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OPTIMAL CONTROL PROBLEM FOR SYSTEMS GOVERNED BY NONLINEAR PARABOLIC EQUATIONS WITHOUT INITIAL CONDITIONS

An optimal control problem for systems described by Fourier problem for nonlinear parabolic equations is studied. Control functions occur in the coefficients of the state equations. The existence of the optimal control in the case of final observation is proved.

Key words and phrases: optimal control, problem without initial conditions, nonlinear parabolic equation.

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INTRODUCTION

Optimal control of determined systems governed by partial differential equations (PDEs) is currently of much interest. Optimal control problems for PDEs are most completely studied for the case in which the control functions occur either on the right-hand sides of the state equations, or the boundary or initial conditions [8, 22, 26]. So far, problems in which control functions occur in the coefficients of the state equations are less studied.

The main ideas and methods of solving different optimal control problems for systems governed by evolutionary equations and variational inequalities are considered in monograph [18]. Problem, where control functions occur in the coefficients of the state equations, is given as only one among many other problems which were considered there by author.

A lot of various generalizations of this problem were investigated in many papers, including [1, 2, 4, 5, 10–13, 15, 20, 21, 24, 25], where the state of controlled system is described by the initial-boundary value problems for parabolic equations.

In [1, 21, 24, 25] the state of controlled system is described by linear parabolic equations and systems, while in [1] and [21] control functions appears as coefficients at lower derivatives, and in [24, 25] the control functions are coefficients at higher derivatives. In [21] the existence and uniqueness of optimal control in the case of final observation was shown and a necessary optimality condition in the form of the generalized rule of Lagrange multipliers was obtained. In paper [1] authors proved the existence of at least one optimal control for system governed by a system of general parabolic equations with degenerate discontinuous parabolicity coefficient. In papers [24, 25] the authors consider cost function in general form, and as special case it includes different kinds of specific practical optimization problems. The well-posedness of the problem statement is investigated and a necessary optimality condition in the form of the generalized principle of Lagrange multiplies is established in this papers.

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In papers [2, 10–13, 15, 20] authors investigate optimal control of systems governed by nonlinear PDEs. In particular, in [2] the problem of allocating resources to maximize the net benefit in the conservation of a single species is studied. The population model is an equation with density dependent growth and spatial-temporal resource control coefficient. The existence of an optimal control and the uniqueness and the characterization of the optimal control are established. Numerical simulations illustrate several cases with Dirichlet and Neumann boundary conditions. In [11] the optimal control problem is converted to an optimization problem which is solved using a penalty function technique. The existence and uniqueness theorems are investigated. The derivation of formula for the gradient of the modified function is explained by solving the adjoint problem. Paper [15] presents analytical and numerical solutions of an optimal control problem for quasilinear parabolic equations. The existence and uniqueness of the solution are shown. The derivation of formula for the gradient of the modified cost function by solving the conjugated boundary value problem is explained. In [16] the authors consider the optimal control of a degenerate parabolic equation governing a diffusive population with logistic growth terms. The optimal control is characterized in terms of the solution of the optimality system, which is the state equation coupled with the adjoint equation. Uniqueness for the solutions of the optimality system is valid for a sufficiently small time interval due to the opposite time orientations of the two equations involved. In paper [20] optimal control for semilinear parabolic equations without Cesari-type conditions is investigated.

In this paper, we study an optimal control problem for systems whose states are described by problems without initial conditions or, other words, Fourier problems for nonlinear parabolic equations.

The problem without initial conditions for evolution equations describes processes that started a long time ago and initial conditions do not affect on them in the actual time moment. Such problem were investigated in the works of many mathematicians (see [3, 7, 23] and bibliography there).

As we know among numerous works devoted to the optimal control problems for PDEs, only in papers [4, 5] the state of controlled system is described by the solution of Fourier problem for parabolic equations. In the current paper, unlike the above two, we consider optimal control problem in case when the control functions occur in the coefficients of the state equation. The main result of this paper is existence of the solution of this problem.

The outline of this paper is as follows. In Section 1, we give notations, definitions of function spaces and auxiliary results. In Section 2, we prove existence and uniqueness of the solutions for the state equations. Furthermore, we construct a priori estimates for the weak solutions of the state equations. In Section 3, we formulate the optimal control problem. Finally, the existence of the optimal control is presented in Section 4.

1 PRELIMINARIES

Let n be a natural number, $\mathbb{R}^n$ be the linear space of ordered collections $x = (x_1, \dots, x_n)$ of real numbers with the norm $|x| := (|x_1|^2 + \ldots + |x_n|^2)^{1/2}$. Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ. Set $S := (-\infty, 0]$, $Q := \Omega \times S$, $\Sigma := \Gamma \times S$.

Denote by $L^\infty_{\mathrm{loc}}(\overline{Q})$ the linear space of measurable functions on Q such that their restrictions to any bounded measurable set $Q' \subset Q$ belong to the space $L^\infty(Q')$.

Let X be an arbitrary Hilbert space with the scalar product $(\cdot, \cdot)_X$ and the norm $\| \cdot \|_X$.

Denote by $L^2_{\rm loc}(S;X)$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[a,b]\subset S$ belong to the space $L^2(a,b;X)$.

Let $\omega\in\mathbb{R}$, $\alpha\in C(S)$ be such that $\alpha(t)>0$ for all $t\in S$, $\gamma=\alpha$ or $\gamma=1/\alpha$, and let X be as above. Put by definition

$$L^2_{\omega,\gamma}(S;X):=\Big\{f\in L^2_{\rm loc}(S;X)\ \Big|\ \int\limits_S\gamma(t)e^{2\omega\int\limits_0^t\alpha(s)ds}\|f(t)\|^2_X dt<\infty\Big\}.$$

This space is a Hilbert space with respect to the scalar product

$$(f,g)_{L^2_{\omega,\gamma}(S;X)}=\int\limits_S\gamma(t)\,e^{2\omega\int\limits_0^t\alpha(s)\,ds}(f(t),g(t))_X\,dt$$

and the norm

$$\|f\|_{L^2_{\omega,\gamma}(S;X)}:=\Big(\int\limits_S\gamma(t)\,e^{2\omega\int\limits_0^t\alpha(s)\,ds}\|f(t)\|^2_X\,dt\Big)^{1/2}.$$

Denote by $C^1_c(a,b)$, where $-\infty\le a<b\le+\infty$, the linear space of continuously differentiable functions on (a,b) with compact supports.

Let $H^1(\Omega):=\{v\in L_2(\Omega)\mid v_{x_i}\in L_2(\Omega)\ (i=\overline{1,n})\}$ be a Sobolev space, which is a Hilbert space with respect to the scalar product $(v,w)_{H^1(\Omega)}:=\int\limits_\Omega\{\sum\limits_{i=1}^n v_{x_i}w_{x_i}+vw\}\,dx$ and the corresponding norm $\|v\|_{H^1(\Omega)}:=\Big(\int\limits_\Omega\{\sum\limits_{i=1}^n|v_{x_i}|^2+|v|^2\}\,dx\Big)^{1/2}$. Under $H^1_0(\Omega)$ we mean the closure in $H^1(\Omega)$ of the space $C^\infty_c(\Omega)$ consisting of infinitely differentiable functions on Ω with compact supports. Denote by

$$K:=\inf_{v\in H^1_0(\Omega),\ v\ne0}\frac{\int\limits_\Omega|\nabla v|^2\,dx}{\int\limits_\Omega|v|^2\,dx},\tag{1}$$

where $\nabla v=(v_{x_1},\dots,v_{x_n}),\ |\nabla v|^2=\sum\limits_{i=1}^n|v_{x_i}|^2$.

It is well known that the constant K is finite and coincides with the first eigenvalue of the following eigenvalue problem:

$$-\Delta v=\lambda v,\quad v|_{\partial\Omega}=0.\tag{2}$$

From (1) it clearly follows the Friedrichs inequality

$$\int\limits_\Omega|\nabla v|^2\,dx\ge K\int\limits_\Omega|v|^2\,dx\ \text{for all}\ \ v\in H^1_0(\Omega).\tag{3}$$

Also define $\partial_0 z=z,\quad \partial_j z=z_{x_j}$ if $j\in\{1,\dots,n\}$. Further, an important role will be played by the following statement.

Lemma 1. *Suppose that a function $z\in L^2(t_1,t_2;H^1_0(\Omega))$, where $t_1,t_2\in\mathbb{R}$ $(t_1<t_2)$, satisfies the identity*

$$\int\limits_{t_1}^{t_2}\!\!\int\limits_\Omega\{-z\psi\varphi'+\sum_{i=0}^n g_i\partial_i\psi\varphi\}\,dxdt=0,\quad \psi\in H^1_0(\Omega),\ \varphi\in C^1_c(t_1,t_2),\tag{4}$$

for some $g_i \in L^2(t_1,t_2;L^2(\Omega))$ $(i=\overline{0,n})$. *Then*

(i) the function z *belongs to the space* $C([t_1,t_2];L^2(\Omega))$ *and for every* $\theta \in C^1([t_1,t_2])$ *and for all* $\tau_1,\tau_2 \in [t_1,t_2]$ $(\tau_1<\tau_2)$ *we have*

$$\frac{1}{2}\theta(t)\int\limits_{\Omega}|z(x,t)|^2\,dx\Big|_{t=\tau_1}^{t=\tau_2}-\frac{1}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}|z|^2\theta'\,dxdt+\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\{\sum_{i=0}^{n}g_i\partial_i z\}\theta\,dxdt=0; \tag{5}$$

(ii) the derivative z_t *of the function* z *in the sense* $D'(t_1,t_2;H^{-1}(\Omega))$ *(the distributions space) belongs to* $L^2(t_1,t_2;H^{-1}(\Omega))$*, furthermore*

$$\int\limits_{t_1}^{t_2}\|z_t(\cdot,t)\|^2_{H^{-1}(\Omega)}\,dt\le\sum_{i=0}^{n}\|g_i\|^2_{L^2(\Omega\times(t_1,t_2))}. \tag{6}$$

Proof. The first statement follows directly from Lemma 2 of [6]. Let us prove the second statement. Firstly note that the following continuous and dense embeddings hold

$$H_0^1(\Omega)\subset L^2(\Omega)\subset H^{-1}(\Omega). \tag{7}$$

Let $C_c^\infty(t_1,t_2)$ be the space of functions on (t_1,t_2) which are infinitely continuously differentiable and have compact supports. Under $D'(t_1,t_2;H^{-1}(\Omega))$ we mean the space of distributions which are defined on $C_c^\infty(t_1,t_2)$ with values in $H^{-1}(\Omega)$ (see, for example, [14]). Since the spaces $L^2(t_1,t_2;H_0^1(\Omega))$, $L^2(t_1,t_2;H^{-1}(\Omega))$ can be identified with subspaces of the space of distributions $D'(t_1,t_2;H^{-1}(\Omega))$, then it allows us to speak about derivatives of functions from $L^2(t_1,t_2;H_0^1(\Omega))$ in the sense $D'(t_1,t_2;H^{-1}(\Omega))$ and their belonging to the space $L^2(t_1,t_2;H^{-1}(\Omega))$.

Let us rewrite equality (4) in the form

$$-\int\limits_{t_1}^{t_2}\int\limits_{\Omega}z\psi\varphi'\,dxdt=-\int\limits_{t_1}^{t_2}\int\limits_{\Omega}\sum_{i=0}^{n}g_i\partial_i\psi\varphi\,dxdt,\quad \psi\in H_0^1(\Omega),\ \varphi\in C_c^1(t_1,t_2). \tag{8}$$

According to the definition of the derivative of distributions from $D'(t_1,t_2;H^{-1}(\Omega))$, (8) implies that z_t belongs to the space $L^2(t_1,t_2;H^{-1}(\Omega))$, and for almost all $t\in(t_1,t_2)$

$$< z_t(\cdot,t),\psi(\cdot)>_{H_0^1(\Omega)}=-\int\limits_{\Omega}\sum_{i=0}^{n}g_i(x,t)\partial_i\psi(x)\,dx,$$

where $<\cdot,\cdot>_{H_0^1(\Omega)}$ denotes the canonical scalar product in $H^{-1}(\Omega)\times H_0^1(\Omega)$. From this, using the Cauchy-Schwarz inequality, for almost all $t\in(t_1,t_2)$ we obtain

$$\begin{aligned}|< z_t(\cdot,t),\psi(\cdot)>_{H_0^1(\Omega)}|&\le\sum_{i=0}^{n}\|g_i(\cdot,t)\|_{L^2(\Omega)}\|\partial_i\psi(\cdot)\|_{L^2(\Omega)}\\&\le\Big(\sum_{i=0}^{n}\|g_i(\cdot,t)\|^2_{L^2(\Omega)}\Big)^{1/2}\|\psi(\cdot)\|_{H^1(\Omega)}.\end{aligned} \tag{9}$$

From (9) it follows that for almost all $t\in(t_1,t_2)$ the following estimate is valid

$$\|z_t(\cdot,t)\|^2_{H^{-1}(\Omega)}\le\sum_{i=0}^{n}\|g_i(\cdot,t)\|^2_{L^2(\Omega)},$$

which easily implies (6). ∎

2 WELL-POSEDNESS OF THE PROBLEM WITHOUT INITIAL CONDITIONS FOR NONLINEAR PARABOLIC EQUATIONS

Consider the equation

$$y_t - \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x,t,y,\nabla y) + a_0(x,t,y,\nabla y) = f(x,t), \quad (x,t) \in Q, \tag{10}$$

where $y : \overline{Q} \to \mathbb{R}$ is an unknown function and data-in satisfies following conditions:

($\mathcal{A}_1$) for every $i \in \{0, 1, \dots, n\}$

$$Q \times \mathbb{R} \times \mathbb{R}^n \ni (x,t,s,\xi) \mapsto a_i(x,t,s,\xi) \in \mathbb{R}$$

is the Caratheodory function, i.e., $a_i(x,t,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the continuous function for a.e. $(x,t) \in Q$, and $a_i(\cdot,\cdot,s,\xi) : Q \to \mathbb{R}$ is the measurable function for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$; moreover, $a_i(x,t,0,0) = 0$ for a. e. $(x,t) \in Q$;

($\mathcal{A}_2$) for every $i \in \{0, 1, \dots, n\}$, for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, and for a.e. $(x,t) \in Q$ the following estimate is valid $|a_i(x,t,s,\xi)| \le C_1\big(|s| + |\xi|\big) + h_i(x,t)$, where $C_1 = \text{const} > 0$, $h_i \in L^2_{\text{loc}}(S; L^2(\Omega))$;

($\mathcal{A}_3$) for every $(s_1, \xi^1), (s_2, \xi^2) \in \mathbb{R} \times \mathbb{R}^n$ and for a.e. $(x,t) \in Q$ the following inequality holds

$$\begin{aligned}&\sum_{i=1}^{n} \big(a_i(x,t,s_1,\xi^1) - a_i(x,t,s_2,\xi^2)\big)(\xi^1_i - \xi^2_i) \\ &\qquad + \big(a_0(x,t,s_1,\xi^1) - a_0(x,t,s_2,\xi^2)\big)(s_1 - s_2) \ge \alpha(t)|\xi^1 - \xi^2|^2,\end{aligned}$$

where $\alpha \in C(S)$ such that $\alpha(t) > 0$ for all $t \in S$;

($\mathcal{F}$) $f \in L^2_{\text{loc}}(S; L^2(\Omega))$.

Additionally, we impose the boundary condition

$$y\big|_{\Sigma} = 0 \tag{11}$$

on a solution of equation (10).

Definition 1. *The function y is called a weak solution of equation* (10) *satisfying boundary condition* (11) *if it belongs to $L^2_{\text{loc}}(S; H^1_0(\Omega)) \cap C(S; L^2(\Omega))$ and the following integral equality holds*

$$\begin{aligned}&\iint_Q \Big\{ -y\psi\varphi' + \sum_{i=0}^{n} a_i(x,t,y,\nabla y)\partial_i\psi\varphi \Big\}\, dxdt \\ &\qquad = \iint_Q f\psi\varphi\, dxdt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty, 0).\end{aligned} \tag{12}$$

There may exist many weak solutions of equation (10) satisfying boundary condition (11). To ensure uniqueness of the weak solution of equation (10) satisfying condition (11), we have to impose some additional conditions on solutions, for instance, some restrictions on their behavior as $t \to -\infty$. We will consider the problem of finding a weak solution of equation (10) satisfying boundary condition (11) and the analogue of the initial condition

$$\lim_{t\to-\infty} e^{\omega\int_0^t \alpha(s)ds} \|y(\cdot,t)\|_{L^2(\Omega)} = 0, \tag{13}$$

where $\omega \in \mathbb{R}$. We will briefly call this problem by problem (10), (11), (13), and the function y is called the weak solution of problem (10), (11), (13).

Lemma 2. *Let $\omega < K$, where K is a constant defined in* (1)*, and conditions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ are satisfied. Then two following statements are true.*

(i) If y is a weak solution of problem (10), (11), (13) *and*

$$f \in L^2_{\omega,1/\alpha}(S; L^2(\Omega)), \tag{14}$$

then $y \in L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ and the following estimates hold:

$$e^{2\omega\int_0^\tau \alpha(s)\,ds} \|y(\cdot,\tau)\|^2_{L^2(\Omega)} \le C_1 \int_{-\infty}^{\tau} [\alpha(t)]^{-1} e^{2\omega\int_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt,\ \tau \in S, \tag{15}$$

$$\|y\|_{L^2_{\omega,\alpha}(S;H^1_0(\Omega))} \le C_2 \|f\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))}, \tag{16}$$

where C_1, C_2 are positive constants depending on K and ω only.

(ii) If y_1 and y_2 are two weak solutions of problem (10), (11), (13) *with $f = f_1$ and $f = f_2$ correspondingly, and*

$$f_k \in L^2_{\omega,1/\alpha}(S; L^2(\Omega)) \quad (k = 1, 2), \tag{17}$$

then the following estimates hold:

$$\begin{aligned} &e^{2\omega\int_0^\tau \alpha(s)\,ds} \|y_1(\cdot,\tau) - y_2(\cdot,\tau)\|^2_{L^2(\Omega)} \\ &\qquad \le C_1 \int_{-\infty}^{\tau} [\alpha(t)]^{-1} e^{2\omega\int_0^t \alpha(s)\,ds} \|f_1(\cdot,t) - f_2(\cdot,t)\|^2_{L^2(\Omega)}\,dt,\ \tau \in S, \end{aligned} \tag{18}$$

$$\|y_1 - y_2\|_{L^2_{\omega,\alpha}(S;H^1_0(\Omega))} \le C_2 \|f_1 - f_2\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))}, \tag{19}$$

where C_1, C_2 are positive constants such as in (15) *and* (16)*.*

Proof. First we prove statement *(ii)*. For function $z : Q \to \mathbb{R}$ let us denote

$$a_i(z)(x,t) := a_i(x,t,z(x,t),\nabla z(x,t)), \quad (x,t) \in Q,\ i = \overline{0,n}. \tag{20}$$

From (12) for difference $y_{12} := y_1 - y_2$ we get such an integral identity

$$\begin{aligned} &\iint_Q \Big\{ -y_{12}\psi\varphi' + \sum_{i=0}^{n} \big(a_i(y_1) - a_i(y_2)\big)\partial_i\psi\varphi \Big\}\,dxdt \\ &\qquad = \iint_Q f_{12}\psi\varphi\,dxdt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty,0), \end{aligned} \tag{21}$$

where $f_{12} := f_1 - f_2$. According to Lemma 1, (21) implies that

$$\begin{aligned}&\frac{1}{2}\theta(t)\int\limits_{\Omega}|y_{12}(x,t)|^2\,dx\Big|_{t=\tau_1}^{t=\tau_2}-\frac{1}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}|y_{12}|^2\theta'\,dxdt\\&+\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\Big[\sum_{i=0}^{n}\big(a_i(y_1)-a_i(y_2)\big)\partial_i y_{12}\Big]\theta\,dxdt=\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}f_{12}y_{12}\theta\,dxdt,\end{aligned}\tag{22}$$

where $\theta \in C^1(S)$ and $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ are arbitrary. Using Cauchy inequality with ε:

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad a, b \in \mathbb{R},\ \varepsilon > 0,\tag{23}$$

let us estimate the right side of equality (22) as follows:

$$\Big|\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}f_{12}y_{12}\theta\,dxdt\Big|\le\frac{\varepsilon}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha|y_{12}|^2\theta\,dxdt+\frac{1}{2\varepsilon}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}[\alpha]^{-1}|f_{12}|^2\theta\,dxdt,\tag{24}$$

where $\varepsilon > 0$ is arbitrary. From condition $(\mathcal{A}_3)$ we obtain following

$$\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\Big[\sum_{i=0}^{n}\big(a_i(y_1)-a_i(y_2)\big)\partial_i y_{12}\Big]\theta\,dxdt\ge\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha|\nabla y_{12}|^2\theta\,dxdt,\tag{25}$$

where $\nabla y := (y_{x_1}, \dots, y_{x_n})$. According to (24) and (25), (22) implies the inequality

$$\begin{aligned}&\frac{1}{2}\theta(\tau_2)\int\limits_{\Omega}|y_{12}(x,\tau_2)|^2\,dx-\frac{1}{2}\theta(\tau_1)\int\limits_{\Omega}|y_{12}(x,\tau_1)|^2\,dx-\frac{1}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}|y_{12}|^2\theta'\,dxdt\\&+\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha|\nabla y_{12}|^2\theta\,dxdt\le\frac{\varepsilon}{2}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha|y_{12}|^2\theta\,dxdt+\frac{1}{2\varepsilon}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}[\alpha]^{-1}|f_{12}|^2\theta\,dxdt,\end{aligned}$$

where $\varepsilon > 0$ is arbitrary.

From this taking $\theta(t) = 2e^{2\omega\int_0^t \alpha(s)\,ds}$, $t \in S$, we obtain

$$\begin{aligned}&e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int\limits_{\Omega}|y_{12}(x,\tau_2)|^2dx-e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int\limits_{\Omega}|y_{12}(x,\tau_1)|^2dx\\&\quad-2\omega\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|y_{12}|^2dxdt+2\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|\nabla y_{12}|^2\,dxdt\\&\quad\le\varepsilon\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|y_{12}|^2\,dxdt+\frac{1}{\varepsilon}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}[\alpha(t)]^{-1}e^{2\omega\int_0^t\alpha(s)\,ds}|f_{12}|^2\,dxdt.\end{aligned}\tag{26}$$

Due to (26) using (3) we obtain

$$\begin{aligned}&e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int\limits_{\Omega}|y_{12}(x,\tau_2)|^2\,dx-e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int\limits_{\Omega}|y_{12}(x,\tau_1)|^2\,dx\\&+\chi(K,\omega,\varepsilon)\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}\alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|\nabla y_{12}|^2\,dxdt\le\frac{1}{\varepsilon}\int\limits_{\tau_1}^{\tau_2}\int\limits_{\Omega}[\alpha(t)]^{-1}e^{2\omega\int_0^t\alpha(s)\,ds}|f_{12}|^2\,dxdt,\end{aligned}\tag{27}$$

where $\chi(K,\omega,\varepsilon) := (2(K-\omega)-\varepsilon)/K$ if $0<\omega<K$, and $\chi(K,\omega,\varepsilon) := (2K-\varepsilon)/K$ if $\omega \le 0$.

Taking $\varepsilon = K$ if $\omega \le 0$, and $\varepsilon = K-\omega$ if $0<\omega<K$ in (27), we obtain

$$\begin{aligned} &e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int_\Omega |y_{12}(x,\tau_2)|^2\,dx - e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int_\Omega |y_{12}(x,\tau_1)|^2\,dx \\ &+ C_3\int_{\tau_1}^{\tau_2}\int_\Omega \alpha(t)e^{2\omega\int_0^{t}\alpha(s)\,ds}|\nabla y_{12}|^2\,dxdt \le C_4\int_{\tau_1}^{\tau_2}\int_\Omega [\alpha(t)]^{-1}e^{2\omega\int_0^{t}\alpha(s)\,ds}|f_{12}|^2\,dxdt, \end{aligned} \tag{28}$$

where C_3, C_4 are positive constants depending on K and ω only.

From (13) it easily follows the condition

$$e^{2\omega\int_0^{t}\alpha(s)ds}\int_\Omega |y_{12}(x,t)|^2\,dx \to 0 \quad \text{as} \quad t\to-\infty. \tag{29}$$

Taking into account (29) and (17), we let $\tau_1 \to -\infty$ in (28). As a result, adopting $\tau_2 = \tau \in S$, we obtain

$$\begin{aligned} &e^{2\omega\int_0^{\tau}\alpha(s)\,ds}\int_\Omega |y_{12}(x,\tau)|^2\,dx + C_3\int_{-\infty}^{\tau}\int_\Omega \alpha(t)e^{2\omega\int_0^{t}\alpha(s)\,ds}|\nabla y_{12}|^2\,dxdt \\ &\le C_4\int_{-\infty}^{\tau}\int_\Omega [\alpha(t)]^{-1}e^{2\omega\int_0^{t}\alpha(s)\,ds}|f_{12}|^2\,dxdt. \end{aligned} \tag{30}$$

Hence, using inequality (3), we easily obtain estimates (18) and (19).

Now let us prove statement *(i)*. Using the condition $(\mathcal{A}_1)$ one can easily see that $y=0$ is a weak solution of problem (10), (11), (13) with $f=0$, thus estimates (18) and (19) with $y_1=y$, $f_1=f$ and $y_2=0$, $f_2=0$ imply estimates (15) and (16). Estimate (16) implies that $y\in L^2_{\omega,\alpha}(S;H^1_0(\Omega))$. □

Lemma 3. *If $\omega \le K$, where K is a constant defined by* (1)*, then problem* (10)*,* (11)*,* (13) *has at most one weak solution.*

Proof. Assume the opposite. Let y_1, y_2 be two weak solutions of problem (10), (11), (13). In case $\omega<K$ according to Lemma 2 we obtain the equality

$$e^{2\omega\int_0^{\tau}\alpha(s)\,ds}\int_\Omega |y_1(x,\tau)-y_2(x,\tau)|^2\,dx = 0 \text{ for all } \tau\in S. \tag{31}$$

From proof of Lemma 2 it follows that estimate (31) is correct in case $\omega=K$ also. Indeed, if $\omega=K$, then in (27) and (30) we have $\chi(K,\omega,\varepsilon)=0$ and $C_3=0$, correspondingly, and its easily follows from the proof that estimate (18) is correct.

Equality (31) implies equality $y_1(x,t)-y_2(x,t)=0$ for a. e. $(x,t)\in Q$, that is, $y_1(x,t)=y_2(x,t)=0$ for a. e. $(x,t)\in Q$. The resulting contradiction proves our statement. □

Remark 1. *Functions* $y_c(x,t) = cv(x)e^{-Kt}$, $(x,t) \in \overline{Q}$ $(c \in \mathbb{R})$, *where* v *is an eigenfunction of problem* (2) *corresponding to the first eigenvalue, are weak solutions of equation* (10) *satisfying condition* (11), *when* $a_i = \xi_i$ $(i = \overline{1,n})$, $a_0 = 0$ *and* $f = 0$. *In this case we have* $\alpha(t) = 1$, *therefore condition* (13) *takes on the form:* $e^{\omega t}\|y(\cdot,t)\|_{L^2(\Omega)} \underset{t\to-\infty}{\longrightarrow} 0$. *Obviously in this case for nonzero solutions we have* $e^{Kt}\|y_c(\cdot,t)\|_{L^2(\Omega)} \underset{t\to-\infty}{\longrightarrow} C = const \neq 0$, $e^{\omega t}\|y_c(\cdot,t)\|_{L^2(\Omega)} \underset{t\to-\infty}{\longrightarrow} +\infty$ *if* $\omega < K$, *and* $e^{\omega t}\|y_c(\cdot,t)\|_{L^2(\Omega)} \underset{t\to-\infty}{\longrightarrow} 0$ *if* $\omega > K$. *This means that the condition* $\omega \leq K$ *is essential for ensuring the uniqueness of the weak solution of problem* (10), (11), (13), *i.e., it cannot be simplified.*

Theorem 1. *Suppose that conditions* $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ *hold, and* $\omega < K$, *where* K *is a constant defined in* (1), *and*

$$f \in L^2_{\omega,1/\alpha}(S; L^2(\Omega)). \tag{32}$$

Then there exists a unique weak solution of problem (10), (11), (13), *it belongs to the space* $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ *and estimates* (15) *and* (16) *are correct.*

Proof. Lemma 3 gives us a uniqueness of a weak solution of problem (10), (11), (13). It remains to prove the existence of a weak solution of this problem.

For each $m \in N$ we define $f_m(\cdot,t) := f(\cdot,t)$, if $-m < t \leq 0$, and $f_m(\cdot,t) := 0$, if $t \leq -m$, and consider the problem of finding a function $y_m \in L^2(-m,0; H^1_0(\Omega)) \cap C([-m,0]; L^2(\Omega))$ satisfying the initial condition

$$y_m(x,-m) = 0, \quad x \in \Omega, \tag{33}$$

(as an element of space $C([-m,0]; L^2(\Omega))$) and equation (10) in Q_m in the sense of the following integral identity

$$\iint\limits_{Q_m} \Big\{ -y_m\psi\varphi' + \sum_{i=0}^{n} a_i(y_m)\partial_i\psi\varphi \Big\}\, dxdt = \iint\limits_{Q_m} f_m\psi\varphi\, dxdt, \quad \psi \in H^1_0(\Omega), \varphi \in C^1_c(-m,0).$$

The existence and uniqueness of the solution of this problem easily follows from the known results (see, for example, [14]). For every $m \in \mathbb{N}$ we extend y_m by zero for the entire set Q and keep the same notation y_m for this extension. Note that for each $m \in N$, the function y_m belongs to $L^2(S; H^1_0(\Omega)) \cap C(S; L^2(\Omega))$ and satisfies integral identity (12) with f_m substituted for f, i.e.,

$$\iint\limits_{Q} \Big\{ -y_m\psi\varphi' + \sum_{i=0}^{n} a_i(y_m)\partial_j\psi\varphi \Big\}\, dxdt = \iint\limits_{Q} f_m\psi\varphi\, dxdt, \psi \in H^1_0(\Omega), \ \varphi \in C^1_c(-\infty,0). \tag{34}$$

Consequently, we have shown that y_m is a weak solution of problem (10), (11), (13) with f_m substituted for f. Then, in particular, statement *(i)* of Lemma 2 implies estimates

$$e^{2\omega \int_0^{\tau} \alpha(s)\, ds}\|y_m(\cdot,\tau)\|^2_{L^2(\Omega)} \leq C_1 \int_{-\infty}^{\tau} [\alpha(t)]^{-1} e^{2\omega \int_0^{t} \alpha(s)\, ds}\|f(\cdot,t)\|^2_{L^2(\Omega)}\, dt, \ \tau \in S, \tag{35}$$

$$\|y_m\|_{L^2_{\omega,\alpha}(S;H^1_0(\Omega))} \leq C_2\|f\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))}, \tag{36}$$

where C_1, C_2 are positive constants such as in estimates (15), (16).

Let us take identity (34) with alternately $m = k$ and $m = l$, where k, l are arbitrary positive integers, $l > k$, and apply statement (ii) of Lemma 2. As a result, we obtain estimates similar to (18), (19), i.e.

$$e^{2\omega \int\limits_0^\tau \alpha(s)\,ds} \|y_k(\cdot,\tau) - y_l(\cdot,\tau)\|^2_{L^2(\Omega)} \le C_1 \int\limits_{-l}^{-k} [\alpha(t)]^{-1} e^{2\omega \int\limits_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt,\ \tau \in S, \tag{37}$$

$$\|y_k - y_l\|_{L^2_{\omega,\alpha}(S;H^1_0(\Omega))} \le C_2 \int\limits_{-l}^{-k} [\alpha(t)]^{-1} e^{2\omega \int\limits_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt. \tag{38}$$

Condition (32) implies that the right-hand sides of inequalities (37) and (38) tend to zero when k and l tend to $+\infty$. This means that the sequence $\{y_m\}_{m=1}^\infty$ is a Cauchy sequence in the space $L^2_{\omega,\alpha}(S;H^1_0(\Omega))$ and $C(S;L^2(\Omega))$. Consequently, we obtain the existence of the function $y \in L^2_{\omega,\alpha}(S;H^1_0(\Omega)) \cap C(S;L^2(\Omega))$ such that

$$y_m \underset{m\to\infty}{\longrightarrow} y \qquad \text{strongly in} \quad L^2_{\omega,\alpha}(S;H^1_0(\Omega)) \quad \text{and} \quad C(S;L^2(\Omega)). \tag{39}$$

Note that (39) implies

$$\partial_i y_m \underset{m\to\infty}{\longrightarrow} \partial_i y \quad \text{strongly in} \quad L^2_{\mathrm{loc}}(S;L^2(\Omega)), \quad i = \overline{0,n}. \tag{40}$$

Condition $(\mathcal{A}_2)$ and estimate (36) gives us for each $t_1, t_2 \in S (t_1 < t_2)$ the following:

$$\int\limits_{t_1}^{t_2}\int\limits_\Omega |a_i(y_m)|^2\,dxdt \le C_5 \int\limits_{t_1}^{t_2}\int\limits_\Omega \left(|y_m|^2 + |\nabla y_m|^2 + |h_i|^2\right)dxdt \le C_6, \tag{41}$$

where C_5 and C_6 are positive constants independent on m.

Hence, from (41) we obtain that $a_i(y_m)$ is bounded in $L^2_{\mathrm{loc}}(S;L^2(\Omega))$. This and (40) yield that there exists a subsequence of $\{y_m\}_{m=1}^\infty$ (still denoted by $\{y_m\}_{m=1}^\infty$) and functions $\chi_i \in L_{2,\mathrm{loc}}(S;L_2(\Omega))$ $(i = \overline{0,n})$ such that

$$\partial_i y_m \underset{m\to\infty}{\longrightarrow} \partial_i y \quad \text{a.e. on} \quad Q, \quad i = \overline{0,n}, \tag{42}$$

$$a_i(y_m) \underset{m\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in} \quad L_{2,\mathrm{loc}}(S;L_2(\Omega)), \quad i = \overline{0,n}. \tag{43}$$

Condition $(\mathcal{A}_1)$ and (42) yield

$$a_i(y_m) \underset{m\to\infty}{\longrightarrow} a_i(y) \quad \text{a.e. on Q}, \quad i = \overline{0,n}. \tag{44}$$

According to [17, Lemma 1.3], from (43) and (44) we obtain

$$a_i(y_m) \underset{m\to\infty}{\longrightarrow} a_i(y) \quad \text{weakly in} \quad L_{2,\mathrm{loc}}(S;L_2(\Omega)), \quad i = \overline{0,n}. \tag{45}$$

Let us show that the function y is a weak solution of problem (10), (11), (13). To do this, we let $m \to \infty$ in identity (34), taking into account (40), (45) and the definition of the function f_m. As a result we obtain identity (12). Now, taking into account (39), we let $m \to +\infty$ in (35). From the resulting inequality and condition (32), we obtain condition (13). Hence, we have proven that y is a weak solution of problem (10), (11), (13). $\square$

3 FORMULATION OF THE OPTIMAL CONTROL PROBLEM AND THE MAIN RESULT

Let $U := L^\infty(Q)$ be a space of controls and $U_\partial := \Big\{ v \in U \,\Big|\, v \geq 0 \quad \text{a. e. in} \quad Q \Big\}$ be the set of admissible controls. We assume that the state of the investigated evolutionary system for a given control $v \in U_\partial$ is described by a weak solution of the equation

$$y_t - \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x,t,y,\nabla y) + a_0(x,t,y,\nabla y) + v(x,t)y = f(x,t), \quad (x,t) \in Q, \tag{46}$$

satisfying conditions (11) and (13) (this problem is similar to problem (10), (11), (13)). This means that y is a function belonging to the space $L^2_{\text{loc}}(S; H^1_0(\Omega)) \cap C(S; L^2(\Omega))$ and satisfying the integral identity

$$\begin{aligned} &\iint\limits_Q \Big\{ - y\psi\varphi' + \sum_{i=0}^{n} a_i(x,t,y,\nabla y)\partial_i\psi\varphi + vy\psi\varphi \Big\}\, dxdt \\ &\qquad = \iint\limits_Q f\psi\varphi\, dxdt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty,0), \end{aligned} \tag{47}$$

and condition (13) under assumptions $(\mathcal{A}_1)$–$(\mathcal{A}_3)$, $(\mathcal{F})$.

A weak solution y of the specified problem will be called a weak solution of problem (46), (11), (13) for control v, and will be denoted by $y(v)$, or $y(x,t)$, $(x,t) \in Q$, or $y(x,t;v)$, $(x,t) \in Q$. Further, we assume that condition (32) and the inequality $\omega < K$ hold. From the previous section (see Theorem 1), we immediately obtain the existence and uniqueness of a weak solution of problem (46), (11), (13) (for a given $v \in U_\partial$) and its estimates (15), (16).

We assume that the cost functional has the form

$$J(v) = \|y(\cdot,0;v) - z_0(\cdot)\|^2_{L^2(\Omega)} + \mu\|v\|_{L^\infty(Q)}, \quad v \in U, \tag{48}$$

where $z_0 \in L^2(\Omega)$, $\mu > 0$ are given.

We consider the following *optimal control problem*: find a control $u \in U_\partial$ such that

$$J(u) = \inf_{v \in U_\partial} J(v). \tag{49}$$

We briefly call this problem (49), and its solutions will be called *optimal controls*.

The main result of this paper is the following theorem.

Theorem 2. *Problem (49) has a solution.*

4 PROOF OF THE MAIN RESULT

Proof of Theorem 2. Since the cost functional J is bounded below, there exists a minimizing sequence $\{v_k\}$ for J in U_∂, i.e., $J(v_k) \underset{k\to\infty}{\longrightarrow} \inf\limits_{v \in U_\partial} J(v)$. This and (48) imply that the sequence $\{v_k\}$ is bounded in the space $L^\infty(Q)$, that is

$$\operatorname*{ess\,sup}_{(x,t)\in Q} |v_k(x,t)| \leq C_7 \text{ for all } k \in \mathbb{N}, \tag{50}$$

where C_7 is a constant, which does not depend on k.

Since for each $k \in \mathbb{N}$ the function $y_k := y(v_k)$ $(k \in \mathbb{N})$ is a weak solution of problem (46), (11), (13) for $v = v_k$, the following identity holds:

$$\iint\limits_Q \Big\{ - y_k\psi\varphi' + \sum_{i=0}^n a_i(y_k)\partial_i\psi\varphi + v_k y_k\psi\varphi \Big\}\, dxdt = \iint\limits_Q f\psi\varphi\, dxdt, \quad \psi \in H_0^1(\Omega),\ \varphi \in C_c^1(-\infty, 0). \tag{51}$$

According to Lemma 2 for each $k \in \mathbb{N}$ we have the estimates

$$e^{2\omega\int\limits_0^\tau \alpha(s)\,ds}\|y_k(\cdot,\tau)\|^2_{L_2(\Omega)} \le C_1 \int\limits_{-\infty}^{\tau} [\alpha(t)]^{-1} e^{2\omega\int\limits_0^t \alpha(s)\,ds}\|f(\cdot,t)\|^2_{L_2(\Omega)}\, dt, \quad \tau \in S, \tag{52}$$

$$\|y_k\|_{L^2_{\omega,\alpha}(S;H_0^1(\Omega))} \le C_2\|f\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))}, \tag{53}$$

where constants C_1, C_2 are independent on $k \in \mathbb{N}$. From $(\mathcal{A}_2)$ and (53) it follows

$$\int\limits_{\tau_1}^{\tau_2}\int\limits_\Omega \sum_{i=0}^n |a_i(y_k)|^2\, dxdt \le C_8 \int\limits_{\tau_1}^{\tau_2}\int\limits_\Omega \left(|y_k|^2 + |\nabla y_k|^2 + |h_i|^2\right) dxdt \le C_9, \tag{54}$$

where $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ are arbitrary, and C_8, C_9 are positive constants independent on k.

Taking into statement *(ii)* of Lemma 1, from (51) for arbitrary $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ and $k \in \mathbb{N}$ we obtain

$$\int\limits_{\tau_1}^{\tau_2} \|y_{k,t}\|^2_{H^{-1}(\Omega)}\, dt \le \int\limits_{\tau_1}^{\tau_2}\int\limits_\Omega \Big(\sum_{i=0}^n |a_i(y_k)|^2 + |v_k y_k - f|^2 \Big) dxdt. \tag{55}$$

Taking into account condition (32), (50) and (54), estimate (55) implies

$$\int\limits_{\tau_1}^{\tau_2} \|y_{k,t}\|^2_{H^{-1}(\Omega)}\, dt \le C_{10} \text{ for all } k \in \mathbb{N}, \tag{56}$$

where $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ are arbitrary, $C_{10} > 0$ is a constant which depends on τ_1 and τ_2, but does not depend on k.

According to the Compactness Lemma (see [19, Proposition 4.2]), and the compactness of the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ (see [18] c. 245), estimates (50), (53), (54), (56) yield that there exists a subsequence of the sequence $\{v_k, y_k\}$ (still denoted by $\{v_k, y_k\}$) and functions $u \in U_\partial$, $y \in L^2_{\omega,\alpha}(S; H_0^1(\Omega))$ and $\chi_i \in L^2_{\rm loc}(S; L_2(\Omega))$ $(i = \overline{0,n})$ such that

$$v_k \underset{k\to\infty}{\longrightarrow} u \quad *\text{-weakly in} \quad L^\infty(Q), \tag{57}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{weakly in} \quad L^2_{\omega,\alpha}(S; H_0^1(\Omega)), \tag{58}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{strongly in} \quad L^2_{\rm loc}(S; L^2(\Omega)), \tag{59}$$

$$a_i(y_k) \underset{k\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in} \quad L_{2,\rm loc}(S; L_2(\Omega)), \quad i = \overline{0,n}. \tag{60}$$

Note that (58) implies the following

$$\partial_i y_k \underset{k\to\infty}{\longrightarrow} \partial_i y \quad \text{weakly in} \quad L^2_{\mathrm{loc}}(S; L^2(\Omega)), \quad i=\overline{0,n}. \tag{61}$$

Let us show that (57) and (59) yield

$$\iint\limits_Q y_k v_k \psi\varphi\, dxdt \underset{k\to\infty}{\longrightarrow} \iint\limits_Q yu\psi\varphi\, dxdt \text{ for all } \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty,0). \tag{62}$$

Indeed, let $g := \psi\varphi$ and $t_1, t_2 \in S$ be such that $\operatorname{supp}\varphi \subset [t_1, t_2]$. Then we have

$$\iint\limits_Q y_k v_k g\, dxdt = \int\limits_{t_1}^{t_2}\int\limits_{\Omega} (y_k v_k - y v_k + y v_k) g\, dxdt = \int\limits_{t_1}^{t_2}\int\limits_{\Omega} y v_k g\, dxdt + \int\limits_{t_1}^{t_2}\int\limits_{\Omega} (y_k - y) v_k g\, dxdt. \tag{63}$$

From (50) and (59) it follows

$$\Big|\int\limits_{t_1}^{t_2}\int\limits_{\Omega} (y_k - y) v_k g\, dxdt\Big| \le \Big(\int\limits_{t_1}^{t_2}\int\limits_{\Omega} |v_k g|^2\, dxdt\Big)^{1/2}\Big(\int\limits_{t_1}^{t_2}\int\limits_{\Omega} |y_k - y|^2\, dxdt\Big)^{1/2} \underset{k\to\infty}{\longrightarrow} 0. \tag{64}$$

Thus, using (64) and (57), (63) implies (62). Similarly to (62) it can be easily shown that (57) and (59) yield

$$\iint\limits_Q |y_k|^2 v_k \varphi\, dxdt \underset{k\to\infty}{\longrightarrow} \iint\limits_Q |y|^2 u\varphi\, dxdt \text{ for all } \varphi \in C^1_c(-\infty,0). \tag{65}$$

Using (61), (62), and letting $k \to \infty$ in (51), we obtain

$$\iint\limits_Q \Big\{ -y\psi\varphi' + \sum_{i=0}^n \chi_i \partial_i\psi\varphi + uy\psi\varphi \Big\}\, dxdt = \iint\limits_Q f\psi\varphi\, dxdt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty,0). \tag{66}$$

According to Lemma 1, identity (66) implies that $y \in C(S; L^2(\Omega))$.

Now let us show that the equality

$$\int\limits_\Omega \Big\{ \sum_{i=0}^n \chi_i \partial_i \psi \Big\}\, dx = \int\limits_\Omega \Big\{ \sum_{i=0}^n a_i(y) \partial_i \psi \Big\}\, dx \tag{67}$$

is valid for every $\psi \in H^1_0(\Omega)$ and for a. e. $t \in S$. For this we use the monotonicity method (see [17]). Let us take an arbitrary functions $w \in L_{2,\mathrm{loc}}(S; H^1(\Omega))$ and $\theta \in C^1_c(-\infty,0)$, $\theta(t) \ge 0$ for all $t \in (-\infty,0)$. Using condition $(\mathcal{A}_3)$ for every $k \in \mathbb{N}$ we have

$$W_k := \iint\limits_Q \Big\{ \sum_{i=0}^n (a_i(y_k) - a_i(w))(\partial_i y_k - \partial_i w) \Big\}\theta\, dxdt \ge 0.$$

From this we obtain

$$W_k = \iint\limits_Q \sum_{i=0}^n a_i(y_k)\partial_i y_k \theta\, dxdt - \iint\limits_Q \sum_{i=0}^n \big[a_i(y_k)\partial_i w + a_i(w)(\partial_i y_k - \partial_i w)\big]\theta \ge 0,\ \ k \in \mathbb{N}. \tag{68}$$

According to Lemma 1, (51) implies

$$-\frac{1}{2}\iint\limits_{Q} |y_k|^2\theta'\,dxdt + \iint\limits_{Q} \Big\{\sum_{i=0}^{n} a_i(y_k)\partial_i y_k + v_k|y_k|^2\Big\}\theta\,dxdt = \iint\limits_{Q} f y_k\theta\,dxdt. \tag{69}$$

From (68), using (69), we obtain

$$\begin{aligned} W_k = &\iint\limits_{Q} \Big\{\frac{1}{2}|y_k|^2\theta' + \big(f y_k - v_k|y_k|^2\big)\theta\Big\} dxdt \\ &- \iint\limits_{Q} \sum_{i=0}^{n} \big[a_i(y_k)\partial_i w + a_i(w)(\partial_i y_k - \partial_i w)\big]\theta\,dxdt \geq 0,\ k\in\mathbb{N}. \end{aligned} \tag{70}$$

Taking into account (59) and (65) we have

$$\lim_{k\to\infty}\iint\limits_{Q}\Big\{\frac{1}{2}|y_k|^2\theta' + \big(f y_k - v_k|y_k|^2\big)\theta\Big\}\,dxdt = \iint\limits_{Q}\Big\{\frac{1}{2}|y|^2\theta' + \big(f y - u|y|^2\big)\theta\Big\}\,dxdt. \tag{71}$$

By (60), (61) and (71) from (70) we get

$$\begin{aligned} 0 \leq \lim_{k\to\infty} W_k = &\iint\limits_{Q} \Big\{\frac{1}{2}|y|^2\theta' + \big(f y - u|y|^2\big)\theta\Big\}\,dxdt \\ &- \iint\limits_{Q}\sum_{i=0}^{n} \big[\chi_i\partial_i w + a_i(w)(\partial_i y - \partial_i w)\big]\theta dxdt. \end{aligned} \tag{72}$$

From (66), using Lemma 1, we obtain

$$\iint\limits_{Q}\sum_{i=0}^{n}\chi_i\partial_i y\theta\,dxdt = \iint\limits_{Q} \Big\{\frac{1}{2}|y|^2\theta' + \big(f y - u|y|^2\big)\theta\Big\}\,dxdt. \tag{73}$$

Thus, (72) and (73) imply that

$$\iint\limits_{Q} \Big\{\sum_{i=0}^{n}(\chi_i - a_i(w))(\partial_i y - \partial_i w)\Big\}\theta\,dxdt \geq 0. \tag{74}$$

Substituting $w = y - \lambda\psi$ in the above inequality, where $\psi \in H_0^1(\Omega)$, $\lambda > 0$ are arbitrary, and dividing the obtained inequality by λ we get

$$\iint\limits_{Q} \Big\{\sum_{i=0}^{n}(\chi_i - a_i(u - \lambda\psi))\partial_i\psi\Big\}\theta\,dxdt \geq 0. \tag{75}$$

Letting $\lambda \to 0+$ in (75), using condition $(\mathcal{A}_2)$ and the Dominated Convergence Theorem (see [9, p. 648]), we have

$$\iint\limits_{Q} \Big\{\sum_{i=1}^{n}(\chi_i - a_i(y))\partial_i\psi\Big\}\theta\,dxdt = 0. \tag{76}$$

Since $\psi \in H_0^1(\Omega)$, $\theta \in C_c^1(-\infty,0)$ are arbitrary functions, then (76) impliest (67).

Therefore y is a weak solution of equation (46), satisfying boundary condition (11). Hence, the function y is a weak solution of equation (46) for $v = u$, satisfying boundary condition (11). Let us show that y satisfies condition (13). First, we prove the following convergence:

$$\text{for all } \tau \in S: \quad y_k(\cdot,\tau) \underset{k\to\infty}{\longrightarrow} y(\cdot,\tau) \quad \text{strongly in} \quad L^2(\Omega). \tag{77}$$

For this purpose, we subtract identity (51) from identity (47) with $v = u$, $\psi \in H_0^1(\Omega)$, $\varphi \in C_c^1(-\infty,0)$:

$$\iint\limits_Q \Big\{ -(y-y_k)\psi\varphi' + \sum_{i=0}^n \big(a_i(y)-a_i(y_k)\big)\partial_i\psi\varphi + uy - v_k y_k \Big\}\, dxdt = 0. \tag{78}$$

To the resulting identity (78), we apply Lemma 1 with $\theta(t) = 2(t-\tau+1)$, $\tau_1 = \tau - 1$, $\tau_2 = \tau$, where $\tau \in S$ is any fixed. Consequently, we get

$$\begin{gathered}\int\limits_\Omega |y(x,\tau)-y_k(x,\tau)|^2\, dx - \int\limits_{\tau-1}^{\tau}\int\limits_\Omega |y-y_k|^2\, dxdt \\ + \int\limits_{\tau-1}^{\tau}\int\limits_\Omega \Big[\sum_{i=0}^n \big(a_i(y)-a_i(y_k)\big)\partial_i(y-y_k) + (uy - v_k y_k)(y-y_k)\Big]\theta\, dxdt = 0.\end{gathered} \tag{79}$$

From (79), taking into account condition $(\mathcal{A}_3)$ we obtain:

$$\int\limits_\Omega |y(x,\tau)-y_k(x,\tau)|^2\, dx \le \int\limits_{\tau-1}^{\tau}\int\limits_\Omega \big[|y-y_k|^2 - (uy - v_k y_k)(y-y_k)\theta\big]\, dxdt. \tag{80}$$

Inequality (80) implies

$$\int\limits_\Omega |y(x,\tau)-y_k(x,\tau)|^2\, dx \le 2\int\limits_{\tau-1}^{\tau}\int\limits_\Omega \big[(1+v_k)|y-y_k|^2 + |y||u-v_k||y-y_k|\big]\, dxdt. \tag{81}$$

Using (50) and Cauchy-Schwarz inequality, from (81) we obtain

$$\int\limits_\Omega |y(x,\tau)-y_k(x,\tau)|^2\, dx \le C_{11}\Big(\Big[\int\limits_{\tau-1}^{\tau}\int\limits_\Omega |y-y_k|^2\, dxdt\Big]^{1/2} + \int\limits_{\tau-1}^{\tau}\int\limits_\Omega |y-y_k|^2\, dxdt\Big), \tag{82}$$

where $C_{11} > 0$ is a constant which does not depend on k. From (82), according to (59), we get (77). Taking into account (77), let $k \to \infty$ in (52). The resulting inequality, according to condition (32), implies

$$\lim_{\tau\to-\infty} e^{2\omega\int_0^\tau \alpha(s)\, ds} \int\limits_\Omega |y(x,\tau|^2\, dx = 0, \tag{83}$$

that is condition (13) holds. Hence, we have shown that $y = y(u) = y(x,t;u)$, $(x,t) \in Q$, is the state of the controlled system for the control u.

It remains to prove that u is a minimizing element of the functional J. Indeed, (77) implies

$$\|y_k(\cdot,0)-z_0(\cdot)\|^2_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} \|y(\cdot,0)-z_0(\cdot)\|^2_{L^2(\Omega)}. \tag{84}$$

Also, (57) and properties of $*$-weakly convergent sequences yield

$$\liminf_{k\to\infty} \|v_k\|_{L^\infty(Q)} \geq \|u\|_{L^\infty(Q)}. \tag{85}$$

From (48), (84) and (85), it easily follows that $\lim_{k\to\infty} J(v_k) \geq J(u)$. Thus, we have shown that u is a solution of problem (49). $\square$

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Бокало М.М., Цебенко А.М. *Задача оптимального керування системами, стан яких описується задачею без початкових умов для нелінійних параболічних рівнянь* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 21–37.

Досліджено задачу оптимального керування системами, стан яких описується задачею Фур'є для нелінійних параболічних рівнянь. Керування входить як коефіцієнт в рівнянні стану системи. Доведено існування оптимального керування у випадку фінального спостереження.

Ключові слова і фрази: оптимальне керування, задача без початкових умов, нелінійне параболічне рівняння.