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A NEW CRITERION OF TESTING HYPOTHESIS ABOUT THE COVARIANCE FUNCTION OF THE HOMOGENEOUS AND ISOTROPIC RANDOM FIELD

In this paper we consider a continuous in mean square homogeneous and isotropic Gaussian random field. A criterion for testing hypotheses about the covariance function of such field using estimates for its norm in the space $L_p(\mathbb{T})$, $p \geq 1$, is constructed.

Key words and phrases: criterion for testing hypotheses, spherical correlogram, isotropic field.

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INTRODUCTION

Since the majority of the papers is devoted to the evaluation of covariance function with given accuracy in the uniform metric that is why in this paper we set the task to estimate the covariance function $B(\tau)$ of a Gaussian homogeneous isotropic random field with given accuracy and reliability in $L_p(T)$, $p \geq 1$. We construct a criterion for testing the hypothesis that the covariance function of homogeneous and isotropic Gaussian random field $\xi(x)$ equals $B(\tau)$. We shall use spherical correlogram

$$\hat{B}(\tau) = \frac{1}{U_n(R)} \int_{V_R(0)} \xi(x)\eta_\tau(x)dx$$

as the estimator of the function $B(\tau)$.

Definition of the square Gaussian random vector was introduced by Yu. Kozachenko and O. Moklyachuk in the paper [9]. They also received estimates for distributions of square Gaussian random vectors. Applications of the theory of square Gaussian random variables and stochastic processes in mathematical statistics were considered in the paper [8] and in the book [3]. A lot of the papers so far have been dedicated to estimation of covariance function of Gaussian random process and field, in particular the books [5], [1] and [15]. The main properties of the correlograms of the stationary Gaussian stochastic processes were studied by V. Buldygin and Yu. Kozachenko in the book [3]. Exponential inequalities for the distribution of the deviations correlograms from respective covariance function in the uniform metric were considered in the papers [8], [10] and [11]. Asymptotic normality of correlograms in the space of continuous functions were given by V. Buldygin and V. Zayats in the paper [4]. Issues of asymptotic normality of correlograms in the certain functional spaces were discussed in the papers by O. Ivanov [6] and V. Buldygin [2]. Leonenko and O. Ivanov in the book [7] considered asymptotic properties for estimates of covariance functions. In the papers [14] and [13]

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Yu. Kozachenko and T. Fedoryanich constructed a criterion for testing hypotheses about the covariance function of a Gaussian stationary process. A criterion for testing hypotheses about the covariance function of a stationary Gaussian stochastic process with given accuracy and reliability in $L_p(\mathbb{T}), p \geq 1$ is constructed in the paper [12].

1 REQUIRED INFORMATION

Definition 1 ([3]). *Let $\mathbb{T}$ be a parametric set and let $\Xi = \{\xi_t : t \in \mathbf{T}\}$ be a family of Gaussian random variables such that $\mathbf{E}\xi_t = 0$. The space $SG_{\Xi}(\Omega)$ is called a space of square Gaussian random variables if any $\zeta \in SG_{\Xi}(\Omega)$ can be represented as*

$$\zeta = \bar{\xi}^T A \bar{\xi} - \mathbb{E}\bar{\xi}^T A \bar{\xi},$$

where $\bar{\xi} = (\xi_1, ..., \xi_N)^T$ with $\xi_k \in \Xi, k = 1, ..., n$, and A is an arbitrary matrix with real-valued entries, or if $\zeta \in SG_{\Xi}(\Omega)$ has the representation

$$\zeta = \lim_{n\to\infty} \left(\bar{\xi}_n^T A \bar{\xi}_n - \mathbb{E}\bar{\xi}_n^T A \bar{\xi}_n \right).$$

Theorem 1 ([12]). *Let $\{\mathbb{T}, \mathfrak{A}, \mu\}$be a measurable space, where $\mathbb{T}$ is a parametric set and let $X = \{X(t), t \in \mathbb{T}\}$ be a square Gaussian stochastic process. Suppose that X is a measurable process. Further, let the Lebesgue integral $\int_{\mathbb{T}} (\mathbf{E}X^2(t))^{\frac{p}{2}} d\mu(t)$ be well defined for $p \geq 1$. Then the integral $\int_{\mathbb{T}} (X(t))^p d\mu(t)$ exists with probability 1 and*

$$P\left\{ \int_{\mathbb{T}} \mid X(t) \mid^p d\mu(t) > \varepsilon \right\} \leq 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{ -\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}} \right\} \tag{1}$$

for all $\varepsilon \geq \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p} \right)^p C_p$, where $C_p = \int_{\mathbb{T}} (\mathbf{E}X^2(t))^{\frac{p}{2}} d\mu(t)$.

Definition 2 ([15]). *Random field $\xi = \{\xi(x), t \in \mathbb{R}^n\}$ is called homogeneous in the wide sense in $\mathbb{R}^n$ if $\mathbb{E}\xi(x) = const, x \in \mathbb{R}^n$ and*

$$\mathbb{E}\xi(x)\xi(y) = B(x-y) = \int_{\mathbb{R}^n} e^{i(\lambda, x-y)} dF(\lambda), x, y \in \mathbb{R}^n.$$

Definition 3 ([15]). *Let $SO(n)$ be a group of rotations $\mathbb{R}^n$ around the origin. Homogeneous random field $\xi(x)$ is called isotropic if $\mathbf{E}\xi(x)\overline{\xi(y)} = \mathbf{E}\xi(gx)\overline{\xi(gy)}$ for all $g \in SO(n)$.*

We denote by $S_R(x)$ and $V_R(x)$ sphere and ball of radius R centered at a point x respectively. Let $m_n^{(R)}(\cdot)$ be a Lebesgue measure on $S_R(x)$. By $U_n(R)$ and $\omega_n(R)$ we denote the volume of ball and the surface area of the sphere of radius R in $\mathbb{R}^n$ respectively.

Consider a random field

$$\eta_R(x) = \frac{1}{\omega_n(R)} \int_{S_R(x)} \xi(y) m_n^{(R)}(dy).$$

Theorem 2 ([15]). *Random field $\eta_R(x)$ is homogeneous and isotropic. Homogeneous and isotropic random fields $\eta_R(x)$ and $\xi(x)$ are related each other and the following equalities hold*

$$\mathbf{E}\eta_{R_1}(x_1)\eta_{R_2}(x_2) = \int_0^\infty Y_n(\lambda R_1)Y_n(\lambda R_2)Y_n(\lambda\tau_{x_1x_2})d\Phi(\lambda), \tag{2}$$

$$\mathbf{E}\eta_{R_1}(x_1)\xi(x_2) = \int_0^\infty Y_n(\lambda R)Y_n(\lambda\tau_{x_1x_2})d\Phi(\lambda), \tag{3}$$

where

- $Y_n(z) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}(z)}{z^{\frac{n-2}{2}}}$ *is a spherical Bessel function,* $\Phi(\lambda) = \int\limits_{\sqrt{v_1^2+\dots+v_n^2}\le\lambda} F(dv)$, $F(\cdot)$ *is a finite measure on σ-algebra B_n Borel sets of $\mathbb{R}^n$.*

- $\tau_{x_1x_2} = |x_1 - x_2|$ *is a distance between the points x_1 and x_2.*

2 CONSTRUCTION CRITERION FOR TESTING HYPOTHESIS ABOUT THE COVARIANCE FUNCTION OF THE HOMOGENEOUS AND ISOTROPIC RANDOM FIELD

Let $\xi(x)$ be a continuous in mean square homogeneous and isotropic Gaussian random field in $\mathbb{R}^n$ with zero-mean. Without any loss of generality, we can assume that the sample paths of the field $\xi(x)$ are continuous with probability one on any bounded and closed set.

Let the random field $\xi(x)$ be observed on the ball $V_{R+\tau}(0)$, $\tau \ge 0$, and let the spectral function of the field $\Phi(\lambda)$ be absolutely continuous.

Theorem 3. *Let a spherical correlogram*

$$\hat{B}(\tau) = \frac{1}{U_n(R)}\int_{V_R(0)}\xi(x)\left(\frac{1}{\omega_n(r)}\int_{S_r(x)}\xi(t)m_n^{(\tau)}(dt)\right)dx = \frac{1}{U_n(R)}\int_{V_R(0)}\xi(x)\eta_\tau(x)dx \tag{4}$$

be an estimator of the covariance function $B(\tau)$. Then the following inequality holds for all $\varepsilon \ge \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$*:*

$$P\left\{\int\limits_0^A(\hat{B}(\tau) - B(\tau))^p d\tau > \varepsilon\right\} \le 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\},$$

where

$$C_p = \frac{1}{U_n^2(R)}\int_0^A\int_{V_R(0)}\int_{V_R(0)}\left(B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right.$$
$$\left. + \left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]\right)dxdyd\tau$$

and $0 < A < \infty$.

Remark 1. *Since the sample paths of the field $\xi(x)$ are continuous with probability one on the ball $V_{R+\tau}(0)$, $\hat{B}(\tau)$ is a Riemann integral.*

Proof. Consider

$$\mathbf{E}(\hat{B}(\tau)-B(\tau))^2=\mathbf{E}(\hat{B}(\tau))^2-B^2(\tau).$$

From the Isserlis equality for jointly Gaussian random variables and relationships (2) and (3) it follows that

$$\begin{aligned}\mathbf{E}\hat{B}^2(\tau)&=\frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}(\mathbf{E}\xi(x)\eta_\tau(x)\mathbf{E}\xi(x)\eta_\tau(x)\\&+\mathbf{E}\xi(x)\xi(y)\mathbf{E}\eta_\tau(x)\eta_\tau(y)+\mathbf{E}\xi(x)\eta_\tau(y)\mathbf{E}\xi(y)\eta_\tau(x))\,dxdy\\&=\frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}\left(\left[\int_0^\infty Y_n(\lambda\tau)Y_n(0)d\Phi(\lambda)\right]^2\right.\\&+B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\\&\left.+\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right)dxdy\\&=\frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}\left(B^2(\tau)+B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right.\\&\left.+\left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]^2\right)dxdy=B^2(\tau)\\&+\frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}\left(B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right.\\&\left.+\left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]\right)dxdy.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E}(\hat{B}(\tau)-B(\tau))^2=\frac{1}{U_n^2(R)}\int_{V_R(0)}\int_{V_R(0)}&\left(B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right.\\&\left.+\left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]\right)dxdy.\end{aligned}\tag{5}$$

Since $\hat{B}(\tau)-B(\tau)$ is a square Gaussian random field (see Lemma 3.1, Chapter 6 in book [3]), then it follows from the Theorem 1 that

$$P\left\{\int_0^A(\hat{B}(\tau)-B(\tau))^p d\tau>\varepsilon\right\}\le 2\sqrt{1+\frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

Applying equality (5) we get

$$\begin{aligned}C_p&=\frac{1}{U_n^2(R)}\int_0^A\int_{V_R(0)}\int_{V_R(0)}\left(B(|x-y|)\int_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right.\\&\left.+\left[\int_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)\right]\right)dxdyd\tau.\end{aligned}$$

□

Denote

$$g(\varepsilon) = 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}}.$$

From the Theorem 3 it follows that if $\varepsilon \geq z_p = C_p \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p$, then

$$P\left\{\int_0^A (\hat{B}(\tau) - B(\tau))^p d\tau > \varepsilon\right\} \leq g(\varepsilon).$$

Let ε_δ be a solution of the equation $g(\varepsilon) = \delta$, $0 < \delta < 1$. Put $S_\delta = \max\{\varepsilon_\delta, z_p\}$. It is obviously that $g(S_\delta) \leq \delta$ and

$$P\left\{\int_0^A (\hat{B}(\tau) - B(\tau))^p d\tau > S_\delta\right\} \leq \delta. \tag{6}$$

Let $\mathbb{H}$ be the hypothesis that the covariance function of homogeneous and isotropic continuous in mean square Gaussian random field $\xi(x)$ equals $B(\tau)$ for $0 \leq \tau \leq A$. From the Theorem 3 and (6) it follows that to test the hypothesis $\mathbb{H}$ one can use the following criterion.

Criterion 1 For a given level of confidence δ the hypothesis $\mathbb{H}$ is accepted if

$$\int_0^A (\hat{B}(\tau) - B(\tau))^p d\mu(\tau) < S_\delta$$

otherwise hypothesis is rejected.

Remark 2. *The equation $g(\varepsilon) = \delta$ has a solution for any $\delta > 0$, since $g(\varepsilon)$ is a monotonically decreasing function. We can find the solution of equation using numerical methods.*

Remark 3. *One can easily see that Criterion 1 can be used if $C_p \to 0$ as $R \to \infty$.*

3 CONCLUSIONS

In this paper, we constructed a new criterion for testing hypothesis about the covariance function of homogeneous and isotropic Gaussian random field. The evaluation is carried out by observing for the random field on the ball. We regard spherical correlogram as the estimator of the covariance function.

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Трошкі В.Б. *Новий критерій для перевірки гіпотези про вигляд коваріаційної функції однорідного та ізотропного випадкового поля* // Карпатські матем. публ. — 2015. — Т.7, №1. — С. 114–119.

В даній роботі розглядаються однорідне та ізотропне неперервне в середньому квадратичному випадкове поле. Тут побудований критерій для перевірки гіпотези про вигляд коваріаційної функції однорідного та ізотропного випадкового поля.

Ключові слова і фрази: критерій для перевірки гіпотези, сферична корелограма, ізотропне поле.