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PROPERTIES OF DISTANCE SPACES WITH POWER TRIANGLE INEQUALITIES

Metric spaces provide a framework for analysis and have several very useful properties. Many of these properties follow in part from the triangle inequality. However, there are several applications in which the triangle inequality does not hold but in which we may still like to perform analysis. This paper investigates what happens if the triangle inequality is removed all together, leaving what is called a distance space, and also what happens if the triangle inequality is replaced with a much more general two parameter relation, which is herein called the "power triangle inequality". The power triangle inequality represents an uncountably large class of inequalities, and includes the triangle inequality, relaxed triangle inequality, and inframetric inequality as special cases. The power triangle inequality is defined in terms of a function that is herein called the power triangle function. The power triangle function is itself a power mean, and as such is continuous and monotone with respect to its exponential parameter, and also includes the operations of maximum, minimum, mean square, arithmetic mean, geometric mean, and harmonic mean as special cases.

Key words and phrases: metric space, distance space, semimetric space, quasi-metric space, triangle inequality, relaxed triangle inequality, inframetric, arithmetic mean, means square, geometric mean, harmonic mean, maximum, minimum, power mean.

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1 INTRODUCTION AND SUMMARY

Metric spaces provide a framework for analysis and have several very useful properties. Many of these properties follow in part from the triangle inequality. However, there are several applications[1] in which the triangle inequality does not hold but in which we would still like to perform analysis. So the questions that natually follow are:

Q1. What happens if we remove the triangle inequality all together?

Q2. What happens if we replace the triangle inequality with a generalized relation?

A distance space is a metric space without the triangle inequality constraint. Section 3 introduces distance spaces and demonstrates that some properties commonly associated with metric spaces also hold in any distance space:

D1. $\varnothing$ and X are open, (Theorem 1),

D2. the intersection of a finite number of open sets is open, (Theorem 1),

D3. the union of an arbitrary number of open sets is open, (Theorem 1),

D4. every Cauchy sequence is bounded, (Proposition 1),

D5. any subsequence of a Cauchy sequence is also Cauchy, (Proposition 2),

D6. the Cantor Intersection Theorem holds, (Theorem 4).

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[1] References for applications in which the triangle inequality may not hold: [21, 32–34, 65, 76, 80, 108, 114–116].

The following five properties (M1–M5) *do* hold in any metric space. However, the examples from Section 3 listed below demonstrate that the five properties do *not* hold in all distance spaces:

M1.	the metric function is continuous	fails to hold in Examples 1–3,
M2.	open balls are open	fails to hold in Examples 1 and 2,
M3.	the open balls form a base for a topology	fails to hold in Examples 1 and 2,
M4.	the limits of convergent sequences are unique	fails to hold in Example 1,
M5.	convergent sequences are Cauchy	fails to hold in Example 2.

Hence, Section 3 answers question Q1.

Section 4 begins to answer question Q2 by first introducing a new function, called the power triangle function (see Definition 21) in a distance space (X, d), as

$$\tau(p, \sigma; x, y, z; \mathsf{d}) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x, z) + \frac{1}{2}\mathsf{d}^p(z, y)\right]^{\frac{1}{p}}$$

for some $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}$. Section 4 then goes on to use this function to define a new relation, called the power triangle inequality in (X, d), and defined as

$$\triangle(p, \sigma; \mathsf{d}) := \left\{(x, y, z) \in X^3 \,|\, \mathsf{d}(x, y) \le \tau(p, \sigma; x, y, z; \mathsf{d})\right\}.$$

The power triangle inequality is a generalized form of the triangle inequality in the sense that the two inequalities coincide at $(p, \sigma) = (1, 1)$. Other special values include $(1, \sigma)$ yielding the relaxed triangle inequality (and its associated near metric space) and (∞, σ) yielding the σ-inframetric inequality (and its associated σ-inframetric space). Collectively, a distance space with a power triangle inequality (see Definition 23) is herein called a power distance space (see Definition 24) and denoted $(X, \mathsf{d}, p, \sigma)$.[2]

The power triangle function, at $\sigma = \frac{1}{2}$, is a special case of the power mean (see Definition 32) with $N = 2$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$. Power means have the elegant properties of being continuous and monontone with respect to a free parameter p. From this it is easy to show that the power triangle function is also continuous and monontone with respect to both p and σ. Special values of p yield operators coinciding with maximum, minimum, mean square, arithmetic mean, geometric mean, and harmonic mean. Power means are briefly described in Appendix B.2 (see also Corollaries 2, 3, 8 and Theorem 18).

Section 4.2 investigates the properties of power distance spaces. In particular, it shows for what values of (p, σ) the properties M1–M5 hold. Here is a summary of the results in a power distance space $(X, \mathsf{d}, p, \sigma)$, for all $x, y, z \in X$:

(M1)	holds for any $(p, \sigma) \in (\mathbb{R}^* \backslash \{0\}) \times \mathbb{R}^+$ such that $2\sigma = 2^{\frac{1}{p}}$,	(Theorem 9),
(M2)	holds for any $(p, \sigma) \in (\mathbb{R}^* \backslash \{0\}) \times \mathbb{R}^+$ such that $2\sigma \le 2^{\frac{1}{p}}$,	(Corollary 7),
(M3)	holds for any $(p, \sigma) \in (\mathbb{R}^* \backslash \{0\}) \times \mathbb{R}^+$ such that $2\sigma \le 2^{\frac{1}{p}}$,	(Corollary 6),
(M4)	holds for any $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$,	(Theorem 10),
(M5)	holds for any $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$,	(Theorem 7).

Appendix A briefly introduces topological spaces. The open balls of any metric space form a base for a topology. This is largely due to the fact that in a metric space, open balls are open. Because of this, in metric spaces it is convenient to use topological structure to define

[2] For examples of power distance spaces see Definition 24.

and exploit analytic concepts such as continuity, convergence, closed sets, closure, interior, and accumulation point. For example, in a metric space, the traditional definition of defining continuity using open balls and the topological definition using open sets, coincide with each other. Again, this is largely because the open balls of a metric space are open.

However, this is not the case for all distance spaces. In general, the open balls of a distance space are not open, and they are not a base for a topology. In fact, the open balls of a distance space are a base for a topology if and only if the open balls are open. While the open sets in a distance space do induce a topology, it's open balls may not (see Theorem 2, Corollary 1).

2 STANDARD DEFINITIONS

2.1 Standard sets

Definition 1. *Let $\mathbb{R}$ be the set of real numbers. Let $\mathbb{R}^{\vdash}$ (resp. $\mathbb{R}^{+}$) be the set of non-negative (resp. postive) real numbers. Let $\mathbb{R}^{*} := \mathbb{R} \cup \{-\infty, \infty\}$ be the set of extended real numbers [95]. Let $\mathbb{Z}$ be the set of integers. Let $\mathbb{N} := \{n \in \mathbb{Z} \mid n \geq 1\}$ be the set of natural numbers. Let $\mathbb{Z}^{*} := \mathbb{Z} \cup \{-\infty, \infty\}$ be the extended set of integers.*

Definition 2. *Let X be a set. The quantity $\mathbb{2}^X$ (the set of all subsets of X) is the power set of X, i.e. $\mathbb{2}^X := \{A \subseteq X\}$.*

2.2 Relations

Definition 3 ([12, 13, 29, 57, 67, 78, 106])**.** *Let X and Y be sets. The Cartesian product $X \times Y$ of X and Y is the set $X \times Y := \{(x, y) \mid x \in X \text{ and } y \in Y\}$. An ordered pair (x, y) on X and Y is any element in $X \times Y$. A relation ® on X and Y is any subset of $X \times Y$ such that $® \subseteq X \times Y$. The set $\mathbb{2}^{XY}$ is the set of all relations in $X \times Y$. A relation $\mathsf{f} \in \mathbb{2}^{XY}$ is a function if $(x, y_1) \in \mathsf{f}$ and $(x, y_2) \in \mathsf{f}$ implies $y_1 = y_2$. The set Y^X is the set of all functions in $\mathbb{2}^{XY}$.*

Note, that the notation Y^X and $\mathbb{2}^{XY}$ is motivated by the fact that for finite X and Y, $\left| Y^X \right| = \left| Y \right|^{|X|}$ and $\left| \mathbb{2}^{XY} \right| = 2^{|X| \cdot |Y|}$.

2.3 Set functions

Definition 4 ([55, 87, 92])**.** *Let $\mathbb{2}^X$ be the power set of a set X. A set $\mathcal{S}(X)$ is a set structure on X if $\mathcal{S}(X) \subseteq \mathbb{2}^X$. A set structure $\mathcal{Q}(X)$ is a paving on X if $\varnothing \in \mathcal{Q}(X)$.*

Definition 5 ([25, 55, 56, 92])**.** *Let $\mathcal{Q}(X)$ be a paving on a set X. Let Y be a set containing the element 0. A function $\mathsf{m} \in Y^{\mathcal{Q}(X)}$ is a set function if $\mathsf{m}(\varnothing) = 0$.*

Definition 6. *The set function $|A| \in \mathbb{Z}^{*\mathbb{2}^X}$ is the cardinality of $A \in \mathbb{2}^X$ such that*

$$|A| := \begin{cases} \text{the number of elements in } A, & \text{for finite } A, \\ \infty, & \text{otherwise.} \end{cases}$$

Definition 7. *Let $|X|$ be the* cardinality *of a set X. The structure $\varnothing$ is the empty set, and is a set such that $|\varnothing| = 0$.*

2.4 Order

Definition 8 ([4,38,70,77]). *Let X be a set. A relation $\le$ is an order relation in 2^{XX} if*

$$\begin{array}{llll} 1. & x \le x & \forall x \in X & \text{(reflexive)} \quad \text{and} \\ 2. & x \le y \text{ and } y \le z \implies x \le z & \forall x, y \in X & \text{(transitive)} \quad \text{and} \\ 3. & x \le y \text{ and } y \le x \implies x = y & \forall x, y \in X & \text{(anti-symmetric)}. \end{array}$$

An ordered set is the pair $(X, \le)$.[3] *A relation $\le$ is a preorder relation in 2^{XX} if only the first two conditions hold.*

We write $x < y$ if $x \le y$ and $x \ne y$ for any x, y from an ordered set $(X, \le)$.

Definition 9 ([2,91]). *In an ordered set $(X, \le)$ the set $[x : y] := \{z \in X \mid x \le z \le y\}$ is a closed interval, the sets $(x : y] := \{z \in X \mid x < z \le y\}$ and $[x : y) := \{z \in X \mid x \le z < y\}$ are half-open intervals, the set $(x : y) := \{z \in X \mid x < z < y\}$ is an open interval.*

Definition 10. *Let $(\mathbb{R}, \le)$ be the ordered set of real numbers. The absolute value $|\cdot| \in \mathbb{R}^{\mathbb{R}}$ is defined as*[4] $|x| := \begin{cases} -x, & \text{for } x \le 0, \\ x, & \text{otherwise.} \end{cases}$

3 Background: distance spaces

A distance space can be defined as a metric space without the triangle inequality constraint. Much of the material in this section about distance spaces is standard in metric spaces. However, this paper works through this material again to demonstrate "how far we can go", and can't go, without the triangle inequality.

3.1 Fundamental structure of distance spaces

3.1.1 Definitions

Definition 11 ([6,9,10,41,50,68,74,82,118]). *A function d in the set $\mathbb{R}^{X \times X}$ is a distance if*

$$\begin{array}{llll} 1. & \mathsf{d}(x,y) \ge 0 & \forall x, y \in X & \text{(non-negative)} \quad \text{and} \\ 2. & \mathsf{d}(x,y) = 0 \iff x = y & \forall x, y \in X & \text{(nondegenerate)} \quad \text{and} \\ 3. & \mathsf{d}(x,y) = \mathsf{d}(y,x) & \forall x, y \in X & \text{(symmetric)}. \end{array}$$

The pair (X, d) is a distance space if d is a distance on a set X.

Definition 12. [5] *Let (X, d) be a distance space and 2^X be the power set of X. The diameter in (X, d) of a set $A \in 2^X$ is*

$$\operatorname{diam} A := \begin{cases} 0, & \text{for } A = \varnothing, \\ \sup\{\mathsf{d}(x,y) \mid x, y \in A\}, & \text{otherwise.} \end{cases}$$

Definition 13 ([16,110]). *A set $A \in 2^X$ is bounded in a distance space (X, d) if $\operatorname{diam} A < \infty$.*

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[3] An order relation is also called a partial order relation. An ordered set is also called a partially ordered set or poset.

[4] A more general definition for absolute value is available for any commutative ring [26]. Let R be a commutative ring. A function $|\cdot|$ in R^R is an absolute value, or modulus, on R if

$$\begin{array}{llll} 1. & |x| \ge 0 & x \in R & \text{(non-negative)} \quad \text{and} \\ 2. & |x| = 0 \iff x = 0 & x \in R & \text{(nondegenerate)} \quad \text{and} \\ 3. & |xy| = |x| \cdot |y| & x, y \in R & \text{(homogeneous/submultiplicative)} \quad \text{and} \\ 4. & |x+y| \le |x| + |y| & x, y \in R & \text{(subadditive/triangle inequality)}. \end{array}$$

[5] For definition in metric space see [30,60,83,87].

3.1.2 Properties

Remark 1. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence in a distance space (X, d). The distance space (X, d) does not necessarily have all the nice properties that a metric space has. In particular, note the following:*

1.	d *is a distance in* (X, d)	$\not\Longrightarrow$	d *is continuous in* (X, d),	*(Example 3),*
2.	B *is an open ball in* (X, d)	$\not\Longrightarrow$	B *is open in* (X, d),	*(Example 2),*
3.	B *is the set of all open balls in* (X, d)	$\not\Longrightarrow$	B *is a base for a topology on* X,	*(Example 2),*[6]
4.	$\{x_n\}$ *is convergent in* (X, d)	$\not\Longrightarrow$	*limit is unique,*	*(Example 1),*
5.	$\{x_n\}$ *is convergent in* (X, d)	$\not\Longrightarrow$	$\{x_n\}$ *is Cauchy in* (X, d),	*(Example 2).*

3.2 Open sets in distance spaces

3.2.1 Definitions

Definition 14 ([1])**.** *Let (X, d) be a distance space. An open (resp. closed) ball centered at x with radius r is the set $\mathsf{B}(x, r) := \{y \in X \mid \mathsf{d}(x, y) < r\}$ $\left(\text{resp. } \overline{\mathsf{B}}(x, r) := \{y \in X \mid \mathsf{d}(x, y) \leq r\}\right)$.*

Definition 15. *Let (X, d) be a distance space. Let $X\backslash A$ be the* set difference *of X and a set A. A set U is open in (X, d) if $U \in 2^X$ and for every x in U there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq U$. A set U is an open set in (X, d) if U is open in (X, d). A set D is closed in (X, d) if $X\backslash D$ is open. A set D is a closed set in (X, d) if D is closed in (X, d).*

3.2.2 Properties

Theorem 1 ([43, 97])**.** *Let (X, d) be a distance space. Let N be any (finite) positive integer. Let Γ be a set possibly with an uncountable number of elements. Then the following statements hold.*

1. *X is open.*
2. *$\varnothing$ is open.*
3. *Each element in $\{U_\gamma \in 2^X \mid \gamma \in \Gamma\}$ is open $\Longrightarrow$ $\bigcup_{\gamma\in\Gamma} U_\gamma$ is open.*
4. *Each element in $\{U_n | n = 1, 2, \ldots, N\}$ is open $\Longrightarrow$ $\bigcap_{n=1}^{N} U_n$ is open.*

Proof. 1. By definition of open set, X is open iff $\forall x \in X\, \exists\, r$ such that $\mathsf{B}(x, r) \subseteq X$. By definition of open ball, it is always true that $\mathsf{B}(x, r) \subseteq X$ in (X, d). Therefore, X is open in (X, d).

2. By definition of open set, $\varnothing$ is open iff $\forall x \in \varnothing\, \exists\, r$ such that $\mathsf{B}(x, r) \subseteq \varnothing$. By definition of empty set $\varnothing$, this is always true because no x is in $\varnothing$. Therefore, $\varnothing$ is open in (X, d).

3. By definition of open set, $\bigcup U_\gamma$ is open iff $\forall x \in \bigcup U_\gamma\, \exists\, r$ such that $\mathsf{B}(x, r) \subseteq \bigcup U_\gamma$. If $x \in \bigcup U_\gamma$, then there is at least one $U \in \bigcup U_\gamma$ that contains x. By the left hypothesis in statement 3, that set U is open and so for that x $\exists r$ such that $\mathsf{B}(x, r) \subseteq U \subseteq \bigcup U_\gamma$. Therefore, $\bigcup U_\gamma$ is open in (X, d).

4. Let us prove that if U_1 and U_2 are open, then $U_1 \cap U_2$ is open. By definition of open set, $U_1 \cap U_2$ is open iff $\forall x \in U_1 \cap U_2\, \exists\, r$ such that $\mathsf{B}(x, r) \subseteq U_1 \cap U_2$. By the left hypothesis above, U_1 and U_2 are open, and by the definition of open sets, there exists r_1 and r_2 such that

[6] See [50, 61].

$\mathsf{B}(x, r_1) \subseteq U_1$ and $\mathsf{B}(x, r_2) \subseteq U_2$. Let $r := \min\{r_1, r_2\}$. Then $\mathsf{B}(x, r) \subseteq U_1$ and $\mathsf{B}(x, r) \subseteq U_2$. By definition of set intersection, $\mathsf{B}(x, r) \subseteq U_1 \cap U_2$. Hence, $U_1 \cap U_2$ is open.

Let us prove that $\bigcap_{n=1}^{N} U_n$ is open by induction. For $N = 1$ case: $\bigcap_{n=1}^{N} U_n = \bigcap_{n=1}^{1} U_n = U_1$ is open by hypothesis. By property of intersection $\bigcap_{n=1}^{N+1} U_n = \left(\bigcap_{n=1}^{N} U_n\right) \cap U_{N+1}$, therefore $\bigcap_{n=1}^{N+1} U_n$ is open via "N case" hypothesis and above proof for two sets. $\square$

Corollary 1. *Let (X, d) be a distance space. The set $T := \left\{U \in 2^X \mid U \text{ is open in } (X, \mathsf{d})\right\}$ is a topology on X, and (X, T) is a topological space.*

Proof. This follows directly from the definition of an open set, Theorem 1, and the definition of topology. $\square$

Of course it is possible to define a very large number of topologies even on a finite set with just a handful of elements;[7] and it is possible to define an infinite number of topologies even on a linearly ordered infinite set like the real line $(\mathbb{R}, \leq)$.[8] Be that as it may, Definition 16 defines a single but convenient topological space in terms of a distance space. Note that every metric space conveniently and naturally induces a topological space because the open balls of the metric space form a base for the topology. This is not the case for all distance spaces. But if the open balls of a distance space are all open, then those open balls induce a topology (next theorem).[9]

Definition 16. *Let (X, d) be a distance space. The set $T := \left\{U \in 2^X \mid U \text{ is open in } (X, \mathsf{d})\right\}$ is the topology induced by (X, d) on X. The pair (X, T) is called the topological space induced by (X, d).*

For any distance space (X, d), no matter how strange, there is guaranteed to be at least one topological space induced by (X, d) — and that is the indiscrete topological space (Example 9) because for any distance space (X, d), $\varnothing$ and X are open sets in (X, d) (Theorem 1).

Theorem 2. *Let B be the set of all open balls in a distance space (X, d). Then every open ball in B is open if and only if B is a base for a topology.*

Proof. Let every open ball in B be open. Then for every x in $B_y \in B$ there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq B_y$ by Definition 15. It implies for every $x \in X$ and for every $B_y \in B$ containing x, there exists $B_x \in B$ such that $x \in B_x \subseteq B_y$, because $\forall\,(x, r) \in X \times \mathbb{R}^+, \mathsf{B}(x, r) \subseteq X$. Hence, B is a base for T by Theorem 11.

Vice versa. Let B is a base for a topology. Then for every $x \in X$ and for every $U \subseteq T$ containing x, there exists $B_x \in B$ such that $x \in B_x \subseteq U$ by Theorem 11. From Definition 26 it follows that for every $x \in X$ and for every $B_y \in B \subseteq T$ containing x, there exists $B_x \in B$ such that $x \in B_x \subseteq B_y$. Therefore for every $x \in B_y \in B \subseteq T$, there exists $B_x \in B$ such that $x \in B_x \subseteq B_y$. Hence, every open ball in B is open (see Definition 15). $\square$

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[7] For a finite set X with n elements, there are 29 topologies on X if $n = 3$; 6942 topologies on X if $n = 5$; and and 8.977.053.873.043 (almost 9 trillion) topologies on X if $n = 10$. See [15, 24, 28, 29, 45, 71, 104].

[8] For examples of topologies on the real line see [27, 66, 90, 99].

[9] Metric space: Definition 24; open ball: Definition 14; base: Definition 26; topology: Definition 25; not all open balls are open in a distance space: Example 1 and Example 2.

3.3 Sequences in distance spaces

3.3.1 Definitions

Definition 17. [10] *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). The sequence $\{x_n\}$ converges to a limit x if for any $\varepsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{Z}$ such that $\mathsf{d}(x_n, x) < \varepsilon$ for all $n > N$. This condition can be expressed in any of the following forms:*

1. *the limit of the sequence $\{x_n\}$ is x;*
2. *the sequence $\{x_n\}$ is convergent with limit x;*
3. $\lim\limits_{n\to\infty} \{x_n\} = x$;
4. $\{x_n\} \to x$.

A sequence that converges is convergent.

Definition 18. [11] *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). The sequence $\{x_n\}$ is a Cauchy sequence in (X, d) if for every $\varepsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{Z}$ such that $\mathsf{d}(x_n, x_m) < \varepsilon$ for all $n, m > N$.*

Definition 19. [12] *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). The sequence $\{x_n\}$ is complete in (X, d) if the following implication holds: $\{x_n\}$ is Cauchy in $(X, \mathsf{d}) \implies \{x_n\}$ is convergent in (X, d).*

3.3.2 Properties

Proposition 1. *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). If $\{x_n\}$ is Cauchy in (X, d), then it is bounded in (X, d).*

Proof. Let $\{x_n\}$ be a Cauchy sequence. It means that for every $\varepsilon \in \mathbb{R}^+$ there exists $N \in \mathbb{Z}$ such that $\forall n, m > N$, $\mathsf{d}(x_n, x_m) < \varepsilon$. Let $\varepsilon = 1$. Then $\exists N \in \mathbb{Z}$ such that $\mathsf{d}(x_n, x_m) < 1$ for all $n, m > N$. It implies $\mathsf{d}(x_n, x_{m+1}) < \max\left\{\{1\} \cup \left\{\mathsf{d}\left(x_p, x_q\right) \mid p, q \not> N\right\}\right\}$. Hence, the sequence $\{x_n\}$ is bounded by Definition 13. □

Proposition 2. *Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in a distance space (X, d). Let $\mathsf{f} \in \mathbb{Z}^{\mathbb{Z}}$ be a strictly monotone function such that $\mathsf{f}(n) < \mathsf{f}(n+1)$. Then if $\{x_n\}_{n\in\mathbb{Z}}$ is a Cauchy sequence, then subsequence $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}}$ is also Cauchy.*

Proof. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a Cauchy sequence. It means that for any given $\varepsilon > 0$, $\exists N$ such that $\forall n, m > N$, $\mathsf{d}(x_n, x_m) < \varepsilon$. Therefore there exists N' such that $\mathsf{d}\left(x_{\mathsf{f}(n)}, x_{\mathsf{f}(m)}\right) < \varepsilon$ for all $\mathsf{f}(n), \mathsf{f}(m) > N'$. So, $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}}$ is Cauchy sequence. □

Theorem 3. [13] *Let (X, d) be a distance space. Let A^- be the closure of a A in a topological space induced by (X, d). If limits are unique in (X, d) and (A, d) is complete in (X, d), then A is closed in (X, d), i.e. $A = A^-$.*

Proof. By Lemma 3 we have $A \subseteq A^-$. Let us prove that $A^- \subseteq A$.

Let x be a point in A^-. Define a sequence of open balls $\left\{\mathsf{B}\left(x, \frac{1}{1}\right), \mathsf{B}\left(x, \frac{1}{2}\right), \mathsf{B}\left(x, \frac{1}{3}\right), \ldots\right\}$. Define a sequence of points $\{x_1, x_2, x_3, \ldots\}$ such that $x_n \in \mathsf{B}\left(x_n, \frac{1}{n}\right) \cap A$. Then $\{x_n\}$ is convergent in X with limit x by Definition 17 and $\{x_n\}$ is Cauchy in A by Definition 18. Since (A, d) is

[10] For definition in metric space see [53,68,75,97].
[11] For definition in metric space see [2,97].
[12] For definition in metric space see [97].
[13] For theorem in metric space see [18,54,72,107].

complete in (X, d), $\{x_n\}$ is therefore also convergent in A. Let this limit be y. Note that $y \in A$. From uniqueness of limits it follows $y = x$, and therefore $x \in A$. Hence $A^- \subseteq A$. $\square$

Proposition 3. *Let $\{x_n\}_{n\in\mathbb{Z}}$ be a sequence in a distance space (X, d). Let $\mathsf{f} : \mathbb{Z} \to \mathbb{Z}$ be a strictly increasing function such that $\mathsf{f}(n) < \mathsf{f}(n+1)$. If the sequence $\{x_n\}_{n\in\mathbb{Z}}$ converges to limit x, then a subsequence $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}}$ converges to the same limit x.*

Proof. By Theorem 6 we have $\forall\varepsilon > 0$, $\exists N$ such that $\forall n > N$, $\mathsf{d}(x_n, x) < \varepsilon$. Therefore $\forall\varepsilon > 0$, $\exists\mathsf{f}(N)$ such that $\forall\mathsf{f}(n) > \mathsf{f}(N)$, $\mathsf{d}\left(x_{\mathsf{f}(n)}, x\right) < \varepsilon$. So, $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}} \to x$ via Theorem 6. $\square$

Theorem 4 (Cantor intersection theorem)**.** *Let (X, d) be a distance space, $\{A_n\}_{n\in\mathbb{Z}}$ a sequence with each $A_n \in 2^X$, and $|\,A\,|$ the number of elements in A. If (X, d) is complete, A_n is closed for all $n \in \mathbb{N}$, $\operatorname{diam} A_n \geq \operatorname{diam} A_{n+1}$ for all $n \in \mathbb{N}$, and $\operatorname{diam}\{A_n\}_{n\in\mathbb{Z}} \to 0$, then $\left|\bigcap_{n\in\mathbb{N}} A_n\right| = 1$.*

Proof. Let us prove that $|\bigcap A_n| < 2$. Let $A := \cap A_n$. For any $x \neq y$ and $\{x, y\} \in A$ we have $\mathsf{d}(x, y) > 0$ and $\{x, y\} \subseteq A_n$ for all n. Since $\operatorname{diam}\{A_n\}_{n\in\mathbb{Z}} \to 0$, there exists n such that $\operatorname{diam} A_n < \mathsf{d}(x, y)$. It implies $\exists n$ such that $\sup\{\mathsf{d}(x, y)\,|\,x, y \in A_n\} < \mathsf{d}(x, y)$. This is a contradiction, so $\{x, y\} \notin A$ and $|\bigcap A_n| < 2$.

Let us prove that $|\bigcap A_n| \geq 1$. Let $x_n \in A_n$ and $x_m \in A_m$. Since $\operatorname{diam}\{A_n\}_{n\in\mathbb{Z}} \to 0$, for all ε there exists $N \in \mathbb{N}$ such that $\operatorname{diam} A_N < \varepsilon$. Therefore $\forall m, n > N$, $x_n \in A_n \subseteq A_N$ and $x_m \in A_m \subseteq A_N$. But $\mathsf{d}(x_n, x_m) \leq \operatorname{diam} A_N < \varepsilon$, it means that $\{x_n\}$ is a Cauchy sequence. Because $\{x_n\}$ is complete, $x_n \to x$. It implies $x \in (A_n)^- = A_n$, and, hence, $|\,A_n\,| \geq 1$. $\square$

Definition 20 ([10])**.** *Let (X, d) be a distance space. Let C be the set of all convergent sequences in (X, d). The distance function d is continuous in (X, d) if*

$$\{x_n\}, \{y_n\} \in C \quad\Longrightarrow\quad \lim_{n\to\infty}\{\mathsf{d}(x_n, y_n)\} = \mathsf{d}\left(\lim_{n\to\infty}\{x_n\}, \lim_{n\to\infty}\{y_n\}\right).$$

A distance function is discontinuous if it is not continuous.

Remark 2. *Rather than defining continuity of a distance function in terms of the* sequential characterization of continuity *as in Definition 20, we could define continuity using an* inverse image characterization of continuity *(see Definition 16). Assuming an equivalent topological space is used for both characterizations, the two characterizations are equivalent (Theorem 15). In fact, one could construct an equivalence such as the following:*

$$\left\{\begin{array}{l}\mathsf{d} \text{ is continuous in } \mathbb{R}^{X^2}\\ \text{(Definition 28)}\\ \text{(inverse image characterization}\\ \text{of continuity)}\end{array}\right\} \iff \left\{\begin{array}{l}\{x_n\}, \{y_n\} \in C \quad\Longrightarrow\\ \lim_{n\to\infty}\{\mathsf{d}(x_n, y_n)\} = \mathsf{d}\left(\lim_{n\to\infty}\{x_n\}, \lim_{n\to\infty}\{y_n\}\right)\\ \text{(Definition 29)}\\ \text{(sequential characterization of continuity)}\end{array}\right\}.$$

Note that just as $\{x_n\}$ is a sequence in X, so the ordered pair $(\{x_n\}, \{y_n\})$ is a sequence in X^2. The remainder follows from Theorem 15. However, use of the inverse image characterization *is somewhat troublesome because we would need a topology on X^2, and we don't immediately have one defined and ready to use. In fact, we don't even immediately have a distance space on X^2 defined or even open balls in such a distance space. The result is, for the scope of this paper, it is arguably not worthwhile constructing the extra structure, but rather instead this paper uses the* sequential characterization *as a definition (as in Definition 20).*

3.4 Examples

Similar distance functions and several of the observations for the examples in this section can be found in [10].

In a metric space, all open balls are open, the open balls form a base for a topology, the limits of convergent sequences are unique, and the metric function is continuous. In the distance space of the next example, none of these properties hold.

Example 1. [14] *Let (x, y) be an ordered pair in $\mathbb{R}^2$. Let $(a : b)$ be an open interval and $(a : b]$ a half-open interval in $\mathbb{R}$. Let $|x|$ be the absolute value of $x \in \mathbb{R}$. The function $\mathsf{d}(x, y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ such that*

$$\mathsf{d}(x,y) := \begin{cases} y, & \forall\,(x,y) \in \{4\} \times (0:2], \\ x, & \forall\,(x,y) \in (0:2] \times \{4\}, \\ |x-y|, & \textit{otherwise}, \end{cases}$$

is a distance on $\mathbb{R}$.

Note some characteristics of the distance space $(\mathbb{R}, \mathsf{d})$.

1. *$(\mathbb{R}, \mathsf{d})$ is not a metric space because d does not satisfy the triangle inequality:*
   $\mathsf{d}(0,4) := |0-4| = 4 \not\leq 2 = |0-1| + 1 := \mathsf{d}(0,1) + \mathsf{d}(1,4)$.

2. *Not every open ball in $(\mathbb{R}, \mathsf{d})$ is open. For example, the open ball $\mathsf{B}(3,2)$ is not open because $4 \in \mathsf{B}(3,2)$ but for all $0 < \varepsilon < 1$*
   $\mathsf{B}(4,\varepsilon) = (4-\varepsilon : 4+\varepsilon) \cup (0 : \varepsilon) \not\subseteq (1:5) = \mathsf{B}(3,2)$.

3. *The open balls of $(\mathbb{R}, \mathsf{d})$ do not form a base for a topology on $\mathbb{R}$. This follows directly from previous item and Theorem 2.*

4. *In the distance space $(\mathbb{R}, \mathsf{d})$, limits are not unique. For example, the sequence $\{1/n\}_1^\infty$ converges both to the limit 0 and the limit 4 in $(\mathbb{R}, \mathsf{d})$:*
$$\begin{array}{lllll} \lim\limits_{n\to\infty} \mathsf{d}(x_n,0) := \lim\limits_{n\to\infty} \mathsf{d}(1/n,0) & := \lim\limits_{n\to\infty} |1/n - 0| & = 0 & \Longrightarrow & \{1/n\} \to 0, \\ \lim\limits_{n\to\infty} \mathsf{d}(x_n,4) := \lim\limits_{n\to\infty} \mathsf{d}(1/n,4) & := \lim\limits_{n\to\infty} \{1/n\} & = 0 & \Longrightarrow & \{1/n\} \to 4. \end{array}$$

5. *The topological space (X, T) induced by $(\mathbb{R}, \mathsf{d})$ also yields limits of 0 and 4 for the sequence $\{1/n\}_1^\infty$, just as it does in previous item. This is largely due to the fact that, for small ε, the open balls $\mathsf{B}(0,\varepsilon)$ and $\mathsf{B}(4,\varepsilon)$ are open.*
$$\begin{array}{lll} \mathsf{B}(0,\varepsilon) \textit{ is open} & \Longrightarrow & \textit{for each } U \in T \textit{ that contains } 0,\ \exists N \in \mathbb{N} \textit{ such that } 1/n \in U \quad \forall n > N \\ & \Longleftrightarrow & \{1/n\} \to 0 \qquad \textit{by definition of convergence.} \\ \mathsf{B}(4,\varepsilon) \textit{ is open} & \Longrightarrow & \textit{for each } U \in T \textit{ that contains } 4,\ \exists N \in \mathbb{N} \textit{ such that } 1/n \in U \quad \forall n > N \\ & \Longleftrightarrow & \{1/n\} \to 4 \qquad \textit{by definition of convergence.} \end{array}$$

6. *The distance function d is discontinuous:*
$$\begin{aligned} \lim_{n\to\infty} \{\mathsf{d}(1-1/n, 4-1/n)\} &= \lim_{n\to\infty} \{|(1-1/n)-(4-1/n)|\} = |1-4| = 3 \neq 4 = \mathsf{d}(0,4) \\ &= \mathsf{d}\left(\lim_{n\to\infty}\{1-1/n\}, \lim_{n\to\infty}\{4-1/n\}\right). \end{aligned}$$

[14] A similar distance function d and item 4 can in essence be found in [10].

In a metric space, all convergent sequences are also Cauchy. However, this is not the case for all distance spaces, as demonstrated next.

Example 2. [15] *The function* $\mathsf{d}(x,y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that*

$$\mathsf{d}(x,y) := \begin{cases} |x-y|, & \text{for } x=0 \text{ or } y=0 \text{ or } x=y, \\ 1, & \text{otherwise,} \end{cases}$$

is a distance on $\mathbb{R}$

Note some characteristics of the distance space $(\mathbb{R},\mathsf{d})$.

1. $(\mathbb{R},\mathsf{d})$ *is not a metric space because the triangle inequality does not hold:*
$$\mathsf{d}\left(\tfrac{1}{4},\tfrac{1}{2}\right) = 1 \not\leq \tfrac{3}{4} = \left|\tfrac{1}{4}-0\right| + \left|0-\tfrac{1}{2}\right| = \mathsf{d}\left(\tfrac{1}{4},0\right) + \mathsf{d}\left(0,\tfrac{1}{2}\right).$$

2. *The open ball* $\mathsf{B}\left(\tfrac{1}{4},\tfrac{1}{2}\right)$ *is not open because for any* $\varepsilon \in \mathbb{R}^+$, *no matter how small,*
$$\mathsf{B}(0,\varepsilon) = (-\varepsilon : +\varepsilon) \not\subseteq \left\{0,\ \tfrac{1}{4}\right\} = \left\{x \in X \,\middle|\, \mathsf{d}\left(\tfrac{1}{4},x\right) < \tfrac{1}{2}\right\} := \mathsf{B}\left(\tfrac{1}{4},\tfrac{1}{2}\right).$$

3. *Even though not all the open balls are open, it is still possible to have an open set in* $(\mathbb{R},\mathsf{d})$. *For example, the set* $U := \{1,\ 2\}$ *is open:*
$$\begin{array}{lclclclcl} \mathsf{B}(1,1) & := & \{x \in X \,|\, \mathsf{d}(1,x) < 1\} & = & \{1\} & \subseteq & \{1,2\} & := & U, \\ \mathsf{B}(2,1) & := & \{x \in X \,|\, \mathsf{d}(2,x) < 1\} & = & \{2\} & \subseteq & \{1,2\} & := & U. \end{array}$$

4. *By item 2 and Theorem 2, the open balls of the distance space* $(\mathbb{R},\mathsf{d})$ *do not form a base for a topology on* $\mathbb{R}$.

5. *Even though the open balls in* $(\mathbb{R},\mathsf{d})$ *do not induce a topology on* $\mathbb{R}$, *it is still possible to find a set of open sets in* $(\mathbb{R},\mathsf{d})$ *that is a topology. For example, the set* $\{\varnothing,\ \{1,2\},\ \mathbb{R}\}$ *is a topology on* $\mathbb{R}$.

6. *In* $(\mathbb{R},\mathsf{d})$ *limits of convergent sequences are unique. Namely,* $\{x_n\} \to x \implies$
$$\lim_{n\to\infty} \mathsf{d}(x_n,x) = \begin{cases} \lim |x_n - 0| & = & 0, & \text{for } x = 0, \\ |x-x| & = & 0, & \text{for constant } \{x_n\} \text{ for } n > N, \\ 1 & \neq & 0, & \text{otherwise,} \end{cases}$$
which says that there are only two ways for a sequence to converge: either $x = 0$ *or the sequence eventually becomes constant (or both). Any other sequence will diverge.*

7. *In* $(\mathbb{R},\mathsf{d})$ *a convergent sequence is not necessarily Cauchy. For example, the sequence* $\{1/n\}_{n\in\mathbb{N}}$ *is convergent with limit* 0
$$\lim_{n\to\infty} \mathsf{d}(1/n,0) = \lim_{n\to\infty} 1/n = 0.$$
However, even though $\{1/n\}$ *is convergent, it is not Cauchy*
$$\lim_{n,m\to\infty} \mathsf{d}(1/n,1/m) = 1 \neq 0.$$

8. *The distance function* d *is discontinuous in* (X,d):
$$\lim_{n\to\infty} \{\mathsf{d}(1/n, 2-1/n)\} = 1 \neq 2 = \mathsf{d}(0,2) = \mathsf{d}\left(\lim_{n\to\infty}\{1/n\},\ \lim_{n\to\infty}\{2-1/n\}\right).$$

[15] The distance function d and item 7 can in essence be found in [10].

Example 3. [16] *The function* $\mathsf{d}(x,y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that*
$$\mathsf{d}(x,y) := \begin{cases} 2|x-y|, & \forall\,(x,y) \in \{(0,1)\,,\,(1,0)\}, \\ |x-y|, & \text{otherwise}, \end{cases}$$
is a distance on $\mathbb{R}$.

Note some characteristics of the distance space $(\mathbb{R},\mathsf{d})$.

1. $(\mathbb{R},\mathsf{d})$ *is not a metric space because* d *does not satisfy the triangle inequality:*
$$\mathsf{d}(0,1) := 2|0-1| = 2 \not\leq 1 = |0-1/2| + |1/2-1| := \mathsf{d}(0,1/2) + \mathsf{d}(1/2,1)\,.$$

2. *The function* d *is discontinuous:*
$$\begin{aligned}\lim_{n\to\infty}\{\mathsf{d}(1-1/n,1/n)\} &:= \lim_{n\to\infty}\{|1-1/n-1/n|\} = 1 \neq 2 \\ &= 2|0-1| := \mathsf{d}(0,1) = \mathsf{d}\Big(\lim_{n\to\infty}\{1-1/n\}\,,\,\lim_{n\to\infty}\{1/n\}\Big)\,.\end{aligned}$$

3. *In* $(\mathbb{R},\mathsf{d})$ *open balls are open:*
   (a) $\mathsf{p}\,(x,y) := |x-y|$ *is a metric and thus all open balls in that do not contain both* 0 *and* 1 *are open;*
   (b) *by Example 14,* $\mathsf{q}\,(x,y) := 2|x-y|$ *is also a metric and thus all open balls containing* 0 *and* 1 *only are open;*
   (c) *the only question remaining is with regards to open balls that contain* $0, 1$ *and some other element(s) in* $\mathbb{R}$. *But even in this case, open balls are still open. For example,* $\mathsf{B}(-1,2) = (-1:2) = (-1:1) \cup (1:2)$. *Note that both* $(-1:1)$ *and* $(1:2)$ *are open, and thus by Theorem 1,* $\mathsf{B}(-1,2)$ *is open as well.*

4. *By previous item and Theorem 2, the open balls of* $(\mathbb{R},\mathsf{d})$ do *form a base for a topology on* $\mathbb{R}$.

5. *In* $(\mathbb{R},\mathsf{d})$ *the limits of convergent sequences are unique. This is demonstrated in Example 7 using additional structure developed in Section 4.*

6. *In* $(\mathbb{R},\mathsf{d})$ *convergent sequences are Cauchy. This is also demonstrated in Example 7.*

The distance functions in Examples 1–3 were all discontinuous. In the absence of the triangle inequality and in light of these examples, one might try replacing the triangle inequality with the weaker requirement of continuity. However, as demonstrated by the next example, this also leads to an arguably disastrous result.

Example 4 ([10,74])**.** *The function* $\mathsf{d} \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that* $\mathsf{d}(x,y) := (x-y)^2$ *is a distance on* $\mathbb{R}$. *Note some characteristics of the distance space* $(\mathbb{R},\mathsf{d})$.

1. $(\mathbb{R},\mathsf{d})$ *is not a metric space because the triangle inequality does not hold:*
$$\mathsf{d}(0,2) := (0-2)^2 = 4 \not\leq 2 = (0-1)^2 + (1-2)^2 := \mathsf{d}(0,1) + \mathsf{d}(1,2)\,.$$

2. *The distance function* d *is continuous in* (X,d). *This is demonstrated in the more general setting of Section 4 in Example 8.*

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[16] The distance function d and item 2 can in essence be found in [10].

3. *Calculating the length of curves in* $(\mathbb{R}, \mathsf{d})$ *leads to a paradox.*[17] *Partition* $[0:1]$ *into* 2^N *consecutive line segments connected at the points* $\left\{0, \frac{1}{2^N}, \frac{2}{2^N}, \frac{3}{2^N}, \ldots, \frac{2^{N-1}1}{2^N}, 1\right\}$. *Then the distance, as measured by* d*, between any two consecutive points is equal to* $\mathsf{d}(p_n, p_{n+1}) := (p_n - p_{n+1})^2 = \left(\frac{1}{2^N}\right)^2 = \frac{1}{2^{2N}}$. *But this leads to the paradox that the total length of* $[0:1]$ *is 0:*

$$\lim_{N\to\infty} \sum_{n=0}^{2^N-1} \frac{1}{2^{2N}} = \lim_{N\to\infty} \frac{2^N}{2^{2N}} = \lim_{N\to\infty} \frac{1}{2^N} = 0.$$

4 Distance spaces with power triangle inequalities

4.1 Definitions

This paper introduces a new relation called the power triangle inequality. It is a generalization of other common relations, including the triangle inequality. The power triangle inequality is defined in terms of a function herein called the power triangle function (next definition). This function is a special case of the power mean with $N = 2$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$. Power means have the attractive properties of being continuous and strictly monotone with respect to a free parameter $p \in \mathbb{R}^*$. This fact is inherited and exploited by the power triangle inequality.

Definition 21. *Let* (X, d) *be a distance space. Let* $\mathbb{R}^+$ *be the set of all positive real numbers and* $\mathbb{R}^*$ *be the set of extended real numbers. The power triangle function* τ *on* (X, d) *is defined as*

$$\tau(p, \sigma; x, y, z; \mathsf{d}) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x,z) + \frac{1}{2}\mathsf{d}^p(z,y)\right]^{\frac{1}{p}}, \qquad \forall (p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^{\vdash}, \quad x, y, z \in X.$$

Remark 3. *In the field of probabilistic metric spaces, a function called the triangle function was introduced by Sherstnev [102]. However, the power triangle function as defined in the present paper is not a special case of (is not compatible with) the triangle function of Sherstnev. Another definition of triangle function has been offered by Bessenyei [6] with special cases of* $\Phi(u,v) := c(u+v)$ *and* $\Phi(u,v) := (u^p + v^p)^{\frac{1}{p}}$*, which are similar to the definition of power triangle function offered in the present paper.*

Definition 22. *Let* (X, d) *be a distance space. Let* 2^{XXX} *be the set of all trinomial relations on* X *(see Definition 3). A relation* $\triangle(p, \sigma; \mathsf{d})$ *in* 2^{XXX} *is a power triangle inequality on* (X, d) *if*

$$\triangle(p, \sigma; \mathsf{d}) := \left\{(x, y, z) \in X^3 \mid \mathsf{d}(x,y) \le \tau(p, \sigma; x, y, z; \mathsf{d})\right\} \qquad \text{for some } (p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+.$$

The tuple $(X, \mathsf{d}, p, \sigma)$ *is a power distance space and* d *a power distance or power distance function if* (X, d) *is a distance space in which the triangle relation* $\triangle(p, \sigma; \mathsf{d})$ *holds.*

The power triangle function can be used to define some standard inequalities (next definition). See Corollary 3 for some justification of the definitions.

[17] This is the method of "inscribed polygons" for calculating the length of a curve and goes back to Archimedes [17, 117].

Definition 23 ([6,36,41,44,46,47,52,62,63,69,119]). *Let $\triangle(p,\sigma;\mathsf{d})$ be a power triangle inequality on a distance space (X,d).*

1. $\triangle(\infty,\sigma/2;\mathsf{d})$ *is the σ-inframetric inequality.*
2. $\triangle(\infty,\frac{1}{2};\mathsf{d})$ *is the inframetric inequality.*
3. $\triangle(2,\sqrt{2}/2;\mathsf{d})$ *is the quadratic inequality.*
4. $\triangle(1,\sigma;\mathsf{d})$ *is the relaxed triangle inequality.*
5. $\triangle(1,1;\mathsf{d})$ *is the triangle inequality.*
6. $\triangle(1/2,2;\mathsf{d})$ *is the square mean root inequality.*
7. $\triangle(0,\frac{1}{2};\mathsf{d})$ *is the geometric inequality.*
8. $\triangle(-1,\frac{1}{4};\mathsf{d})$ *is the harmonic inequality.*
9. $\triangle(-\infty,\frac{1}{2};\mathsf{d})$ *is the minimal inequality.*

Definition 24. [18] *Let (X,d) be a distance space.*

1. (X,d) *is a metric space if the triangle inequality holds in X.*
2. (X,d) *is a near metric space if the relaxed triangle inequality holds in X.*
3. (X,d) *is an inframetric space if the inframetric inequality holds in X.*
4. (X,d) *is a σ-inframetric space if the σ-inframetric inequality holds in X.*

4.2 Properties

4.2.1 Relationships of the power triangle function

Corollary 2. *Let $\tau(p,\sigma;x,y,z;\mathsf{d})$ be the power triangle function in the distance space (X,d). Let $(\mathbb{R},|\cdot|,\leq)$ be the ordered metric space with the usual ordering relation $\leq$ and usual metric $|\cdot|$ on $\mathbb{R}$. The function $\tau(p,\sigma;x,y,z;\mathsf{d})$ is continuous and strictly monotone in $(\mathbb{R},|\cdot|,\leq)$ with respect to both the variables p and σ.*

Proof. The function $\tau(p,\sigma;x,y,z;\mathsf{d})$ is continuous and strictly monotone with respect to p via Theorem 18. By definition 21 of τ we have

$$\tau(p,\sigma;x,y,z;\mathsf{d}) := 2\sigma\underbrace{\left[\frac{1}{2}\mathsf{d}^p(x,z)+\frac{1}{2}\mathsf{d}^p(z,y)\right]^{\frac{1}{p}}}_{\mathsf{f}(p,x,y,z)} = 2\sigma\mathsf{f}(p,x,y,z),$$

where f is defined as above. Therefore τ is affine with respect to σ, and, hence, $\tau(p,\sigma;x,y,z;\mathsf{d})$ is continuous and strictly monotone with respect to σ. □

Corollary 3. *Let $\tau(p,\sigma;x,y,z;\mathsf{d})$ be the power triangle function in the distance space (X,d).*

$$\tau(p,\sigma;x,y,z;\mathsf{d}) = \begin{cases} 2\sigma\max\{\mathsf{d}(x,z),\,\mathsf{d}(z,y)\} & \text{for } p=\infty, & (\text{maximum}),^{19} \\ 2\sigma\left[1/2\mathsf{d}^2(x,z)+1/2\mathsf{d}^2(z,y)\right]^{\frac{1}{2}} & \text{for } p=2, & (\text{quadratic mean}), \\ \sigma[\mathsf{d}(x,z)+\mathsf{d}(z,y)] & \text{for } p=1, & (\text{arithmetic mean}),^{20} \\ 2\sigma\sqrt{\mathsf{d}(x,z)}\,\sqrt{\mathsf{d}(z,y)} & \text{for } p=0, & (\text{geometric mean}), \\ 4\sigma\left[\frac{1}{\mathsf{d}(x,z)}+\frac{1}{\mathsf{d}(z,y)}\right]^{-1} & \text{for } p=-1, & (\text{harmonic mean}), \\ 2\sigma\min\{\mathsf{d}(x,z),\,\mathsf{d}(z,y)\} & \text{for } p=-\infty, & (\text{minimum}). \end{cases}$$

Proof. These follow directly from Theorem 18. □

[18] For definitions in metric space see [30,43,48,49,60]; in near metric space see [36,41,46,47,62,69,119].
[19] The maximum $\tau(\infty,\sigma;x,y,z;d)$ corresponds to the inframetric space.
[20] The arithmetic mean $\tau(1,\sigma;x,y,z;d)$ corresponds to the near metric space.

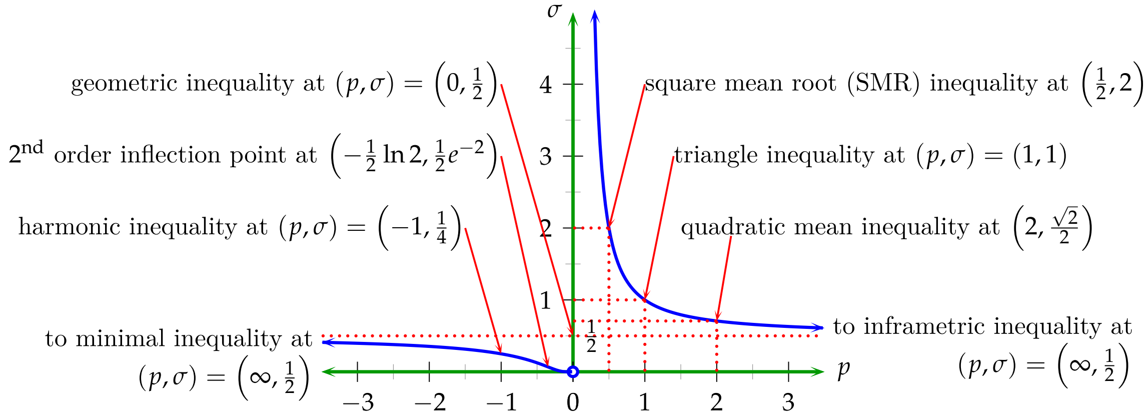

Figure 1: $\sigma = \frac{1}{2}(2^{\frac{1}{p}}) = 2^{\frac{1}{p}-1}$ or $p = \frac{\ln 2}{\ln(2\sigma)}$ (see Lemma 1, Lemma 2, Corollary 6, Corollary 7, and Theorem 9).

Corollary 4. *Let* (X, d) *be a distance space. Then*

$$\begin{aligned} 2\sigma \min\{\mathsf{d}(x,z), \mathsf{d}(z,y)\} &\le 4\sigma\left[\frac{1}{\mathsf{d}(x,z)} + \frac{1}{\mathsf{d}(z,y)}\right]^{-1} &&\le 2\sigma\sqrt{\mathsf{d}(x,z)}\,\sqrt{\mathsf{d}(z,y)} \\ &\le \sigma[\mathsf{d}(x,z) + \mathsf{d}(z,y)] &&\le 2\sigma \max\{\mathsf{d}(x,z), \mathsf{d}(z,y)\}. \end{aligned}$$

Proof. These follow directly from Corollary 8. □

4.2.2 Properties of power distance spaces

The power triangle inequality property of a power distance space axiomatically endows a metric with an upper bound. Lemma 1 demonstrates that there is a complementary lower bound somewhat similar in form to the power triangle inequality upper bound. In the special case where $2\sigma = 2^{\frac{1}{p}}$, the lower bound helps provide a simple proof of the continuity of a large class of power distance functions (Theorem 9). The inequality $2\sigma \le 2^{\frac{1}{p}}$ is a special relation in this paper and appears repeatedly in this paper; it appears as an inequality in Lemma 2, Corollaries 6 and 7, and as an equality in Lemma 1 and Theorem 9. It is plotted in Figure 1.

Lemma 1. [21] *Let* $(X, \mathsf{d}, p, \sigma)$ *be a power triangle triangle space. Let* $|\cdot|$ *be the absolute value function. Let* $\max\{x,y\}$ *be the maximum and* $\min\{x,y\}$ *the minimum of any* $x, y \in \mathbb{R}^*$. *Then, for all* $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$,

1. $\mathsf{d}^p(x,y) \ge \max\left\{0, \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y), \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x)\right\} \quad \forall x,y,z \in X,$
2. $\mathsf{d}(x,y) \ge |\mathsf{d}(x,z) - \mathsf{d}(z,y)|$ *if* $p \ne 0$ *and* $2\sigma = 2^{\frac{1}{p}} \quad \forall x,y,z \in X.$

Proof. From power triangle inequality and symmetric property of d we obtain

$$\begin{aligned} \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y) &\le \frac{2}{(2\sigma)^p}\left[2\sigma[1/2\mathsf{d}^p(x,y) + 1/2\mathsf{d}^p(y,z)]^{\frac{1}{p}}\right]^p - \mathsf{d}^p(z,y) \\ &= \frac{2(2\sigma)^p}{(2\sigma)^p}[1/2\mathsf{d}^p(x,y) + 1/2\mathsf{d}^p(y,z)] - \mathsf{d}^p(z,y) \\ &= [\mathsf{d}^p(x,y) + \mathsf{d}^p(y,z)] - \mathsf{d}^p(y,z) = \mathsf{d}^p(x,y). \end{aligned}$$

[21] For assertion in metric space, i.e. $(p,\sigma) = (1,1)$ see [5,43,83].

Using commutative and non-negative properties of d, for $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$ one can derive

$$\mathsf{d}^p(x,y) \geq \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y)\,, \qquad \mathsf{d}^p(y,x) \geq \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x)\,, \qquad \mathsf{d}^p(x,y) \geq 0.$$

The rest follows because $\mathsf{g}(x) := x^{\frac{1}{p}}$ is strictly monotone in $\mathbb{R}^{\mathbb{R}}$.

In case $2\sigma = 2^{\frac{1}{p}}$ we have

$$\begin{aligned}
\mathsf{d}(x,y) &\geq \max\left\{0,\, \frac{2}{(2\sigma)^p}\mathsf{d}^p(x,z) - \mathsf{d}^p(z,y)\,,\, \frac{2}{(2\sigma)^p}\mathsf{d}^p(y,z) - \mathsf{d}^p(z,x)\right\}^{\frac{1}{p}} \\
&= \max\{0,\, \mathsf{d}(x,z) - \mathsf{d}(z,y)\,,\, \mathsf{d}(y,z) - \mathsf{d}(z,x)\} \\
&= \max\{0,\, (\mathsf{d}(x,z) - \mathsf{d}(z,y)),\, -(\mathsf{d}(x,z) - \mathsf{d}(z,y))\} = |(\mathsf{d}(x,z) - \mathsf{d}(z,y))|.
\end{aligned}$$

□

Theorem 5. *Let (X,d,p,σ) be a power distance space. Let B be an open ball on (X,d). Then for all $(p,\sigma) \in (\mathbb{R}^*\backslash\{0\}) \times \mathbb{R}^+$ the following implications hold:*

1. *if $2\sigma \leq 2^{\frac{1}{p}}$ and $q \in \mathsf{B}(\theta,r)$ then there exists $r_q \in \mathbb{R}^+$ such that $\mathsf{B}(q,r_q) \subseteq \mathsf{B}(\theta,r)$;*
2. *if there exists $r_q \in \mathbb{R}^+$ such that $\mathsf{B}(q,r_q) \subseteq \mathsf{B}(\theta,r)$ then $q \in \mathsf{B}(\theta,r)$.*

Proof. Using the Archimedean Property[22] we obviously obtain

$$q \in \mathsf{B}(\theta,r) \iff \mathsf{d}(\theta,q) < r \iff 0 < r - \mathsf{d}(\theta,q) \iff \exists r_q \in \mathbb{R}^+,\, 0 < r_q < r - \mathsf{d}(\theta,q)\,.$$

Therefore

$$\begin{aligned}
\mathsf{B}(q,r_q) &:= \left\{x \in X \,\middle|\, \mathsf{d}(q,x) < r_q\right\} = \left\{x \in X \,\middle|\, \mathsf{d}^p(q,x) < r_q^p \in \mathbb{R}^+\right\} \\
&\subseteq \{x \in X \,|\, \mathsf{d}^p(q,x) < r^p - \mathsf{d}^p(\theta,q)\} = \{x \in X \,|\, \mathsf{d}^p(\theta,q) + \mathsf{d}^p(q,x) < r^p\} \\
&= \left\{x \in X \,\middle|\, [\mathsf{d}^p(\theta,q) + \mathsf{d}^p(q,x)]^{\frac{1}{p}} < r\right\} \subseteq \left\{x \in X \,\middle|\, 2^{1-1/p}\sigma[\mathsf{d}^p(\theta,q) + \mathsf{d}^p(q,x)]^{\frac{1}{p}} < r\right\} \\
&= \left\{x \in X \,\middle|\, 2\sigma[1/2\mathsf{d}(\theta,q)\,x + 1/2\mathsf{d}^p(q,x)]^{\frac{1}{p}} < r\right\} := \{x \in X \,|\, \tau(p,\sigma,\theta,x,q) < r\} \\
&\subseteq \{x \in X \,|\, \mathsf{d}(\theta,x) < r\} := \mathsf{B}(\theta,r)\,.
\end{aligned}$$

Here we used the fact that the functions $\mathsf{f}(x) := x^p$ and $\mathsf{f}(x) := x^{\frac{1}{p}}$ are monotone. So, the first implication is proved.

The second implication follows from

$$q \in \{x \in X \,|\, \mathsf{d}(q,x) = 0\} \subseteq \left\{x \in X \,\middle|\, \mathsf{d}(q,x) < r_q\right\} := \mathsf{B}(q,r_q) \subseteq \mathsf{B}(\theta,r)\,.$$

□

The next assertion follows from Theorem 2 and Theorem 5.

Corollary 5. *Let (X,d,p,σ) be a power distance space. If the inequality $2\sigma \leq 2^{\frac{1}{p}}$ holds for all $(p,\sigma) \in (\mathbb{R}^*\backslash\{0\}) \times \mathbb{R}^+$ then every open ball in (X,d) is open.*

Corollary 6. *Let (X,d,p,σ) be a power distance space. Let B be the set of all open balls in (X,d). If the inequality $2\sigma \leq 2^{\frac{1}{p}}$ holds for all $(p,\sigma) \in (\mathbb{R}^*\backslash\{0\}) \times \mathbb{R}^+$ then B is a base for (X,T).*

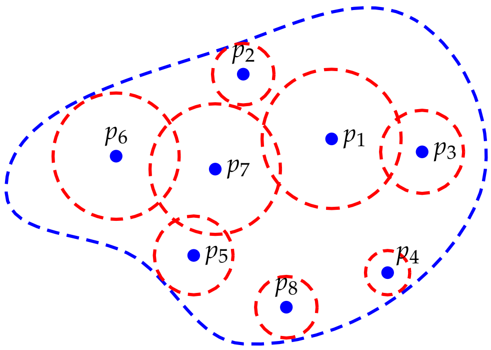

Figure 2: open set (see Lemma 2) .

Proof. The set of all open balls in (X, d) is a base for (X, T) by Corollary 5 and Theorem 11. T is a topology on X by Definition 26. $\square$

The next assertion demonstrates that every point in an open set is contained in an open ball that is contained in the original open set (see also Figure 2).

Lemma 2. *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space. Let B be an open ball on (X, d). Then for all $(p, \sigma) \in (\mathbb{R}^*\backslash\{0\}) \times \mathbb{R}^+$ the following implications hold:*

1. *if $2\sigma \leq 2^{\frac{1}{p}}$ and U is open in (X, d) then for all $x \in U$ there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq U$;*
2. *if for all $x \in U$ there exists $r \in \mathbb{R}^+$ such that $\mathsf{B}(x, r) \subseteq U$ then U is open in (X, d).*

Proof. From Corollary 6 we have

$$U = \bigcup \{\mathsf{B}(x_\gamma, r_\gamma) \,|\, \mathsf{B}(x_\gamma, r_\gamma) \subseteq U\} \supseteq \mathsf{B}(x, r)\,,$$

because x must be in one of those balls in U. So, the first implication is proved.

The second implication follows from

$$U = \bigcup \{x \in X \,|\, x \in U\} = \bigcup \{\mathsf{B}(x, r) \,|\, x \in U \text{ and } \mathsf{B}(x, r) \subseteq U\} \implies U \text{ is open}$$

by Corollary 6 and Corollary 1. $\square$

Corollary 7. [23] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space. Let B be an open ball on (X, d). If $2\sigma \leq 2^{\frac{1}{p}}$ for all $(p, \sigma) \in (\mathbb{R}^*\backslash\{0\}) \times \mathbb{R}^+$ then every open ball $\mathsf{B}(x, r)$ in (X, d) is open.*

Proof. The union of any set of open balls is open by Corollary 6, therefore the union of a set of just one open ball is open. Hence, every open ball is open. $\square$

Theorem 6. [24] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space. Let (X, T) be a topological space induced by (X, d). Let $\{x_n\}_{n \in \mathbb{Z}} \subset X$ be a sequence in (X, d). The sequence $\{x_n\}$ converges to a limit x iff for any $\varepsilon \in \mathbb{R}^+$ there exists $N \in \mathbb{Z}$ such that for all $n > N$, $\mathsf{d}(x_n, x) < \varepsilon$.*

[22] See [1, 121].
[23] For assertion in metric space see [1, 97].
[24] For theorem in metric space see [53, 97].

Proof. The sequence $\{x_n\}$ converges to x if and only if $x_n \in U\ \forall U \in N_x$, $n > N$. By Lemma 2 $\exists \mathsf{B}(x,\varepsilon)$ such that $x_n \in \mathsf{B}(x,\varepsilon)\ \forall n > N$. So, $\mathsf{d}(x_n, x) < \varepsilon$. ☐

In distance spaces not all convergent sequences are Cauchy (see Example 2). However in a distance space with any power triangle inequality all convergent sequences are Cauchy.

Theorem 7. [25] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space with any $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$. Let $\{x_n\}_{n\in\mathbb{Z}} \subset X$ be a sequence in (X, d). Every convergent sequence $\{x_n\}$ is a Cauchy sequence and therefore it is bounded in (X, d).*

Proof. Let $\{x_n\}_{n\in\mathbb{Z}}$ be a convergent sequence in (X, d). Then we have

$$\mathsf{d}(x_n, x_m) \le \tau(p,\sigma; x_n, x_m, x) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x_n, x) + \frac{1}{2}\mathsf{d}^p(x_m, x)\right]^{\frac{1}{p}} < 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} = 2\sigma\varepsilon.$$

By Corollary 3 and definitions of power triangle inequality at $p = \infty$, $p = -\infty$ and $p = 0$ we have

$$\begin{aligned}
\mathsf{d}(x_n, x_m) &\le \tau(\infty,\sigma; x_n, x_m, x) = 2\sigma \max\{\mathsf{d}(x_n, x),\ \mathsf{d}(x_m, x)\} = 2\sigma \max\{\varepsilon,\ \varepsilon\} = 2\sigma\varepsilon;\\
\mathsf{d}(x_n, x_m) &\le \tau(-\infty,\sigma; x_n, x_m, x) = 2\sigma \min\{\mathsf{d}(x_n, x),\ \mathsf{d}(x_m, x)\} = 2\sigma \min\{\varepsilon,\ \varepsilon\} = 2\sigma\varepsilon;\\
\mathsf{d}(x_n, x_m) &\le \tau(0,\sigma; x_n, x_m, x) = 2\sigma\sqrt{\mathsf{d}(x_n, x)}\ \sqrt{\mathsf{d}(x_m, x)} = 2\sigma\sqrt{\varepsilon}\,\sqrt{\varepsilon} = 2\sigma\varepsilon.
\end{aligned}$$

Therefore the sequence $\{x_n\}$ is Cauchy. By Proposition 1 every Cauchy sequence is bounded. ☐

Theorem 8. [26] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space with any $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$. Let $\mathsf{f} \in \mathbb{Z}^{\mathbb{Z}}$ be a strictly monotone function such that $\mathsf{f}(n) < \mathsf{f}(n+1)$. If $\{x_n\}_{n\in\mathbb{Z}}$ is a Cauchy sequence and $\left\{x_{\mathsf{f}(n)}\right\}_{n\in\mathbb{Z}}$ is convergent then $\{x_n\}_{n\in\mathbb{Z}}$ is convergent.*

Proof. It is easy to see that

$$\begin{aligned}
\mathsf{d}(x_n, x) = \mathsf{d}(x, x_n) &\le \tau(p,\sigma; x, x_n, x_{\mathsf{f}(n)}) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p\left(x, x_{\mathsf{f}(n)}\right) + \frac{1}{2}\mathsf{d}^p\left(x_{\mathsf{f}(n)}, x_n\right)\right]^{\frac{1}{p}}\\
&= 2\sigma\left[\frac{1}{2}\varepsilon + \frac{1}{2}\mathsf{d}^p\left(x_{\mathsf{f}(n)}, x_n\right)\right]^{\frac{1}{p}} = 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} = 2\sigma\varepsilon,
\end{aligned}$$

so, the sequence $\{x_n\}_{n\in\mathbb{Z}}$ is convergent. ☐

Theorem 9. [27] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space. Let $(\mathbb{R}, \mathsf{q})$ be a metric space of real numbers with the usual metric $\mathsf{q}(x,y) := |x - y|$. If $2\sigma = 2^{\frac{1}{p}}$ then d is continuous in $(\mathbb{R}, \mathsf{q})$.*

Proof. Using triangle inequality of $(\mathbb{R}, |x-y|)$ and Lemma 1 we obtain

$$\begin{aligned}
|\mathsf{d}(x,y) - \mathsf{d}(x_n, y_n)| &\le |\mathsf{d}(x,y) - \mathsf{d}(x_n, y)| + |\mathsf{d}(x_n, y) - \mathsf{d}(x_n, y_n)|\\
&= |\mathsf{d}(x,y) - \mathsf{d}(y, x_n)| + |\mathsf{d}(y, x_n) - \mathsf{d}(x_n, y_n)|\\
&\le \mathsf{d}(x, x_n) + \mathsf{d}(y, y_n) \to 0 \qquad \text{as } n \to \infty.
\end{aligned}$$

☐

[25] For theorem in metric space see [2,53,97].
[26] For theorem in metric space see [97].
[27] For theorem in metric space see [5].

In distance spaces and topological spaces, limits of convergent sequences are in general not unique (see Example 1, Example 12). However the next theorem demonstrates that in a power distance space limits are unique.

Theorem 10 (Uniqueness of limit). [28] *Let $(X, \mathsf{d}, p, \sigma)$ be a power distance space with any $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$. Let $x, y \in X$ and let $\{x_n\} \subset X$ be an X-valued sequence.*
If $(\{x_n\}, \{y_n\}) \to (x, y)$ then $x = y$.

Proof. Let us prove that for all $(p, \sigma) \in \mathbb{R}^* \times \mathbb{R}^+$ and for any $\varepsilon \in \mathbb{R}^+$, there exists N such that $\mathsf{d}(x, y) < 2\sigma\varepsilon$. For $p \in \mathbb{R}^* \backslash \{-\infty, 0, \infty\}$ we have

$$\mathsf{d}(x,y) \leq \tau(p,\sigma;x,y,x_n) := 2\sigma\left[\frac{1}{2}\mathsf{d}^p(x,x_n) + \frac{1}{2}\mathsf{d}^p(x_n,y)\right]^{\frac{1}{p}} < 2\sigma\left[\frac{1}{2}\varepsilon^p + \frac{1}{2}\varepsilon^p\right]^{\frac{1}{p}} = 2\sigma\varepsilon.$$

By Corollary 3 and definition of power triangle inequality at $p = \infty$, $p = -\infty$, $p = 0$ we have

$$\begin{aligned}
\mathsf{d}(x,y) &\leq \tau(\infty,\sigma;x,y,x_n) = 2\sigma \max\{\mathsf{d}(x,x_n), \mathsf{d}(x_n,y)\} < 2\sigma\varepsilon,\\
\mathsf{d}(x,y) &\leq \tau(-\infty,\sigma;x,y,x_n) = 2\sigma \min\{\mathsf{d}(x,x_n), \mathsf{d}(x_n,y)\} < 2\sigma\varepsilon,\\
\mathsf{d}(x,y) &\leq \tau(0,\sigma;x,y,x_n) = 2\sigma\sqrt{\mathsf{d}(x,x_n)}\sqrt{\mathsf{d}(x_n,y)} = 2\sigma\sqrt{\varepsilon}\sqrt{\varepsilon} < 2\sigma\varepsilon
\end{aligned}$$

respectively.

Suppose that $x \neq y$. Then $\mathsf{d}(x, y) \neq 0$, and therefore $\mathsf{d}(x, y) > 0$. It implies that there exists ε such that $\mathsf{d}(x, y) > 2\sigma\varepsilon$, which contradicts the proved above inequality $\mathsf{d}(x, y) < 2\sigma\varepsilon$. □

4.3 Examples

It is not always possible to find a triangle relation $\triangle(p, \sigma; \mathsf{d})$ that holds in every distance space, as demonstrated by Example 5 and Example 6 (next two examples).

Example 5. *Let $\mathsf{d}(x, y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ be defined as follows*

$$\mathsf{d}(x,y) := \begin{cases} y, & \forall\,(x,y) \in \{4\} \times (0:2], \\ x, & \forall\,(x,y) \in (0:2] \times \{4\}, \\ |x-y|, & \text{otherwise.} \end{cases}$$

Note the following about the pair $(\mathbb{R}, \mathsf{d})$.

1. *By Example 1, $(\mathbb{R}, \mathsf{d})$ is a distance space, but not a metric space, that is, the triangle relation $\triangle(1, 1; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$.*

2. *Observe further that $(\mathbb{R}, \mathsf{d})$ is not a power distance space. In particular, the triangle relation $\triangle(p, \sigma; \mathsf{d})$ does not hold in $(\mathbb{R}, \mathsf{d})$ for any finite value of σ (does not hold for any $\sigma \in \mathbb{R}^+$)*

$$\begin{aligned}
\mathsf{d}(0,4) = 4 \nleq 0 = \lim_{\varepsilon\to 0} 2\sigma\varepsilon &= \lim_{\varepsilon\to 0} 2\sigma\left[1/2|0-\varepsilon|^p + 1/2\varepsilon^p\right]^{\frac{1}{p}}\\
&:= \lim_{\varepsilon\to 0} 2\sigma[1/2\mathsf{d}^p(0,\varepsilon) + 1/2\mathsf{d}^p(\varepsilon,4)]^{\frac{1}{p}} := \lim_{\varepsilon\to 0} \triangle(p,\sigma;0,4,\varepsilon;\mathsf{d}).
\end{aligned}$$

[28] For theorem in metric space see [97, 109].

Example 6. *Let* $\mathsf{d}(x,y) \in \mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *be defined as follows*

$$\mathsf{d}(x,y) := \begin{cases} |x-y|, & \text{for } x = 0 \text{ or } y = 0 \text{ or } x = y, \\ 1, & \text{otherwise.} \end{cases}$$

Note the following about the pair $(\mathbb{R}, \mathsf{d})$.

1. *By Example 2,* $(\mathbb{R}, \mathsf{d})$ *is a distance space, but not a metric space, that is, the triangle relation* $\triangle(1,1;\mathsf{d})$ *does not hold in* $(\mathbb{R}, \mathsf{d})$.

2. *Observe further that* $(\mathbb{R}, \mathsf{d})$ *is not a power distance space, that is, the triangle relation* $\triangle(p,\sigma;\mathsf{d})$ *does not hold in* $(\mathbb{R}, \mathsf{d})$ *for any value of* $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$.

   Let us prove that $\triangle(p,\sigma;\mathsf{d})$ *does not hold for any* $(p,\sigma) \in \{\infty\} \times \mathbb{R}^+$. *Indeed, Corollary 3 and Corollary 2 imply*

$$\lim_{n,m\to\infty} \mathsf{d}(1/n, 1/m) := 1 \not\leq 0 = 2\sigma \max\{0,0\} = 2\sigma \lim_{n,m\to\infty} \max\{\mathsf{d}(1/n, 0), \mathsf{d}(0, 1/m)\}$$
$$\geq \lim_{n,m\to\infty} 2\sigma[1/2\mathsf{d}^p(1/n,0) + 1/2\mathsf{d}^p(0,1/m)]^{\frac{1}{p}} := \lim_{n,m\to\infty} \tau(p,\sigma,1/n,1/m,0).$$

   The triangle relation $\triangle(p,\sigma;\mathsf{d})$ *does not hold for any* $(p,\sigma) \in \mathbb{R}^* \times \mathbb{R}^+$ *also. The triangle function* $\tau(p,\sigma;x,y,z;\mathsf{d})$ *is continuous and strictly monotone in* $(\mathbb{R}, |\cdot|, \leq)$ *with respect to the variable* p *via Corollary 2. From proved above it follows that* $\triangle(p,\sigma;\mathsf{d})$ *fails to hold at the best case of* $p = \infty$*, and so by Corollary 2, it doesn't hold for any other value of* $p \in \mathbb{R}^*$ *either.*

Example 7. *Let* d *be a function in* $\mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that*

$$\mathsf{d}(x,y) := \begin{cases} 2|x-y|, & \forall (x,y) \in \{(0,1), (1,0)\}, \\ |x-y|, & \text{otherwise.} \end{cases}$$

Note the following about the pair $(\mathbb{R}, \mathsf{d})$.

1. *By Example 3,* $(\mathbb{R}, \mathsf{d})$ *is a distance space, but not a metric space, that is, the triangle relation* $\triangle(1,1;\mathsf{d})$ *does not hold in* $(\mathbb{R}, \mathsf{d})$.

2. *But observe further that* $(\mathbb{R}, \mathsf{d}, 1, 2)$ *is a power distance space. Let us prove that* $\triangle(1,2;\mathsf{d})$ *holds for all* $(x,y) \in \{(0,1), (1,0)\}$. *Indeed, for any* $z \in \mathbb{R}$ *we have*

$$\mathsf{d}(1,0) = \mathsf{d}(0,1) := 2|0-1| = 2 \leq 2 \leq 2(|0-z| + |z-1|)$$
$$= 2\sigma\left(1/2|0-z|^p + 1/2|z-1|^p\right)^{\frac{1}{p}} := 2\sigma(1/2\mathsf{d}^p(0,z) + \mathsf{d}^p(z,1))^{\frac{1}{p}} := \tau(1,2;0,1,z).$$

   Let us show that $\triangle(1,2;\mathsf{d})$ *holds for all other* $(x,y) \in \mathbb{R}^* \times \mathbb{R}^+$. *Using Corollary 2 we obtain*

$$\mathsf{d}(x,y) := 2|x-y| \leq (|x-z| + |z-y|) = 2\sigma\left(1/2|0-z|^p + 1/2|z-1|^p\right)^{\frac{1}{p}}$$
$$:= \tau(1,1;x,y,z) \leq \tau(1,2;x,y,z).$$

3. *In* (X, d)*, the limits of convergent sequences are unique. This follows directly from the fact that* $(\mathbb{R}, \mathsf{d}, 1, 2)$ *is a power distance space and by Theorem 10.*

4. *In* (X, d)*, convergent sequences are Cauchy. This follows directly from the fact that* $(\mathbb{R}, \mathsf{d}, 1, 2)$ *is a power distance space and by Theorem 7.*

Example 8. *Let* d *be a function in* $\mathbb{R}^{\mathbb{R}\times\mathbb{R}}$ *such that* $\mathsf{d}(x,y) := (x-y)^2$. *Note the following about the pair* $(\mathbb{R},\mathsf{d})$.

1. *It was demonstrated in Example 4 that* $(\mathbb{R},\mathsf{d})$ *is a distance space, but that it is not a metric space because the triangle inequality does not hold.*
2. *However, the tuple* $(\mathbb{R},\mathsf{d},p,\sigma)$ *is a power distance space for any* $(p,\sigma) \in \mathbb{R}^* \times [2:\infty)$. *In particular, for all* $x,y,z \in \mathbb{R}$, *the power triangle inequality must hold. The "worst case" for this is when a third point* z *is exactly "halfway between"* x *and* y *in* $\mathsf{d}(x,y)$; *that is, when* $z = \frac{x+y}{2}$:

$$\begin{aligned}
(x-y)^2 &:= \mathsf{d}(x,y) \le \tau(p,\sigma;x,y,z;\mathsf{d}) := 2\sigma[1/2\mathsf{d}^p(x,z) + 1/2\mathsf{d}^p(z,y)]^{\frac{1}{p}} \\
&:= 2\sigma\left[1/2(x-z)^{2p} + 1/2(z-y)^{2p}\right]^{\frac{1}{p}} = 2\sigma\left[1/2|x-z|^{2p} + 1/2|z-y|^{2p}\right]^{\frac{1}{p}} \\
&= 2\sigma\left[1/2\left|x - \frac{x+y}{2}\right|^{2p} + 1/2\left|\frac{x+y}{2} - y\right|^{2p}\right]^{\frac{1}{p}} \\
&= 2\sigma\left[1/2\left|\frac{y-x}{2}\right|^{2p} + 1/2\left|\frac{x-y}{2}\right|^{2p}\right]^{\frac{1}{p}} = 2\sigma\left[\left|\frac{x-y}{2}\right|^{2p}\right]^{\frac{1}{p}} = \frac{2\sigma}{4}|x-y|^2.
\end{aligned}$$

   It follows $(p,\sigma) \in \mathbb{R}^* \times [2:\infty)$.

3. *The power distance function* d *is continuous in* $(\mathbb{R},\mathsf{d},p,\sigma)$ *for any* (p,σ) *such that* $\sigma \ge 2$ *and* $2\sigma = p^{\frac{1}{p}}$. *This follows directly from Theorem 9.*

APPENDIX A TOPOLOGICAL SPACES

Definition 25 ([59, 60, 89, 96, 111]). *Let* Γ *be a set with an arbitrary (possibly uncountable) number of elements. Let* 2^X *be the power set of a set* X. *A family of sets* $T \subseteq 2^X$ *is a topology on* X *if*

1. $\varnothing \in T$ *and*
2. $X \in T$ *and*
3. $U, V \in T \implies U \cap V \in T$ *and*
4. $\{U_\gamma \mid \gamma \in \Gamma\} \subseteq T \implies \bigcup_{\gamma\in\Gamma} U_\gamma \in T$.

The ordered pair (X,T) *is a topological space if* T *is a topology on* X. *A set* U *is open in* (X,T) *if* U *is any element of* T. *A set* D *is closed in* (X,T) *if* D^c *is open in* (X,T).

Just as the power set 2^X and the set $\{\varnothing, X\}$ are algebras of sets on a set X, so also are these sets topologies on X.

Example 9 ([42, 73, 89, 105]). *Let* $\mathcal{T}(X)$ *be the set of topologies on a set* X *and* 2^X *the power set on* X. *Then* $\{\varnothing, X\}$ *is a topology in* $\mathcal{T}(X)$, *which is called indiscrete topology or trivial topology;* 2^X *is a topology in* $\mathcal{T}(X)$, *which is called discrete topology.*

Definition 26 ([37, 66]). *Let* (X,T) *be a topological space. A set* $B \subseteq 2^X$ *is a base for* T *if* $B \subseteq T$ *and for all* $U \in T$ *there exist* $\{B_\gamma \in B\}$ *such that* $U = \bigcup_\gamma B_\gamma$.

Theorem 11 ([37,66]). *Let (X, T) be a topological space. Let B be a subset of 2^X. If B is a base for (X, T) then for every $x \in X$ and for every open set U containing x, there exists $B_x \in B$ such that $x \in B_x \subseteq U$.*

Theorem 12 ([11]). *Let (X, T) be a topological space and $B \subseteq 2^X$. If B is a base for (X, T) then*
1. $x \in X \implies \exists B_x \in B$ *such that* $x \in B_x$ *and*
2. $B_1, B_2 \in B \implies B_1 \cap B_2 \in B$.

Example 10 ([37]). *Let (X, d) be a metric space. The set $B := \{\mathsf{B}(x,r) \mid x \in X,\ r \in \mathbb{N}\}$ (the set of all open balls in (X, d)) is a base for a topology on (X, d).*

Example 11 (the standard topology on the real line). [29] *The set $B := \{(a : b) \mid a, b \in \mathbb{R},\ a < b\}$ is a base for the metric space $(\mathbb{R}, |b - a|)$ (the usual metric space on $\mathbb{R}$).*

Definition 27 ([51,67,72,81,89,110]). *Let (X, T) be a topological space. Let 2^X be the power set of X. The set A^- is the closure of $A \in 2^X$ if $A^- := \bigcap \left\{ D \in 2^X \mid A \subseteq D \text{ and } D \text{ is closed} \right\}$. The set A° is the interior of $A \in 2^X$ if $A^\circ := \bigcup \left\{ U \in 2^X \mid U \subseteq A \text{ and } U \text{ is open} \right\}$. A point x is a closure point of A if $x \in A^-$. A point x is an interior point of A if $x \in A^\circ$. A point x is an accumulation point of A if $x \in (A \backslash \{x\})^-$. A point x in A^- is a point of adherence in A or is adherent to A if $x \in A^-$.*

Lemma 3 ([1,81]). *Let A^- be the closure, and A° the interior of a set $A \in 2^X$ in a topological space (X, T). Then $A^\circ \subseteq A \subseteq A^-$; $A = A^\circ$ iff A is open; $A = A^-$ iff A is closed.*

Definition 28 ([37]). *Let (X, T_x) and (Y, T_y) be topological spaces. Let f be a function in Y^X. A function $\mathsf{f} \in Y^X$ is continuous if for any open set $U \in T_y$ in (Y, T_y) the set $\mathsf{f}^{-1}(U) \in T_x$ is open in (X, T_x). A function is discontinuous in $(X, T_y)^{(X, T_x)}$ if it is not continuous in $(X, T_y)^{(X, T_x)}$.*

Definition 28 defines continuity using open sets. Continuity can alternatively be defined using closed sets or closure.

Theorem 13 ([81,101]). *Let (X, T) and (Y, S) be topological spaces. Let f be a function in Y^X. The following are equivalent:*
1. f *is continuous;*
2. *if B is closed in (Y, S) then $\mathsf{f}^{-1}(B)$ is closed in (X, T);*
3. $\mathsf{f}(A^-) \subseteq \mathsf{f}(A)^-$;
4. $\mathsf{f}^{-1}(B) \subseteq \mathsf{f}^{-1}(B^-)$.

Remark 4. *A word of warning about defining continuity in terms of topological spaces — continuity is defined in terms of a pair of topological spaces, and whether function is continuous or discontinuous in general depends very heavily on the selection of these spaces. This is illustrated in Proposition 4. The ramification of this is that when declaring a function to be continuous or discontinuous, one must make clear the assumed topological spaces.*

Proposition 4 ([35,94]). *Let (X, T) and (Y, S) be topological spaces. Let f be a function in $(Y, S)^{(X, T)}$. If T is the discrete topology then f is continuous. If S is the indiscrete topology then f is continuous.*

[29] See [37,89].

Definition 29 ([66,75]). *Let (X,T) be a topological space. A sequence $\{x_n\}_{n\in\mathbb{Z}}$ converges in (X,T) to a point x if for each open set $U \in T$ that contains x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n > N$. This condition can be expressed in any of the following forms:*

1. *The limit of the sequence $\{x_n\}$ is x.*
2. *The sequence $\{x_n\}$ is convergent with limit x.*
3. $\lim_{n\to\infty} \{x_n\} = x$.
4. $\{x_n\} \to x$.

A sequence that converges is convergent. A sequence that does not converge is said to diverge, or is divergent. An element $x \in A$ is a limit point of A if it is the limit of some A-valued sequence $\{x_n\} \subset A$.

Example 12 ([89]). *Let $X := \{x,y,z\}$ and $T_{31} := \{\varnothing, \{x\}, \{x,y\}, \{x,z\}, \{x,y,z\}\}$. Then (X,T_{31}) is a topological space. In this space, the sequence $\{x,x,x,\ldots\}$ converges to x. But this sequence also converges to both y and z because x is in every open set that contains y and x is in every open set that contains z. So, the limit of the sequence is not unique.*

Example 13. *In contrast to the low resolution topological space of Example 12, the limit of the sequence $\{x,x,x,\ldots\}$ is unique in a topological space with sufficiently high resolution with respect to y and z such as the following. Define a topological space (X,T_{56}) where $X := \{x,y,z\}$ and $T_{56} := \{\varnothing, \{y\}, \{z\}, \{x,y\}, \{y,z\}, \{x,y,z\}\}$. In this space, the sequence $\{x,x,x,\ldots\}$ converges to x only. The sequence does not converge to y or z because there are open sets containing y or z that do not contain x (the open sets $\{y\}$, $\{z\}$, and $\{y,z\}$).*

Theorem 14 (The Closed Set Theorem). [30] *Let (X,T) be a topological space. Let A be a subset of X. Then A is closed in (X,T) if and only if every A-valued sequence $\{x_n\}_{n\in\mathbb{Z}} \subset A$ that converges in (X,T) has its limit in A.*

Theorem 15 ([94]). *Let (X,T) and (Y,S) be topological spaces. Let f be a function in $(Y,S)^{(X,T)}$. Then inverse image characterization of continuity (see Definition 28) is equivalent to sequential characterization of continuity (see Definition 29).*

APPENDIX B FINITE SUMS

B.1 Convexity

Definition 30 ([3,11,64,103]). *A function $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ is said to be*

convex if $\mathsf{f}\big(\lambda x + [1-\lambda]y\big) \le \lambda \mathsf{f}(x) + (1-\lambda)\,\mathsf{f}(y)$, $\forall x,y \in \mathbb{R}$, $\forall \lambda \in (0:1)$;
strictly convex if $\mathsf{f}\big(\lambda x + [1-\lambda]y\big) = \lambda \mathsf{f}(x) + (1-\lambda)\,\mathsf{f}(y)$, $\forall x,y \in \mathbb{R}$, $x \ne y$, $\forall \lambda \in (0:1)$;
concave if $-\mathsf{f}$ is convex;
affine if f is convex and concave.

Theorem 16 (Jensen's Inequality). [31] *Let $\mathsf{f} \in \mathbb{R}^{\mathbb{R}}$ be a function. If f is convex and $\sum_{n=1}^{N} \lambda_n = 1$ then $\mathsf{f}\left(\sum_{n=1}^{N} \lambda_n\, x_n\right) \le \sum_{n=1}^{N} \lambda_n\, \mathsf{f}(x_n)$ for all $x_n \in \mathbb{R}$ and $N \in \mathbb{N}$.*

[30] See [54,72,97].
[31] See [11,64,86].

B.2 Power means

Definition 31 ([11]). *The $(\!|\lambda_n|\!)_1^N$ weighted* φ***-mean*** *of a tuple $(\!|x_n|\!)_1^N$ is defined as*

$$\mathsf{M}_\varphi((\!|x_n|\!)) := \varphi^{-1}\left(\sum_{n=1}^{N} \lambda_n \varphi(x_n)\right)$$

where φ is a continuous and strictly monotonic function in $\mathbb{R}^{\mathbb{R}^{\vdash}}$ and $(\!|\lambda_n|\!)_{n=1}^N$ is a sequence of weights for which $\sum_{n=1}^{N} \lambda_n = 1$.

Lemma 4 ([11, 58, 93]). *Let $\mathsf{M}_\varphi((\!|x_n|\!))$ be the $(\!|\lambda_n|\!)_1^N$ weighted φ-mean and $\mathsf{M}_\psi((\!|x_n|\!))$ the $(\!|\lambda_n|\!)_1^N$ weighted ψ-mean of a tuple $(\!|x_n|\!)_1^N$.*

If $\varphi\psi^{-1}$ is convex and φ is increasing then $\mathsf{M}_\varphi((\!|x_n|\!)) \geq \mathsf{M}_\psi((\!|x_n|\!))$.
If $\varphi\psi^{-1}$ is convex and φ is decreasing then $\mathsf{M}_\varphi((\!|x_n|\!)) \leq \mathsf{M}_\psi((\!|x_n|\!))$.
If $\varphi\psi^{-1}$ is concave and φ is increasing then $\mathsf{M}_\varphi((\!|x_n|\!)) \leq \mathsf{M}_\psi((\!|x_n|\!))$.
If $\varphi\psi^{-1}$ is concave and φ is decreasing then $\mathsf{M}_\varphi((\!|x_n|\!)) \geq \mathsf{M}_\psi((\!|x_n|\!))$.

One of the most well known inequalities in mathematics is Minkowski's Inequality. In 1946, H.P. Mulholland submitted a result that generalizes Minkowski's Inequality to an equal weighted φ-mean. In 1979, G.V. Milovanović and I. Milovanović generalized this even further to a *weighted* φ-mean.[32]

Theorem 17 ([20, 84]). *Let φ be a convex strictly monotone function in $\mathbb{R}^{\mathbb{R}}$, such that $\varphi(0) = 0$ and $\log \circ \varphi \circ \exp$ is convex. Then*

$$\varphi^{-1}\left(\sum_{n=1}^{N} \lambda_n \varphi(x_n + y_n)\right) \leq \varphi^{-1}\left(\sum_{n=1}^{N} \lambda_n \varphi(x_n)\right) + \varphi^{-1}\left(\sum_{n=1}^{N} \lambda_n \varphi(y_n)\right).$$

Definition 32 ([11, 20]). *Let $\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!))$ be the $(\!|\lambda_n|\!)_1^N$ weighted φ-mean of a non-negative tuple $(\!|x_n|\!)_1^N$. A mean $\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!))$ is a power mean with parameter p if $\varphi(x) := x^p$. That is,*

$$\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!)) = \left(\sum_{n=1}^{N} \lambda_n (x_n)^p\right)^{\frac{1}{p}}.$$

Theorem 18 ([7, 8, 11, 14, 19, 20]). *Let $\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!))$ be the power mean with parameter p of an N-tuple $(\!|x_n|\!)_1^N$ in which the elements are not all equal. Then $\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!)) := \left(\sum_{n=1}^{N} \lambda_n (x_n)^p\right)^{\frac{1}{p}}$ is continuous and strictly monotone in $\mathbb{R}^*$ and*

$$\mathsf{M}_{\varphi(x;p)}((\!|x_n|\!)) = \begin{cases} \max\limits_{n=1,2,\ldots,N} (\!|x_n|\!), & \text{for } p = +\infty, \\ \prod\limits_{n=1}^{N} x_n^{\lambda_n}, & \text{for } p = 0, \\ \min\limits_{n=1,2,\ldots,N} (\!|x_n|\!), & \text{for } p = -\infty. \end{cases}$$

[32] See also [20, 22, 58, 79, 85, 88, 112].

Proof. Let p and s be such that $-\infty < p < s < \infty$. Let $\varphi_p := x^p$ and $\varphi_s := x^s$. Then $\varphi_p \varphi_s^{-1} = x^{\frac{p}{s}}$. The composite function $\varphi_p \varphi_s^{-1}$ is convex or concave depending on the values of p and s:

	$p < 0$ (φ_p decreasing)	$p > 0$ (φ_p increasing)
$s < 0$	convex	(not possible)
$s > 0$	convex	concave

Therefore by Lemma 4, we obtain $\mathsf{M}_{\varphi(x;p)}(⦇x_n⦈) < \mathsf{M}_{\varphi(x;s)}(⦇x_n⦈)$. So, $M_{\varphi(x;p)}$ is strictly monotone in p.

The sum of continuous functions is continuous. Therefore, $\mathsf{M}_{\varphi(x;p)}$ is continuous in p for $p \in \mathbb{R}\backslash 0$. The cases of $p \in \{-\infty, 0, \infty\}$ we consider below.

Note that using the definition of M_φ we obtain

$$\left\{\mathsf{M}_{\varphi(x;p)}\left(⦇x_n^{-1}⦈\right)\right\}^{-1} = \left\{\left(\sum_{n=1}^{N} \lambda_n \left(x_n^{-1}\right)^p\right)^{\frac{1}{p}}\right\}^{-1} = \left(\sum_{n=1}^{N} \lambda_n (x_n)^{-p}\right)^{\frac{1}{-p}} = \mathsf{M}_{\varphi(x;-p)}(⦇x_n⦈). \quad (1)$$

Denote $x_m := \max_{n\in\mathbb{Z}}⦇x_n⦈$. Note that $\lim_{p\to\infty} \mathsf{M}_\varphi \le \max_{n\in\mathbb{Z}}⦇x_n⦈$. Indeed, using the definition of M_φ, we obtain

$$\begin{aligned}
\lim_{p\to\infty} \mathsf{M}_\varphi(⦇x_n⦈) = \lim_{p\to\infty}\left(\sum_{n=1}^{N} \lambda_n x_n^p\right)^{\frac{1}{p}} &\le \lim_{p\to\infty}\left(\sum_{n=1}^{N} \lambda_n x_m^p\right)^{\frac{1}{p}} \\
&= \lim_{p\to\infty}\left(x_m^p \underbrace{\sum_{n=1}^{N} \lambda_n}_{1}\right)^{\frac{1}{p}} = \lim_{p\to\infty}\left(x_m^p \cdot 1\right)^{\frac{1}{p}} = x_m = \max_{n\in\mathbb{Z}}⦇x_n⦈.
\end{aligned}$$

But also note that $\lim_{p\to\infty} \mathsf{M}_\varphi \ge \max_{n\in\mathbb{Z}}⦇x_n⦈$ because

$$\lim_{p\to\infty} \mathsf{M}_\varphi(⦇x_n⦈) = \lim_{p\to\infty}\left(\sum_{n=1}^{N} \lambda_n x_n^p\right)^{\frac{1}{p}} \ge \lim_{p\to\infty}\left(w_m x_m^p\right)^{\frac{1}{p}} = \lim_{p\to\infty} w_m^{\frac{1}{p}} x_m^{\frac{p}{p}} = x_m = \max_{n\in\mathbb{Z}}⦇x_n⦈.$$

Here we used the fact, that $\varphi(x) := x^p$ and φ^{-1} are both increasing or both decreasing. So, $\lim_{p\to\infty} \mathsf{M}_\varphi(⦇x_n⦈) = \max_{n\in\mathbb{Z}}⦇x_n⦈$.

Let us prove that $\lim_{p\to-\infty} \mathsf{M}_\varphi(⦇x_n⦈) = \min_{n\in\mathbb{Z}}⦇x_n⦈$. From the equation (1) it follows

$$\begin{aligned}
\lim_{p\to-\infty} \mathsf{M}_{\varphi(x;p)}(⦇x_n⦈) &= \lim_{p\to\infty} \mathsf{M}_{\varphi(x;-p)}(⦇x_n⦈) = \lim_{p\to\infty}\left\{\mathsf{M}_{\varphi(x;p)}\left(⦇x_n^{-1}⦈\right)\right\}^{-1} = \lim_{p\to\infty} \frac{1}{\mathsf{M}_{\varphi(x;p)}\left(⦇x_n^{-1}⦈\right)} \\
&= \frac{\lim_{p\to\infty} 1}{\lim_{p\to\infty} \mathsf{M}_{\varphi(x;p)}\left(⦇x_n^{-1}⦈\right)} = \frac{1}{\max_{n\in\mathbb{Z}}⦇x_n^{-1}⦈} = \frac{1}{\left(\min_{n\in\mathbb{Z}}⦇x_n⦈\right)^{-1}} = \min_{n\in\mathbb{Z}}⦇x_n⦈.
\end{aligned}$$

It remains to prove that $\lim_{p\to 0} \mathsf{M}_\varphi(⦇x_n⦈) = \prod_{n=1}^{N} x_n^{\lambda_n}$. Using the definition of M_φ and l'Hôpital's rule[33] we obtain

$$
\begin{aligned}
\lim_{p\to 0} \mathsf{M}_{\varphi}(\lparen x_n \rparen) &= \lim_{p\to 0} \exp\left\{\ln\left\{\mathsf{M}_{\varphi}(\lparen x_n \rparen)\right\}\right\} = \lim_{p\to 0} \exp\left\{\ln\left\{\left(\sum_{n=1}^{N} \lambda_n \left(x_n^p\right)\right)^{\frac{1}{p}}\right\}\right\} \\
&= \exp\left\{\frac{\frac{\partial}{\partial p}\ln\left(\sum_{n=1}^{N}\lambda_n\left(x_n^p\right)\right)}{\frac{\partial}{\partial p}p}\right\}_{p=0} = \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial p}\left(x_n^p\right)}{\sum_{n=1}^{N}\lambda_n\left(x_n^p\right)}\right\}_{p=0} \\
&= \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial p}\exp\left(\ln\left(x_n^p\right)\right)}{\sum_{n=1}^{N}\lambda_n}\right\}_{p=0} = \exp\left\{\frac{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial p}\exp\left(r\ln\left(x_n\right)\right)}{1}\right\}_{p=0} \\
&= \exp\left\{\sum_{n=1}^{N}\lambda_n\frac{\partial}{\partial p}\exp\left(p\ln\left(x_n\right)\right)\right\}_{p=0} = \exp\left\{\sum_{n=1}^{N}\lambda_n\exp\left\{p\ln x_n\right\}\ln\left(x_n\right)\right\}_{p=0} \\
&= \exp\left\{\sum_{n=1}^{N}\lambda_n\ln\left(x_n\right)\right\} = \exp\left\{\sum_{n=1}^{N}\ln\left(x_n^{\lambda_n}\right)\right\} = \exp\left\{\ln\prod_{n=1}^{N}x_n^{\lambda_n}\right\} = \prod_{n=1}^{N}x_n^{\lambda_n}.
\end{aligned}
$$

□

Corollary 8 ([11,20,23,63,64]). *Let $\lparen x_n \rparen_1^N$ be a tuple. Let $\lparen \lambda_n \rparen_1^N$ be a tuple of weighting values such that $\sum_{n=1}^{N} \lambda_n = 1$. Then*

$$
\min\lparen x_n \rparen \le \underbrace{\left(\sum_{n=1}^{N}\lambda_n\frac{1}{x_n}\right)^{-1}}_{\text{harmonic mean}} \le \underbrace{\prod_{n=1}^{N}x_n^{\lambda_n}}_{\text{geometric mean}} \le \underbrace{\sum_{n=1}^{N}\lambda_n x_n}_{\text{arithmetic mean}} \le \max\lparen x_n \rparen.
$$

Proof. These five means are all special cases of the power mean $\mathsf{M}_{\varphi(x:p)}$, namely

$p=\infty$:	$\max\lparen x_n \rparen$,	
$p=1$:	arithmetic mean,	
$p=0$:	geometric mean,	So, the inequalities follow directly from Theorem 18.
$p=-1$:	harmonic mean,	
$p=-\infty$:	$\min\lparen x_n \rparen$.	

If one is only concerned with the arithmetic mean and geometric mean, their relationship can be established directly using Jensen's Inequality (Theorem 16):

$$
\sum_{n=1}^{N}\lambda_n x_n = b^{\log_b\left(\sum_{n=1}^{N}\lambda_n x_n\right)} \ge b^{\left(\sum_{n=1}^{N}\lambda_n \log_b x_n\right)} = \prod_{n=1}^{N} b^{\left(\lambda_n \log_b x_n\right)} = \prod_{n=1}^{N} b^{\left(\log_b x_n\right)\lambda_n} = \prod_{n=1}^{N} x_n^{\lambda_n}.
$$

□

[33] See [98].

B.3 Inequalities

Lemma 5 (Young's Inequality). [34]

$$\begin{aligned} xy &< \frac{x^p}{p} + \frac{y^q}{q} \quad \text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \text{but } y \neq x^{p-1}, \\ xy &= \frac{x^p}{p} + \frac{y^q}{q} \quad \text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad \forall 1 < p < \infty,\ x, y \geq 0, \quad \text{and } y = x^{p-1}. \end{aligned}$$

Theorem 19 (Minkowski's Inequality for sequences). [35] *Let* $(\!(x_n)\!)_1^N \subset \mathbb{C}$ *and* $(\!(y_n)\!)_1^N \subset \mathbb{C}$ *be complex* N*-tuples. Then*

$$\left(\sum_{n=1}^{N} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{N} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} |y_n|^p\right)^{\frac{1}{p}} \qquad \forall\, 1 < p < \infty.$$

APPENDIX C METRIC PRESERVING FUNCTIONS

Definition 33 ([31, 40, 113]). *Let* $\mathbb{M}$ *be the set of all metric spaces on a set* X. *A function* $\varphi \in \mathbb{R}^{\vdash\mathbb{R}^{\vdash}}$ *is a metric preserving function if* $\mathsf{d}(x, y) := \varphi \circ \mathsf{p}(x, y)$ *is a metric on* X *for all* $(X, \mathsf{p}) \in \mathbb{M}$.

Theorem 20 (necessary conditions). [36] *Let* $\mathcal{R}\varphi$ *be the range of a function* φ. *If* φ *is a metric preserving function then* $\varphi^{-1}(0) = \{0\}$, $\mathcal{R}\varphi \subseteq \mathbb{R}^{\vdash}$, *and the function* φ *is subadditive, i.e.* $\varphi(x + y) \leq \varphi(x) + \varphi(y)$.

Theorem 21 (sufficient conditions). [37] *Let* φ *be a function in* $\mathbb{R}^{\mathbb{R}}$. *If the conditions*

1. $x \geq y \implies \varphi(x) \geq \varphi(y), \quad \forall x, y \in \mathbb{R}^{\vdash}$,
2. $\varphi(0) = 0$,
3. $\varphi(x + y) \leq \varphi(x) + \varphi(y), \quad \forall x, y \in \mathbb{R}^{\vdash}$,

hold, then φ *is a metric preserving function.*

The proofs for Example 14–Example 19 follow from Theorem 21.

Example 14 (α-scaled metric/dilated metric). [38] *Let* (X, d) *be a metric space. The function* $\varphi(x) := \alpha x$, $\alpha \in \mathbb{R}^{+}$, *is a metric preserving function (see Figure 3 (A)).*

Example 15 (power transform metric/snowflake transform metric). [39] *Let* (X, d) *be a metric space. The function* $\varphi(x) := x^{\alpha}$, $\alpha \in (0 : 1]$, *is a metric preserving function (see Figure 3 (B)).*

Example 16 (α-truncated metric/radar screen metric). [40] *Let* (X, d) *be a metric space. The function* $\varphi(x) := \min\{\alpha,\ x\}$, $\alpha \in \mathbb{R}^{+}$, *is a metric preserving function (see Figure 3 (C)).*

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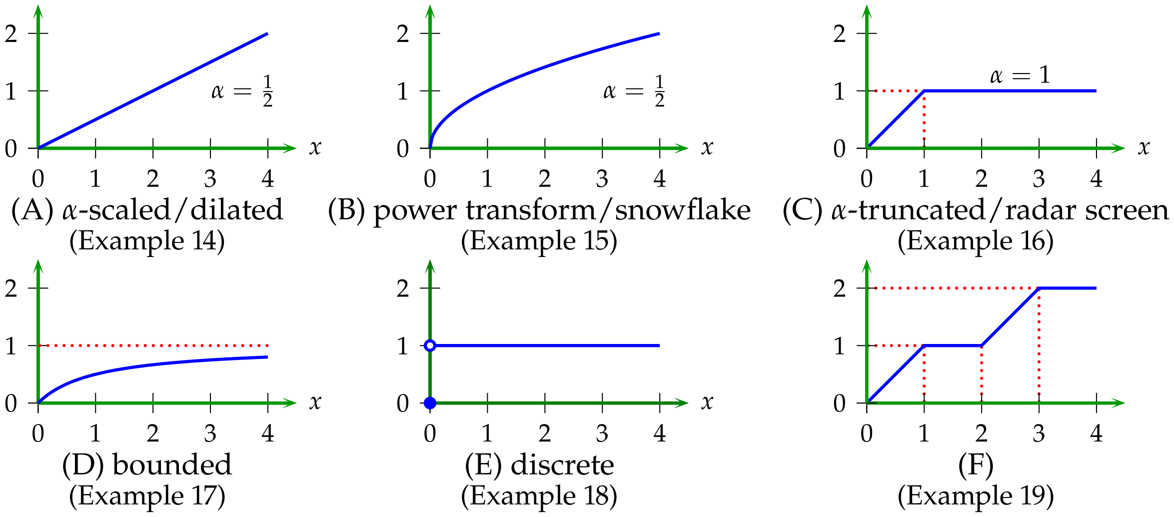
[34] See [22, 58, 79, 112, 120].
[35] See [20, 22, 58, 79, 85, 112].
[36] See [31, 40].
[37] See [31, 40, 67].
[38] See [39].
[39] See [39, 40].
[40] See [39, 53].



(A) α-scaled/dilated (Example 14)

(B) power transform/snowflake (Example 15)

(C) α-truncated/radar screen (Example 16)

(D) bounded (Example 17)

(E) discrete (Example 18)

(F) (Example 19)

Figure 3: metric preserving functions.

Example 17 (bounded metric)**.** [41] *Let* (X, d) *be a metric space. The function* $\varphi(x) := \dfrac{x}{1+x}$ *is a metric preserving function (see Figure 3 (D)).*

Example 18 (discrete metric preserving function)**.** [42] *The function* $\varphi(x) := \begin{cases} 0, & \text{for } x \leq 0, \\ 1, & \text{otherwise,} \end{cases}$ *from* $\mathbb{R}^{\mathbb{R}}$ *is a metric preserving function (see Figure 3 (E)).*

Example 19. *The function*

$$\varphi(x) := \begin{cases} x, & \text{for } 0 \leq x < 1, \\ 1, & \text{for } 1 \leq x \leq 2, \\ x-1, & \text{for } 2 < x < 3, \\ 2, & \text{for } x \geq 3, \end{cases}$$

from $\mathbb{R}^{\mathbb{R}}$ *is a metric preserving function (see Figure 3 (F)).*

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[41] See [1, 113].
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Ґрінхое Д.Дж. *Властивості просторів з відстанню, що задовольняють степеневі нерівності трикутника* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 51–82.

Метричні простори забезпечують основу для математичного аналізу і мають ряд дуже корисних властивостей. Багато з цих властивостей випливають зокрема з нерівності трикутника. Однак є багато застосувань, в яких нерівність трикутника не справджується, але в яких ми все ще можемо здійснювати аналіз. У цій статті досліджуємо, що трапиться, якщо нерівність трикутника вилучено з переліку аксіом метрики, при цьому метричний простір стає так званим простором з відстанню. Також нас цікавить, що буде коли нерівність трикутника замінена на більш загальне двохпараметричне співвідношення, яке ми називаємо степеневою нерівністю трикутника. Таке узагальнення нерівності трикутника дає незліченно великий клас нерівностей, і включає при цьому звичайну нерівність трикутника, слабку нерівність трикутника та інфраметричну нерівність як частинні випадки. Степенева нерівність трикутника визначена в термінах функції, яку ми називаємо степеневою трикутною функцією. Ця функція є неперервною і монотонною відносно свого експоненціального параметру, є степеневим середнім, і також включає як частинні випадки максимум, мінімум, середнє квадратичне, середнє арифметичне, середнє геометричне і середнє гармонійне.

Ключові слова і фрази: метричний простір, простір з відстанню, напівметричний простір, квазі-метричний простір, нерівність трикутника, слабка нерівність трикутника, інфраметрика, середнє арифметичне, середнє квадратичне, середнє геометричне, середнє гармонійне, максимум, мінімум, середнє степеневе.