\in\Gamma(TM)$ *we have*

$$\begin{aligned}\left(\nabla_X h\right)Y=&\left(\kappa+1\right)g\left(\varphi X,Y\right)\overline{\xi}-g\left(hX,Y\right)\overline{\xi}-\eta\left(Y\right)h\left(X+h\varphi X\right)\\&-\mu\eta\left(X\right)\varphi hY+\left(\nu-2\right)\eta\left(X\right)hY.\end{aligned}\tag{34}$$

Proof. Let be $X, Y \in \Gamma(\mathcal{D})$. From Proposition 2, $i)$ we get $g\left(h_i Y, \xi_j\right) = 0$. Taking the derivative of this equality of the direction X we obtain

$$(\nabla_X h) Y = -g\left(Y, h_i X + h_i^2 \varphi X\right) \xi_j.$$

Then, we write any vector field X on M as $X = X_+ + \eta_i(X)\xi_j$, X_+ denoting positive component of X in $\mathcal{D}$, and, using (18) and (8), we have

$$\begin{aligned}(\nabla_X h) Y = &\left(\nabla_{X_+} h\right) Y_+ + \overline{\eta}(Y)\left(\nabla_{X_+} h\right)\overline{\xi} + \overline{\eta}(X)(-\mu\varphi h + \nu h - 2h) Y \\ &- g\left(Y, hX + h^2\varphi X\right)\overline{\xi} - \overline{\eta}(Y)\left(hX + h^2\varphi X\right) + \overline{\eta}(X)(-\mu\varphi hY + \nu hY - 2hY).\end{aligned}$$

□

Remark 4.4. *Let M be an almost Kenmotsu f-manifold verifying the (κ, μ, ν)-nullity condition. Then using (27), (34) and (8) we get, for all $X, Y \in \Gamma(TM)$*

$$\begin{aligned}(\nabla_X \varphi h) Y = &(\kappa+1) g\left(\varphi^2 X, Y\right)\overline{\xi} + g(\varphi X, hY)\overline{\xi} - \overline{\eta}(Y)\varphi hX \\ &+ (\kappa+1)\overline{\eta}(Y)\varphi^2 X + \mu\overline{\eta}(X) hY + (\nu-2)\eta(X)\varphi hY.\end{aligned} \tag{35}$$

Lemma 4.7. *Let M be an almost Kenmotsu f-manifold verifying the (κ, μ, ν)-nullity condition. Then for each $X, Y, Z \in \Gamma(\mathcal{D})$ we have*

$$\begin{aligned}R_{XY}hZ - hR_{XY}Z = s[&\kappa\{g(Z,\varphi Y)X - g(Z,\varphi Y)\varphi hX - g(Z,\varphi X)Y + g(Z,\varphi X)\varphi hY \\ &+ g(Z,X)\varphi Y - g(Z,\varphi hX)\varphi Y - g(Z,Y)\varphi X + g(Z,\varphi hY)\varphi X\} \\ &+ g(Z,\varphi Y)X - g(Z,\varphi Y)\varphi hX - g(Z,\varphi X)Y + g(Z,\varphi X)\varphi hY \\ &- g(Z,hY)X + g(Z,hY)\varphi hX + g(Z,hX)Y - g(Z,hX)\varphi hY \\ &- g(Z,X)hY + g(Z,X)\varphi Y + g(Z,\varphi hX)hY - g(Z,\varphi hX)\varphi Y \\ &+ g(Z,Y)hX - g(Z,Y)\varphi X - g(Z,\varphi hY)hX + g(Z,\varphi hY)\varphi X].\end{aligned} \tag{36}$$

Proof. Let $X, Y, Z \in \Gamma(TM)$. Then by a direct computation we get

$$\begin{aligned}(\nabla_X\nabla_Y h) Z = &(\kappa+1)[g(\nabla_X Z, \varphi Y)\overline{\xi} + g(Z, (\nabla_X\varphi)Y)\overline{\xi} + g(Z, \varphi(\nabla_X Y))\overline{\xi} \\ &+ g(Z,\varphi Y)\left(-\varphi^2 X - \varphi hX\right)] - g(\nabla_X Z, hY)\overline{\xi} - g(Z, (\nabla_X h)Y)\overline{\xi} \\ &- g(Z, h(\nabla_X Y))\overline{\xi} + g(Z, hY)\left(\varphi^2 X + \varphi hX\right) - g\left(\nabla_X Z, \overline{\xi}\right)\left(hY + h^2\varphi Y\right) \\ &- g\left(Z, \nabla_X\overline{\xi}\right)\left(hY + h^2\varphi Y\right) - \overline{\eta}(Z)(\nabla_X h)Y - \overline{\eta}(Z)h(\nabla_X Y) \\ &(\kappa+1)[\overline{\eta}(Z)(\nabla_X\varphi)Y + \overline{\eta}(Z)\varphi(\nabla_X Y)] - \mu[g\left(\nabla_X Y, \overline{\xi}\right)\varphi hZ \\ &- g\left(Y, \nabla_X\overline{\xi}\right)\varphi hZ - \overline{\eta}(Y)(\nabla_X\varphi h)Z - \overline{\eta}(Y)\varphi h(\nabla_X Z)] \\ &+ (\nu-2)[g\left(\nabla_X Y, \overline{\xi}\right)hZ + g\left(Y, \nabla_X\overline{\xi}\right)hZ + \overline{\eta}(Y)(\nabla_X h)Z + \overline{\eta}(Y)h(\nabla_X Z)],\end{aligned}$$

where we used (34), (8) and the antisymmetry of $\nabla_X\varphi$. Hence, using the Ricci identity

$$R_{XY}hZ - hR_{XY}Z = (\nabla_X\nabla_Y h)Z - (\nabla_Y\nabla_X h)Z - \left(\nabla_{[X,Y]}h\right)Z,$$

formulas (34) and (3), the symmetry of $\nabla_X (h \circ \varphi)$, we obtain

$$
\begin{aligned}
R_{XY}hZ - hR_{XY}Z &= (\kappa+1)\left[g\left(Z,(\nabla_X\varphi)Y-(\nabla_Y\varphi)X\right)\overline{\xi} - g(Z,\varphi Y)\left(\varphi^2X+\varphi hX\right)\right.\\
&\quad \left.+g(Z,\varphi X)\left(\varphi^2Y+\varphi hY\right)\right] - g\left(Z,(\nabla_Xh)Y-(\nabla_Yh)X\right)\overline{\xi} + g(Z,hY)\left(\varphi^2X+\varphi hX\right)\\
&\quad - g(Z,hX)\left(\varphi^2Y+\varphi hY\right) - g\left(Z,\nabla_X\overline{\xi}\right)\left(hY+h^2\varphi Y\right) + g\left(Z,\nabla_Y\overline{\xi}\right)\left(hX+h^2\varphi X\right)\\
&\quad - \overline{\eta}(Z)\left((\nabla_Xh)Y-(\nabla_Yh)X\right) + (\kappa+1)\overline{\eta}(Z)\left((\nabla_X\varphi)Y-(\nabla_Y\varphi)X\right)\\
&\quad + \mu\left[\left(g\left(X,\nabla_Y\overline{\xi}\right)-g\left(Y,\nabla_X\overline{\xi}\right)\right)\varphi hZ - \overline{\eta}(Y)(\nabla_X\varphi h)Z + \overline{\eta}(X)(\nabla_Y\varphi h)Z\right]\\
&\quad + (\nu-2)\left[\left(g\left(Y,\nabla_X\overline{\xi}\right)-g\left(X,\nabla_Y\overline{\xi}\right)\right)hZ + \overline{\eta}(Y)(\nabla_Xh)Z - \overline{\eta}(X)(\nabla_Yh)Z\right].
\end{aligned} \tag{37}
$$

If we take $X, Y, Z \in \Gamma(\mathcal{D})$ then from (37), using identities (35), (27) and (8), we get (36). □

Lemma 4.8. *Let M be an almost Kenmotsu f-manifold verifying the (κ, μ, ν)-nullity condition. Then for each $X, Y, Z \in \Gamma(TM)$ we have*

$$
\begin{aligned}
R_{XY}\varphi Z - \varphi R_{XY}Z &= [\kappa(\overline{\eta}(Y)g(\varphi X,Z)-\overline{\eta}(X)g(\varphi Y,Z))\\
&\quad + \mu(\overline{\eta}(Y)g(\varphi hX,Z)-\overline{\eta}(X)g(\varphi hY,Z)) - \nu(\overline{\eta}(Y)g(hX,Z)-\overline{\eta}(X)g(hY,Z))]\overline{\xi}\\
&\qquad s\left[-g(Z,\varphi Y+hY)\left(\varphi^2X+\varphi hX\right)+g(Z,\varphi X+hX)\left(\varphi^2Y+\varphi hY\right)\right.\\
&\quad \left.+g\left(Z,\varphi^2X+\varphi hX\right)(\varphi Y+hY) - g\left(Z,\varphi^2Y+\varphi hY\right)(\varphi X+hX)\right]\\
&\quad - \overline{\eta}(Z)\left[\kappa(\overline{\eta}(Y)\varphi X-\overline{\eta}(X)\varphi Y)+\mu(\overline{\eta}(Y)\varphi hX-\overline{\eta}(X)\varphi hY)\right.\\
&\quad \left.-\nu(\overline{\eta}(Y)hX-\overline{\eta}(X)hY)\right].
\end{aligned}
$$

Proof. We proceed fixing a point $x \in M$ and local vector fields $X,\ Y,\ Z$ such that $\nabla X,\ \nabla Y$ and ∇Z vanish at x. Applying several times (27), using (8) and the symmetry of $\nabla\varphi^2$, we get in x

$$
\begin{aligned}
&\nabla_X((\nabla_Y\varphi)Z) - \nabla_Y((\nabla_X\varphi)Z)\\
&\quad = [g((\nabla_X\varphi)Y-(\nabla_Y\varphi)X,Z) + g((\nabla_Xh)Y-(\nabla_Yh)X,Z)]\overline{\xi}s\\
&\quad \times\left[g(Z,\varphi X+hX)\left(\varphi^2Y+\varphi hY\right) - g(Z,\varphi Y+hY)\left(\varphi^2X+\varphi hX\right)\right.\\
&\quad \left.+g\left(Z,\varphi^2X+\varphi hX\right)(\varphi Y+hY) - g\left(Z,\varphi^2Y+\varphi hY\right)(\varphi X+hX)\right]\\
&\quad - \overline{\eta}(Z)\left[((\nabla_X\varphi)Y-(\nabla_Y\varphi)X)+((\nabla_Xh)Y-(\nabla_Yh)X)\right].
\end{aligned}
$$

From the last identity, using $R_{XY}\varphi Z - \varphi R_{XY}Z = \nabla_X(\nabla_Y\varphi)Z - \nabla_Y(\nabla_X\varphi)Z$ and (28), we get the claimed identity. □

Remark 4.5. *In particular, from Lemma 9 it follows that for a Kenmotsu f-manifold $(M, \varphi, \xi_i, \eta_j, g)$ the following formula holds, for all $X, Y, Z \in \Gamma(TM)$,*

$$
\begin{aligned}
R_{XY}\varphi Z - \varphi R_{XY}Z &= (\overline{\eta}(X)g(\varphi Y,Z)-\overline{\eta}(Y)g(\varphi X,Z))\\
&\qquad s\left[-g(Z,\varphi Y)\varphi^2X+g(Z,\varphi X)\varphi^2Y+g\left(Z,\varphi^2X\right)\varphi Y-g\left(Z,\varphi^2Y\right)\varphi X\right]\\
&\quad - \overline{\eta}(Z)\left[(\overline{\eta}(Y)\varphi X-\overline{\eta}(X)\varphi Y)\right].
\end{aligned}
$$

Theorem 2. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition with $\kappa<-1$. Then for each $X_+,Y_+,Z_+\in\Gamma(\mathcal{D}_+)$, $X_-,Y_-,Z_-\in\Gamma(\mathcal{D}_-)$, we have*

$$\begin{aligned}R_{X_-Y_-}Z_+ &= s(\kappa+1)\left[g(\varphi Y_-,Z_+)\varphi X_- - g(\varphi X_-,Z_+)\varphi Y_-\right]\\ &\quad + s\lambda\left[g(\varphi X_-,Z_+)Y_- - g(\varphi Y_-,Z_+)X_-\right],\end{aligned}$$

$$\begin{aligned}R_{X_+Y_+}Z_+ &= s\left[g(X_+,Z_+)Y_+ - g(Y_+,Z_+)X_+\right]\\ &\quad + s\lambda\left[g(Y_+,Z_+)\varphi X_+ - g(X_+,Z_+)\varphi Y_+\right],\end{aligned} \tag{38}$$

$$\begin{aligned}R_{X_+Y_+}Z_- &= s\lambda\left[g(Z_-,\varphi Y_+)X_+ - g(Z_-,\varphi X_+)Y_+\right]\\ &\quad + s(\kappa+1)\left[g(Z_-,\varphi Y_+)\varphi X_+ - g(Z_-,\varphi X_+)\varphi Y_+\right],\end{aligned}$$

$$\begin{aligned}R_{X_+Y_-}Z_- &= -sg(Y_-,Z_-)X_+ + s(\kappa+1)g(\varphi X_+,Z_-)\varphi Y_-\\ &\quad + s\lambda\left[g(Y_-,Z_-)\varphi X_+ - g(\varphi X_+,Z_-)Y_-\right],\end{aligned} \tag{39}$$

$$\begin{aligned}R_{X_+Y_-}Z_+ &= sg(X_+,Z_+)Y_- - s(\kappa+1)g(\varphi Y_-,Z_+)\varphi X_+\\ &\quad + s\lambda\left[g(X_+,Z_+)\varphi Y_- - g(\varphi Y_-,Z_+)X_+\right],\end{aligned} \tag{40}$$

$$\begin{aligned}R_{X_-Y_-}Z_- &= s\left[g(X_-,Z_-)Y_- - g(Y_-,Z_-)X_-\right]\\ &\quad - s\lambda\left[g(Y_-,Z_-)\varphi X_- - g(X_-,Z_-)\varphi Y_-\right].\end{aligned} \tag{41}$$

Proof. First of all, for any $X_+,Y_+,Z_+\in\mathcal{D}_+$, applying Lemma 7, we get

$$\lambda R_{X_+Y_+}Z_+ - hR_{X_+Y_+}Z_+ = 2s\lambda^2\left(g(Z_+,Y_+)\varphi X_+ - g(Z_+,X_+)\varphi Y_+\right)$$

and by scalar multiplication with $W_-\in\mathcal{D}_-$, one has

$$2\lambda\left(R_{X_+Y_+}Z_+,W_-\right) = 2s\lambda^2\left(g(Z_+,Y_+)g(\varphi X_+,W_-) - g(Z_+,X_+)g(\varphi Y_+,W_-)\right)$$

from which, being $\lambda\neq 0$,

$$\left(R_{X_+Y_+}Z_+,W_-\right) = s\lambda\left(g(Z_+,Y_+)g(\varphi X_+,W_-) - g(Z_+,X_+)g(\varphi Y_+,W_-)\right). \tag{42}$$

With a similar argument, for any $X_+,W_+\in\mathcal{D}_+$ and $Y_-,Z_-\in\mathcal{D}_-$, we also obtain

$$\left(R_{X_+Y_-}Z_-,W_+\right) = (\kappa+1)s\left(g(Z_-,\varphi X_+)g(\varphi Y_-,W_+) - g(Z_-,Y_-)g(X_+,W_+)\right) \tag{43}$$

and, from (42), by symmetries of the tensor field R, for any $X_+,Y_+,W_+\in\mathcal{D}_+$ and $Z_-\in\mathcal{D}_-$

$$\left(R_{X_+Y_+}Z_-,W_+\right) = s\lambda\left(g(Z_-,\varphi Y_+)g(X_+,W_+) - g(Z_-,\varphi X_+)g(Y_+,W_+)\right). \tag{44}$$

Next, fixed a local φ-basis $\{e_1,\ldots,e_n,\varphi e_1,\ldots,\varphi e_n,\xi_1,\ldots,\xi_s\}$, with $e_i\in\mathcal{D}_+$ we compute $R_{X_+Y_+}Z_-$. The nullity condition implies $g\left(R_{X_+Y_+}Z_-,\xi_i\right)=0$, while using the first Bianchi identity, (43) and (44), we get

$$\begin{aligned}g\left(R_{X_+Y_+}Z_-,e_i\right) &= \lambda s\left(g(Z_-,\varphi Y_+)g(X_+,e_i) - g(Z_-,\varphi X_+)g(Y_+,e_i)\right),\\ g\left(R_{X_+Y_+}Z_-,\varphi e_i\right) &= (\kappa+1)s\left(g(\varphi Z_-,X_+)g(Y_+,e_i) - g(\varphi Z_-,Y_+)g(X_+,e_i)\right),\end{aligned}$$

so that, summing on i, the expression for $R_{X_+Y_+}Z_-$ follows.

The terms $R_{X_-Y_-}Z_+$ and $R_{X_+Y_-}Z_-$ are computed in a similar maner. Now, acting by φ on the formula just proved and using Lemma 10, we get

$$R_{X_+Y_+}\varphi Z_- = s\left(g(\varphi Y_+,Z_-)X_+ - g(\varphi X_+,Z_-)Y_+\right) - s\lambda\left(g(\varphi Y_+,Z_-)\varphi X_+ - g(\varphi X_+,Z_-)\varphi Y_+\right).$$

Writing this formula for φZ_-, by the compatibility condition, we have the result for $R_{X_+Y_+}Z_+$. Similar computation yields $R_{X_-Y_-}Z_-$. Analogously, using the third formula and Lemma 10 we obtain $R_{X_+Y_-}Z_+$. □

Now we are able to compute sectional curvature.

Theorem 3. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition with $\kappa<-1$. Then the sectional curvature K of M is determined by*

$$K(X,\xi_i)=\kappa g(X,X)+\mu g(hX,X)+\nu g(\varphi hX,X)=\begin{cases}\kappa+\mu\lambda & \text{if } X\in\mathcal{D}_+,\\ \kappa-\mu\lambda & \text{if } X\in\mathcal{D}_-,\end{cases} \tag{45}$$

$$K(X,Y)=\begin{cases} s & \text{if } X,Y\in\mathcal{D}_+,\\ s & \text{if } X,Y\in\mathcal{D}_-,\\ -s-s(\kappa+1)(g(X,\varphi Y)) & \text{if } X\in\mathcal{D}_+,\ Y\in\mathcal{D}_-.\end{cases} \tag{46}$$

Proof. Identities (45) follow directly from (5), while identities (46) are consequences of (38), (41) and (39) respectively. □

Corollary 4.1. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition with $\kappa<-1$. Then the Ricci operator verifies the following identities*

$$Q=s\left[(-2)\varphi^2+\mu h+(2(n-1)+\nu)(\varphi\circ h)\right]+2n\kappa\overline{\eta}\otimes\overline{\xi}, \tag{47}$$

$$Q\circ\varphi-\varphi\circ Q=2s\left[\mu h\circ\varphi+((n-1)+\nu)h\right]. \tag{48}$$

Proof. Let $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi_1,\dots,\xi_s\}$ be a local φ-basis such that $\{e_1,\dots,e_n\}$ is a basis of $\mathcal{D}_+$ and let $X=X_++X_-\in\mathcal{D}_+\oplus\mathcal{D}_-$. From (38), (39) and (10) we get

$$QX_+=s(-2+\mu\lambda)X_++s(2\lambda(n-1)+\nu)\varphi X_+. \tag{49}$$

On the other hand from (40) and (41) we obtain

$$QX_+=s(-2-\mu\lambda)X_+-s(2\lambda(n-1)+\nu)\varphi X_+. \tag{50}$$

Taking into account (49), (50) and $Q\xi_i=2n\kappa\overline{\xi}$, we get (47). Finally, identity (48) easily follows from (47). □

Corollary 4.2. *Let M be an almost Kenmotsu f-manifold verifying the (κ,μ,ν)-nullity condition with $\kappa<-1$. Then the scalar curvature of (M,g) is constant and verifies the following identity*

$$S=2ns(\kappa(2-n)-2n). \tag{51}$$

Proof. Let $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi_1,\dots,\xi_s\}$ be a local φ-basis such that $\{e_1,\dots,e_n\}$ is a basis of $\mathcal{D}_+$ Then from (38), (39) and (5) we have

$$g(Qe_i,e_i)=ksn+\mu\lambda sn-s(\kappa+1)n^2-sn^2. \tag{52}$$

Furthermore, from (40), (41) and (5) we get

$$g(Q\varphi e_i,\varphi e_i)=ksn-\mu\lambda sn-s(\kappa+1)n^2-sn^2. \tag{53}$$

Then (52), (53) and (21) yield (51). □

5 EXAMPLES

Example 1. *Let R^{2n+s} be $(2n+s)$-dimensional real vector space with standard coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_s)$ and*

$$M=\{(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_s)\,|z_i\neq 0,\ 1\leq i\leq s,\ n\in N,\ n\geq 1\}$$

be a $(2n+s)$-dimensional manifold. For each $i=1,\ldots,n$ and $k=1,\ldots,s$

$$\begin{aligned}
X_i&=\left(-(z_i+1)\pm\sqrt{(z_i+1)^2+e^{2z_i}}\right)\frac{\partial}{\partial x_i}+e^{z_i}\frac{\partial}{\partial z_i},\\
Y_i&=\left(z_i+1\pm\sqrt{(z_i+1)^2+e^{2z_i}}\right)\frac{\partial}{\partial y_i},\\
\xi_i&=\frac{\partial}{\partial z_i},
\end{aligned}$$

is a basis of M.

Then, for each $i,j=1,\ldots,n$ and $k=1,\ldots,s$ we obtain

$$\begin{aligned}
&\left[X_i,Y_j\right]=e^{z_i}\left(2z_i+3+2e^{2z_i}\right)\frac{\partial}{\partial y_i},\quad \left[Y_i,Y_j\right]=0,\\
&[X_i,\xi_i]=\left(2z_i+3+2e^{2z_i}\right)\frac{\partial}{\partial x_i}-e^{z_i}\frac{\partial}{\partial z_i},\quad [Y_i,\xi_i]=\left(2z_i+1+2e^{2z_i}\right)\frac{\partial}{\partial y_i},\\
&\left[X_i,X_j\right]=-e^{z_i}\left(2z_i+3-2e^{2z_i}\right)\frac{\partial}{\partial x_j}+e^{z_i}\left(2z_i+3-2e^{2z_i}\right)\frac{\partial}{\partial x_i}.
\end{aligned}$$

If we take $\eta_i=\dfrac{\partial}{\partial z_i}$, we get

$$\begin{aligned}
g&=\sum_{i=1}^{n}\left(\frac{-1}{(z_i+1)+\sqrt{(z_i+1)^2+e^{2z_i}}}dx_i^2+\frac{1}{(z_i+1)+\sqrt{(z_i+1)^2+e^{2z_i}}}dy_i^2\right)+\sum_{j=1}^{s}dz_j^2,\\
\varphi\xi_i&=0,\quad \varphi\left(\frac{\partial}{\partial x_i}\right)=-\frac{\partial}{\partial y_i},\\
\varphi\left(\frac{\partial}{\partial y_i}\right)&=\frac{\partial}{\partial x_i}-\frac{e^{z_i}}{2(z_i+1)\pm\sqrt{(2z_i+2)^2+4e^{2z_i}}}\frac{\partial}{\partial z_i}.
\end{aligned}$$

Then, we have an almost metric f-structure $\left(\varphi,\xi_j,\eta_i,g\right)$ on M. On the other hand, for each $i=1,\ldots,s$ we obtain $d\eta_i=0$. Moreover

$$\Phi_{ii}:=g\left(\frac{\partial}{\partial x_i},\varphi\frac{\partial}{\partial y_i}\right)=-\frac{1}{\left(-(z_i+1)\pm\sqrt{(z_i+1)^2+e^{2z_i}}\right)\left((z_i+1)+\sqrt{(z_i+1)^2+e^{2z_i}}\right)},$$

and for each $i,j=1,\ldots,s$ $\Phi_{ij}=0$. Then we get

$$\begin{aligned}
\Phi_{ii}:&=g\left(\frac{\partial}{\partial x_i},\varphi\frac{\partial}{\partial y_i}\right)=\\
&=-\frac{1}{\left(-(z_i+1)\pm\sqrt{(z_i+1)^2+e^{2z_i}}\right)\left((z_i+1)+\sqrt{(z_i+1)^2+e^{2z_i}}\right)}dx_i\wedge dyi,
\end{aligned}$$

and

$$d\Phi = 2\sum_{j=1}^{s} dz_j \wedge \left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = 2\overline{\eta} \wedge \Phi.$$

Since the Nijenhuis torsion tensor of this manifold is not equal to zero and in view of this expression we get an almost Kenmotsu f-manifold.

Example 2. *Let R^{2n+s} be $(2n+s)$-dimensional real vector space with standard coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_s)$ and*

$$M = \{(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_s) \,|z_i \neq 0,\ 1 \leq i \leq s,\ n \in N,\ n \geq 1\}$$

be a $(2n+s)$-dimensional manifold. For each $i = 1, \ldots, n$ and $k = 1, \ldots, s$

$$\begin{aligned} X_i &= \left(-1 \pm \sqrt{1+e^{2z_i}}\right)\frac{\partial}{\partial x_i} + z_i^2 \frac{\partial}{\partial z_i}, \\ Y_i &= \left(1 \pm \sqrt{1+e^{2z_i}}\right)\frac{\partial}{\partial y_i}, \\ \xi_i &= \frac{\partial}{\partial z_i}, \end{aligned}$$

is a basis of M.

Then, for each $i, j = 1, \ldots, n$ and $k = 1, \ldots, s$, we obtain

$$\begin{aligned} [X_i, Y_j] &= 2z_i^2 e^{2z_i}\frac{\partial}{\partial y_i}, \quad [Y_i, Y_j] = 0, \\ [X_i, \xi_i] &= -2e^{2z_i}\frac{\partial}{\partial x_i} - z_i^2\frac{\partial}{\partial z_i}, \quad [Y_i, \xi_i] = \pm 2e^{2z_i}\frac{\partial}{\partial y_i}, \\ [X_i, X_j] &= 2z_i^2 e^{2z_i}\frac{\partial}{\partial x_j} - 2z_i^2 e^{2z_i}\frac{\partial}{\partial x_i}. \end{aligned}$$

If we take $\eta_i = \frac{\partial}{\partial z_i}$, we get

$$\begin{aligned} g &= \sum_{i=1}^{n}\left(\frac{-1}{1 \pm \sqrt{1+e^{2z_i}}}dx_i^2 + \frac{1}{1 \pm \sqrt{1+e^{2z_i}}}dy_i^2\right) + \sum_{j=1}^{s} dz_j^2, \\ \varphi\xi_i &= 0, \quad \varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \\ \varphi\left(\frac{\partial}{\partial y_i}\right) &= \frac{\partial}{\partial x_i} - \frac{z_i^2}{2 \pm \sqrt{4+4e^{2z_i}}}\frac{\partial}{\partial z_i}. \end{aligned}$$

Then, we have a metric f-structure $(\varphi, \xi_j, \eta_i, g)$ on M. On the other hand, for each $i = 1, \ldots, s$ we obtain $d\eta_i = 0$. Moreover

$$\Phi_{ii} := g\left(\frac{\partial}{\partial x_i}, \varphi\frac{\partial}{\partial y_i}\right) = -\frac{1}{\left(-1 \pm \sqrt{1+e^{2z_i}}\right)\left(1 \pm \sqrt{1+e^{2z_i}}\right)},$$

and for each $i, j = 1, \ldots, s$ $\Phi_{ij} = 0$. Then we get

$$\Phi_{ii} := g\left(\frac{\partial}{\partial x_i}, \varphi\frac{\partial}{\partial y_i}\right) = -\frac{1}{\left(-1 \pm \sqrt{1+e^{2z_i}}\right)\left(1 \pm \sqrt{1+e^{2z_i}}\right)}dx_i \wedge dyi,$$

and

$$d\Phi = 2\sum_{j=1}^{s} dz_j \wedge \left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = 2\overline{\eta} \wedge \Phi.$$

Since the Nijenhuis torsion tensor of this manifold is equal to 0 *and in view of these expressions we get a Kenmotsu f-manifold.*

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В статті розглядаються узагальнення майже Кенмотсу f-многовидів. Отримано основні властивості Рiманової кривизни, секцiйних кривин i скалярної кривизни для таких типів многовидів. Насамкінець наведено два приклади.

Ключові слова і фрази: f-структура, майже Кенмотсу f-многовиди.