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AN INVERSE PROBLEM FOR A 2D PARABOLIC EQUATION WITH NONLOCAL OVERDETERMINATION CONDITION

We consider an inverse problem of identifying the time-dependent coefficient $a(t)$ in a two-dimensional parabolic equation:

$$u_t = a(t)\Delta u + b_1(x,y,t)u_x + b_2(x,y,t)u_y + c(x,y,t)u + f(x,y,t),\ (x,y,t) \in Q_T,$$

with the initial condition, Neumann boundary data and the nonlocal overdetermination condition

$$\nu_1(t)u(0,y_0,t) + \nu_2(t)u(h,y_0,t) = \mu_3(t), \quad t \in [0,T],$$

where y_0 is a fixed number from $[0,l]$.

The conditions of existence and uniqueness of the classical solution to this problem are established. For this purpose the Green function method, Schauder fixed point theorem and the theory of Volterra intergral equations are utilized.

Key words and phrases: inverse problem, determining coefficients, parabolic equation, nonlocal overdetermination condition, rectangular domain.

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INTRODUCTION

This paper discusses the problem of identifying an unknown pair of functions $(a(t), u(x,y,t))$ for the equation

$$u_t = a(t)\Delta u + b_1(x,y,t)u_x + b_2(x,y,t)u_y + c(x,y,t)u + f(x,y,t),$$
$$(x,y,t) \in Q_T := \{(x,y,t) : 0 < x < h, 0 < y < l, 0 < t < T\} \tag{1}$$

with the initial condition

$$u(x,y,0) = \varphi(x,y), \quad (x,y) \in [0,h] \times [0,l], \tag{2}$$

boundary conditions

$$u_x(0,y,t) = \mu_{11}(y,t), \quad u_x(h,y,t) = \mu_{12}(y,t), \quad (y,t) \in [0,l] \times [0,T], \tag{3}$$

$$u_y(x,0,t) = \mu_{21}(x,t), \quad u_y(x,l,t) = \mu_{22}(x,t), \quad (x,t) \in [0,h] \times [0,T]. \tag{4}$$

With the only above data this problem is underdetermined and we are forced to impose an additional condition to determine $a(t)$. In particular, we shall take a nonlocal overdetermination condition, that arises in practical applications [15]:

$$\nu_1(t)u(0,y_0,t) + \nu_2(t)u(h,y_0,t) = \mu_3(t), \quad t \in [0,T], \tag{5}$$

where y_0 is a fixed number from $[0, l]$.

In the past few decades a great deal of interest has been directed towards the coefficient inverse problems. In 1993 Ivanchov M. considered nonlocal inverse problems of determining a leading time-dependent coefficient in a 1D heat equation [8, 9, 10]. For parabolic equations in one space variable, Bereznytska I. [1] considered the problem of determining conductivity $a(t)$ in a general parabolic equation subject to the Neumann boundary data and nonlocal overdetermination condition. Analogous problem with the Dirichlet boundary data was investigated in [12]. Later Huzyk N. investigated the problem of identifying time-dependent coefficients in a degenerate parabolic equation also subjected to Neumann boundary data and nonlocal overdetermination condition [5], [6]. All these papers are united by the approach utilized to proof the existence of solution: the inverse problem is reformulated as a fixed point problem for a certain nonlinear map, so that the Schauder theorem can be applied to it.

The other approaches to this problem addressing the question of existence and uniqueness are the Fourier method utilized by Ismailov M.I., Kanca F. [11], Oussaeif T.-E., Bouziani A. [16] and the theory of reproducing kernels used by Mohammadi M., Mokhtari R. and Isfahani F.T. [14].

The numerical results to nonlocal inverse problems have been obtained in works of Lesnic D. *et al* [13] with the help of Ritz-Galerkin method. A numerical marching scheme based on the discrete mollification for the recovery of the diffusivity coefficient in the two-dimensional inverse heat conduction problem has been presented by Coles C., Murio D.A. [2, 3].

Since the satisfactory results to the nonlocal coefficient inverse problems were successfully obtained in one-dimensional case, this paper represents an attempt to extend these results to multidimensional case, which is more interesting for its applications.

1   Notations and assumptions

Let $G_k(x, t, \xi, \tau)$ be the Green function of a 1D problem for the equation $u_t = a(t)u_{xx}$ with a Dirichlet boundary condition, when $k = 1$, Neumann bondary condition, when $k = 2$. These functions are defined by the equality

$$G_k(x, t, \xi, \tau) = \frac{1}{2\sqrt{\pi(\theta(t) - \theta(\tau))}} \sum_{n=-\infty}^{+\infty} \left(\exp\left(-\frac{(x - \xi + 2nh)^2}{4(\theta(t) - \theta(\tau))}\right) \right.$$
$$\left. + (-1)^k \exp\left(-\frac{(x + \xi + 2nh)^2}{4(\theta(t) - \theta(\tau))}\right) \right), \quad k = 1, 2, \quad \theta(t) = \int_0^t a(\tau)d\tau. \tag{6}$$

At the same time we define the function $G_m(y, t, \eta, \tau)$ analogously to $G_k(x, t, \xi, \tau)$.

Now, let us introduce the 2D heat equation

$$u_t = a(t)\Delta u + f(x, y, t), \quad (x, y, t) \in Q_T. \tag{7}$$

Green functions for (7) are determined as follows

$$G_{km}(x, y, t, \xi, \eta, \tau) = G_k(x, t, \xi, \tau)G_m(y, t, \eta, \tau), \quad k, m = 1, 2. \tag{8}$$

The Green function of the problem (7), (2)-(4) is defined by (8), when $k = m = 2$.

For $\alpha \in (0,1)$ we denote

$$C^{\alpha,0}(\overline{Q}_T) := \{f \in C(\overline{Q}_T) | \, |f(x_2, y_2, t) - f(x_1, y_1, t)| \leqslant C(|x_2 - x_1|^\alpha + |y_2 - y_1|^\alpha),$$
$$(x_i, y_i, t) \in \overline{Q}_T, \, i = 1, 2\}.$$

Throughout this paper, we assume that:

(A1) $f \in C^{\alpha,0}(\overline{Q}_T)$, $\quad b_1, b_2, c \in C^{1,0}(\overline{Q}_T)$, $\quad \varphi \in C^2([0,h] \times [0,l])$, $\quad \mu_3, \nu_1, \nu_2 \in C^1([0,T])$,
$\mu_{11}, \mu_{12} \in C^{2,1}([0,l] \times [0,T])$, $\quad \mu_{21}, \mu_{22} \in C^{2,1}([0,h] \times [0,T])$;

(A2) $\mu_3'(t) - \nu_1(t)b_1(0, y_0, t)\mu_{11}(y_0, t) - \nu_2(t)b_1(h, y_0, t)\mu_{12}(y_0, t) - \nu_1(t)f(0, y_0, t) - \nu_2(t)$
$\times f(h, y_0, t) > 0$, $\quad \nu_1'(t) + \nu_1(t)c(0, y_0, t) \leqslant 0$, $\quad \nu_2'(t) + \nu_2(t)c(h, y_0, t) \leqslant 0$,
$\nu_k(t) \geqslant 0, k = 1, 2$, $\quad b_2(0, y_0, t) \leqslant 0, b_2(h, y_0, t) \leqslant 0$, $\quad t \in [0, T]$, $\quad \varphi(x, y) \geqslant 0$,
$\varphi_y(x, y) \geqslant 0, (x, y) \in [0, h] \times [0, l]$, $\quad \mu_{21}(x, t) \geqslant 0, \mu_{22}(x, t) \geqslant 0$, $\quad (x, t) \in [0, h] \times [0, T]$;

(A3) $\nu_1(t) + \nu_2(t) > 0$, $\quad t \in [0, T]$, $\quad \Delta\varphi(x, y) > 0, (x, y) \in [0, h] \times [0, l]$;

(A4) $\varphi_x(0, y) = \mu_{11}(y, 0)$, $\varphi_x(h, y) = \mu_{12}(y, 0), y \in [0, l]$, $\quad \varphi_y(x, 0) = \mu_{21}(x, 0)$, $\varphi_y(x, h)$
$= \mu_{22}(x, 0), x \in [0, h]$, $\quad \nu_1(0)\varphi(0, y_0) + \nu_2(0)\varphi(h, y_0) = \mu_3(0)$.

2 EXISTENCE OF A SOLUTION

Theorem 1. *Provided that **(A1)–(A4)** hold, the problem (1)–(5) has at least one solution $(a, u) \in C([0, t^*]) \times C^{2,1}(\overline{Q}_{t^*})$, $a(t) > 0, t \in [0, t^*]$, where $t^* \in (0, T]$ is determined from the input data.*

Proof. To proof the existence of the solution to (1)-(5) we are first going to reduce it to an equivalent in a certain sense operator equation with respect to a and afterwards to proof the existence of the operator equation solution by the Schauder fixed point theorem.

In order to obtain an equation with respect to $a(t)$, (1) is applied to the overdetermination condition (5) previously differentiated:

$$a(t) = [\mu_3'(t) - \nu_1(t)b_1(0, y_0, t)\mu_{11}(y_0, t) - \nu_2(t)b_1(h, y_0, t)\mu_{12}(y_0, t) - \nu_1(t)$$
$$\times f(0, y_0, t) - \nu_2(t)f(h, y_0, t) - (\nu_1'(t) + \nu_1(t)c(0, y_0, t))u(0, y_0, t) - (\nu_2'(t)$$
$$+ \nu_2(t)c(h, y_0, t))u(h, y_0, t) - \nu_1(t)b_2(0, y_0, t)u_y(0, y_0, t) - \nu_2(t)b_2(h, y_0, t)$$
$$\times u_y(h, y_0, t)][\nu_1(t)\Delta u(0, y_0, t) + \nu_2(t)\Delta u(h, y_0, t)]^{-1}, \quad t \in [0, T].$$

To continue the investigation of the equation (9), it is necessary to get some representation of the terms $u(0, y_0, t)$, $u(h, y_0, t)$, $u_y(0, y_0, t)$, $u_y(h, y_0, t)$, $\Delta u(0, y_0, t)$, $\Delta u(h, y_0, t)$.

The solution to the problem (7), (2)–(4) is denoted as $u_0(x, y, t)$ under the temporary assumption that $a \in C([0, T])$, $a(t) > 0, t \in [0, T]$ is a known function. Therefore, taking advan-

tage of (8) we represent u_0 as the solution to (7), (2)–(4)

$$u_0(x,y,t) = \int\limits_0^l \int\limits_0^h G_{22}(x,y,t,\xi,\eta,0)\varphi(\xi,\eta)d\xi d\eta - \int\limits_0^t \int\limits_0^h G_{22}(x,y,t,\xi,0,\tau)a(\tau)$$

$$\times \mu_{21}(\xi,\tau)d\xi d\tau + \int\limits_0^t \int\limits_0^h G_{22}(x,y,t,\xi,l,\tau)a(\tau)\mu_{22}(\xi,\tau)d\xi d\tau$$

$$-\int\limits_0^t \int\limits_0^l G_{22}(x,y,t,0,\eta,\tau)a(\tau)\mu_{11}(\eta,\tau)d\eta d\tau + \int\limits_0^t \int\limits_0^l G_{22}(x,y,t,h,\eta,\tau)a(\tau)$$

$$\times \mu_{12}(\eta,\tau)d\eta d\tau + \int\limits_0^t \int\limits_0^l \int\limits_0^h G_{22}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t)\in \overline{Q}_T.$$

(9)

Denote by

$$v(x,y,t) := (b_1 u_x + b_2 u_y + cu)(x,y,t),$$
$$w_1(x,y,t) := v_x(x,y,t) = (b_1 u_{xx} + b_2 u_{xy} + b_{2x} u_y + (b_{1x} + c)u_x + c_x u)(x,y,t),$$
$$w_2(x,y,t) := v_y(x,y,t) = (b_1 u_{xy} + b_2 u_{yy} + (b_{2y} + c)u_y + b_{1y} u_x + c_y u)(x,y,t),$$
$$(x,y,t)\in \overline{Q}_T.$$

(10)

Problem (1)-(4) is reduced to the equation

$$u(x,y,t) = u_0(x,y,t) + \int\limits_0^t \int\limits_0^l \int\limits_0^h G_{22}(x,y,t,\xi,\eta,\tau)v(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t)\in \overline{Q}_T.$$

(11)

Thus, from (11) we obtain

$$v(x,y,t) = (b_1 u_{0x} + b_2 u_{0y} + cu_0)(x,y,t) + \int\limits_0^t \int\limits_0^l \int\limits_0^h (b_1(x,y,t)G_{22x}(x,y,t,\xi,\eta,\tau)$$

$$+ b_2(x,y,t)G_{22y}(x,y,t,\xi,\eta,\tau) + c(x,y,t)G_{22}(x,y,t,\xi,\eta,\tau))v(\xi,\eta,\tau)d\xi d\eta d\tau,$$
$$(x,y,t)\in \overline{Q}_T.$$

(12)

By differentiating (12) with respect to x, applying the Green function properties and integration by parts we obtain the equation

$$w_1(x,y,t) = (b_1 u_{0xx} + b_2 u_{0xy} + b_{2x} u_{0y} + (b_{1x} + c)u_{0x} + c_x u_0)(x,y,t)$$

$$+ \int\limits_0^t \int\limits_0^l \int\limits_0^h (b_{1x}(x,y,t)G_{22x}(x,y,t,\xi,\eta,\tau) + b_{2x}(x,y,t)G_{22y}(x,y,t,\xi,\eta,\tau)$$

$$+ c_x(x,y,t)G_{22}(x,y,t,\xi,\eta,\tau))v(\xi,\eta,\tau)d\xi d\eta d\tau + \int\limits_0^t \int\limits_0^l \int\limits_0^h (b_1(x,y,t)$$

$$\times G_{12x}(x,y,t,\xi,\eta,\tau) + b_2(x,y,t)G_{12y}(x,y,t,\xi,\eta,\tau) + c(x,y,t)$$

$$\times G_{12}(x,y,t,\xi,\eta,\tau))w_1(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t)\in \overline{Q}_T.$$

(13)

Analogously to (13), by differentiating (12) with respect to y, we obtain

$$
\begin{aligned}
w_2(x,y,t) &= (b_1 u_{0xy} + b_2 u_{0yy} + (b_{2y} + c)u_{0y} + b_{1y}u_{0x} + c_y u_0)(x,y,t) \\
&+ \int_0^t \int_0^l \int_0^h (b_{1y}(x,y,t)G_{22x}(x,y,t,\xi,\eta,\tau) + b_{2y}(x,y,t)G_{22y}(x,y,t,\xi,\eta,\tau) \\
&+ c_y(x,y,t)G_{22}(x,y,t,\xi,\eta,\tau))v(\xi,\eta,\tau)d\xi d\eta d\tau + \int_0^t \int_0^l \int_0^h (b_1(x,y,t) \\
&\times G_{21x}(x,y,t,\xi,\eta,\tau) + b_2(x,y,t)G_{21y}(x,y,t,\xi,\eta,\tau) + c(x,y,t) \\
&\times G_{21}(x,y,t,\xi,\eta,\tau))w_2(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t) \in \overline{Q}_T.
\end{aligned}
\tag{14}
$$

We find from (11)

$$
u_y(x,y,t) = u_{0y}(x,y,t) + \int_0^t \int_0^l \int_0^h G_{22y}(x,y,t,\xi,\eta,\tau)v(\xi,\eta,\tau)d\xi d\eta d\tau,
\tag{15}
$$

$$
\begin{aligned}
\Delta u(x,y,t) &= \Delta u_0(x,y,t) + \int_0^t \int_0^l \int_0^h G_{12x}(x,y,t,\xi,\eta,\tau)w_1(\xi,\eta,\tau)d\xi d\eta d\tau \\
&+ \int_0^t \int_0^l \int_0^h G_{21y}(x,y,t,\xi,\eta,\tau)w_2(\xi,\eta,\tau)d\xi d\eta d\tau, \quad (x,y,t) \in \overline{Q}_T,
\end{aligned}
\tag{16}
$$

where $u_{0y}, \Delta u_0$ are calculated from (9):

$$
\begin{aligned}
u_{0y}(x,y,t) &= \int_0^l \int_0^h G_{21}(x,y,t,\xi,\eta,0)\varphi_\eta(\xi,\eta)d\xi d\eta + \int_0^t \int_0^h G_{21\eta}(x,y,t,\xi,0,\tau)a(\tau) \\
&\times \mu_{21}(\xi,\tau)d\xi d\tau - \int_0^t \int_0^h G_{21\eta}(x,y,t,\xi,l,\tau)a(\tau)\mu_{22}(\xi,\tau)d\xi d\tau \\
&- \int_0^t \int_0^l G_{21}(x,y,t,0,\eta,\tau)a(\tau)\mu_{11\eta}(\eta,\tau)d\eta d\tau + \int_0^t \int_0^l G_{21}(x,y,t,h,\eta,\tau)a(\tau) \\
&\times \mu_{12\eta}(\eta,\tau)d\eta d\tau + \int_0^t \int_0^l \int_0^h G_{22y}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta d\tau,
\end{aligned}
\tag{17}
$$

$$\Delta u_0(x,y,t) = \int_0^l \int_0^h G_{22}(x,y,t,\xi,\eta,0)\Delta\varphi(\xi,\eta)d\xi d\eta - \int_0^t \int_0^h G_{22}(x,y,t,\xi,0,\tau)$$

$$\times \mu_{21\tau}(\xi,\tau)d\xi d\tau + \int_0^t \int_0^h G_{22}(x,y,t,\xi,l,\tau)\mu_{22\tau}(\xi,\tau)d\xi d\tau$$

$$-\int_0^t \int_0^l G_{22}(x,y,t,0,\eta,\tau)\mu_{11\tau}(\eta,\tau)d\eta d\tau + \int_0^t \int_0^l G_{22}(x,y,t,h,\eta,\tau)$$ (18)

$$\times \mu_{12\tau}(\eta,\tau)d\eta d\tau + \int_0^t d\tau \int_0^l \int_0^h \Delta G_{22}(x,y,t,\xi,\eta,\tau)f(\xi,\eta,\tau)d\xi d\eta, \quad (x,y,t) \in \overline{Q}_T.$$

By substituting (11), (16), (15) into (9) we obtain:

$$a(t) = \frac{Q_1(a,v)(t)}{Q_2(a,w_1,w_2)(t)}, \tag{19}$$

where

$$
\begin{aligned}
Q_1(a,v)(t) = {}& \mu_3'(t) - \nu_1(t)b_1(0,y_0,t)\mu_{11}(y_0,t) - \nu_2(t)b_1(h,y_0,t)\mu_{12}(y_0,t) - \nu_1(t) \\
& \times f(0,y_0,t) - \nu_2(t)f(h,y_0,t) - (\nu_1'(t) + \nu_1(t)c(0,y_0,t))u_0(0,y_0,t) - (\nu_2'(t) \\
& + \nu_2(t)c(h,y_0,t))u_0(h,y_0,t) - \nu_1(t)b_2(0,y_0,t)u_{0y}(0,y_0,t) - \nu_2(t)b_2(h,y_0,t) \\
& \times u_{0y}(h,y_0,t) + \int_0^t \int_0^l \int_0^h v(\xi,\eta,\tau)(-(\nu_1'(t) + \nu_1(t)c(0,y_0,t))G_{22}(0,y_0,t,\xi,\eta,\tau) \\
& - (\nu_2'(t) + \nu_2(t)c(h,y_0,t))G_{22}(h,y_0,t,\xi,\eta,\tau) - \nu_1(t)b_2(0,y_0,t) \\
& \times G_{22y}(0,y_0,t,\xi,\eta,\tau) - \nu_2(t)b_2(h,y_0,t)G_{22y}(h,y_0,t,\xi,\eta,\tau))d\xi d\eta d\tau,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
Q_2(a,w_1,w_2)(t) = {}& \nu_1(t)\Delta u_0(0,y_0,t) + \nu_2(t)\Delta u_0(h,y_0,t) \\
& + \int_0^t \int_0^l \int_0^h (\nu_1(t)G_{12x}(0,y_0,t,\xi,\eta,\tau) + \nu_2(t)G_{12x}(h,y_0,t,\xi,\eta,\tau))w_1(\xi,\eta,\tau)d\xi d\eta d\tau \\
& + \int_0^t \int_0^l \int_0^h (\nu_1(t)G_{21y}(0,y_0,t,\xi,\eta,\tau) + \nu_2(t)G_{21y}(h,y_0,t,\xi,\eta,\tau))w_2(\xi,\eta,\tau)d\xi d\eta d\tau,
\end{aligned}
\tag{21}
$$

where v, w_1, w_2 are solutions to the system of integral equations (12)–(14).

Denote

- $\mathcal{N} := \{a \in C([0,t^*]) : A_0 \leqslant a(t) \leqslant A_1\}$, where the constants $A_0, A_1 \in \mathbb{R}_+$, $t^* \in (0,T]$ are to be established below;

- $\widehat{P} : \mathcal{N} \times \left(C(\overline{Q}_T)\right)^3 \to \mathcal{N}$, such that $\widehat{P}(a,v,w_1,w_2) = \dfrac{Q_1(a,v)}{Q_2(a,w_1,w_2)}$;

- $\widetilde{P} : \mathcal{N} \to \left(C(\overline{Q}_T)\right)^3$ an operator that maps each element $a \in \mathcal{N}$ into the solution of the system of integral equations (12)–(14).

Since the functions v, w_1, w_2 in (19) are now defined by $\widetilde{P}$, the equation (19) can be rewritten as the following operator equation:

$$a = Pa, \text{ where } Pa := \widehat{P}(a, \widetilde{P}(a)), \quad a \in \mathcal{N}. \tag{22}$$

The problem (1)–(5) is equivalent to the equation (22) in the following sense: if (a, u) is a solution to problem (1)–(5), then a is a solution of (22) and, on the other hand, if $a \in C([0, T])$ is a solution of (22), then (a, u) is a solution to the problem (1)–(5), where u is determined by the equations (11).

From the way the equation (22) has been obtained it follows, that if (a, u) is the solution to (1)–(5), then a satisfies (22).

Reciprocally, for any $a \in \mathcal{N}$ functions u, v are uniquely determined from (11), (12) and such a system of integral equations is equivalent to the direct problem (1)–(4). Thus, it is left to be shown that (5) follows from (22). By implementing all the substitutions in the reverse order we move from (22) to (9). After (9) is multiplied by its denominator and integrated with respect to time, regarding **(A4)**, the overdetermination condition (5) is obtained.

Consequently, the existence of solution to (1)—(5) is equivalent to the existence of solution to the operator equation (22).

In order to apply the Schauder fixed point theorem we show that P is compact and that it maps $\mathcal{N}$ into itself.

Since for each $a \in \mathcal{N}$ $u_{0x}, u_{0y}, u_{0xx}, u_{0xy}, u_{0yy}$ are continuous functions according to **(A1)**, it follows from the properties of the systems of Volterra integral equations that $\widetilde{P}$ is a bounded operator. The compactness of the operator $\widehat{P}$ follows from [7]. Therefore P is compact as the composition of bounded operator $\widetilde{P}$ and compact operator $\widehat{P}$.

Thus, the next goal is to establish $A_0, A_1 \in \mathbb{R}_+$, such that $A_0 \leqslant (Pa)(t) \leqslant A_1$, $t \in [0, t^*]$, $a \in \mathcal{N}$.

From the explicit representation of u_0 and its derivative u_{0y} (9), (17), the Green function properties and **(A2)** it follows that

$$\lim_{t \to 0} u_0(x, y, t) = \varphi(x, y),$$

$$\lim_{t \to 0} u_{0y}(x, y, t) = \lim_{t \to 0} \left(\int_0^l G_1(y, t, \eta, 0) \varphi_\eta(x, \eta) d\eta + \int_0^t G_{1\eta}(y, t, 0, \tau) a(\tau) \mu_{21}(x, \tau) d\tau \right.$$

$$\left. - \int_0^t G_{1\eta}(y, t, l, \tau) a(\tau) \mu_{22}(x, \tau) d\tau \right).$$

Then for any $(x, y) \in [0, h] \times [0, l]$

$$0 \leqslant \min_{[0,h] \times [0,l]} \varphi(x, y) \leqslant \lim_{t \to 0} u_0(x, y, t) \leqslant \max_{[0,h] \times [0,l]} \varphi(x, y),$$

$$0 \leqslant \min\{ \min_{[0,h] \times [0,l]} \varphi_y(x, y), \min_{[0,h] \times [0,T]} \mu_{21}(x, t), \min_{[0,h] \times [0,T]} \mu_{22}(x, t) \} \leqslant \lim_{t \to 0} u_{0y}(x, y, t)$$

$$\leqslant \max\{ \max_{[0,h] \times [0,l]} \varphi_y(x, y), \max_{[0,h] \times [0,T]} \mu_{21}(x, t), \max_{[0,h] \times [0,T]} \mu_{22}(x, t) \}.$$

The last term in (20) vanishes, when $t \to 0$, according to the properties of Newtonian potentials.

Therefore, thanks to **(A2)** there are such constants m_1, M_1 that

$$0 < m_1 \leqslant \lim_{t \to 0} Q_1(t) \leqslant M_1.$$

Namely,

$$
\begin{aligned}
m_1 :=\ &\min_{[0,T]}(\mu_3'(t) - v_1(t)b_1(0, y_0, t)\mu_{11}(y_0, t) - v_2(t)b_1(h, y_0, t)\mu_{12}(y_0, t) - v_1(t) \\
&\times f(0, y_0, t) - v_2(t)f(h, y_0, t)),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
M_1 :=\ &\max_{[0,T]}(\mu_3'(t) - v_1(t)b_1(0, y_0, t)\mu_{11}(y_0, t) - v_2(t)b_1(h, y_0, t)\mu_{12}(y_0, t) - v_1(t) \\
&\times f(0, y_0, t) - v_2(t)f(h, y_0, t)) + \max_{[0,T]}(-(v_1'(t) + v_1(t)c(0, y_0, t)) - (v_2'(t) + v_2(t) \\
&\times c(h, y_0, t))) \max_{[0,h] \times [0,l]} \varphi(x, y) + \max_{[0,T]}(-v_1(t)b_2(0, y_0, t) - v_2(t)b_2(h, y_0, t)) \\
&\times \max\{\max_{[0,h] \times [0,l]} \varphi_y(x, y), \max_{[0,h] \times [0,T]} \mu_{21}(x, t), \max_{[0,h] \times [0,T]} \mu_{22}(x, t)\}.
\end{aligned}
\tag{24}
$$

Thus from the definition of limit it derives that for $\varepsilon = \frac{1}{2}m_1$ there is such a value $t_1 \in (0, T]$, that

$$\frac{1}{2}m_1 \leqslant Q_1(t) \leqslant M_1 + \frac{1}{2}m_1, \quad t \in [0, t_1]. \tag{25}$$

Similarly, from the explicit representation (18) of Δu_0

$$\lim_{t \to 0} \Delta u_0(x, y, t) = \Delta \varphi(x, y).$$

Denote

$$m_2 := \min_{[0,T]}(v_1(t) + v_2(t)) \min_{[0,h] \times [0,l]} \Delta \varphi(x, y), \tag{26}$$

$$M_2 := \max_{[0,T]}(v_1(t) + v_2(t)) \max_{[0,h] \times [0,l]} \Delta \varphi(x, y). \tag{27}$$

Then $0 < m_2 \leqslant \lim_{t \to 0} Q_2(t) \leqslant M_2$. Analogously, there is such a value $t_2 \in (0, T]$, that

$$\frac{1}{2}m_2 \leqslant Q_2(t) \leqslant M_2 + \frac{1}{2}m_2, \quad t \in [0, t_2]. \tag{28}$$

Define

$$A_0 := \frac{\frac{1}{2}m_1}{M_2 + \frac{1}{2}m_2}, \quad A_1 := \frac{M_1 + \frac{1}{2}m_1}{\frac{1}{2}m_2}, \quad t^* := \min\{t_1, t_2\}.$$

and make sure that: if $a \in \mathcal{N}$, then $A_0 \leqslant (Pa)(t) \leqslant A_1$, $t \in [0, t^*]$.

From the Schauder fixed point theorem follows the existence of the solution to (22), and, hence, for the problem (1)–(5). $\qquad \square$

3   UNIQUENESS OF A SOLUTION

Theorem 2. *Under the condition (A2) the problem* (1)–(5) *cannot have more than one solution* (a, u) *in the space* $C([0, t_1]) \times C^{2,1}(\overline{Q}_{t_1})$, *such that* $\Delta u \in C^{\alpha,0}(\overline{Q}_{t_1})$ *and* $a(t) > 0, t \in [0, t_1]$, *where* $t_1 \in (0, T]$ *is determined from the input data.*

Proof. Suppose that there exist two solutions $(a_1(t), u_1(x, y, t))$ and $(a_2(t), u_2(x, y, t))$ of the problem (1)–(5). Denote

$$a_3(t) := a_1(t) - a_2(t), \quad t \in [0, T], \tag{29}$$

$$u_3(x, y, t) := u_1(x, y, t) - u_2(x, y, t), \quad (x, y, t) \in \overline{Q}_T. \tag{30}$$

Then $(a_3(t), u_3(x, y, t))$ is solution of the problem

$$u_{3t} = a_1(t)\Delta u_3 + b_1(x, y, t)u_{3x} + b_2(x, y, t)u_{3y} + c(x, y, t)u_3 + a_3(t)\Delta u_2, \quad (x, y, t) \in Q_T, \tag{31}$$

$$u_3(x, y, 0) = 0, \quad (x, y) \in [0, h] \times [0, l], \tag{32}$$

$$u_{3x}(0, y, t) = 0, \quad u_{3x}(h, y, t) = 0, \quad (y, t) \in [0, l] \times [0, T], \tag{33}$$

$$u_{3y}(x, 0, t) = 0, \quad u_{3y}(x, l, t) = 0, \quad (x, t) \in [0, h] \times [0, T], \tag{34}$$

$$v_1(t)u_3(0, y_0, t) + v_2(t)u_3(h, y_0, t) = 0, \quad t \in [0, T]. \tag{35}$$

By calculating the derivative of (35) and applying (31) to it, we obtain for $t \in [0, T]$

$$\begin{aligned}
(v_1(t)\Delta u_2(0, y_0, t) + v_2(t)\Delta u_2(h, y_0, t))a_3(t) &= -(v_1'(t) + v_1(t)c(0, y_0, t)) \\
&\times u_3(0, y_0, t) - (v_2'(t) + v_2(t)c(h, y_0, t))u_3(h, y_0, t) - v_1(t)b_2(0, y_0, t)u_{3y}(0, y_0, t) \\
&- v_2(t)b_2(h, y_0, t)u_{3y}(h, y_0, t) - v_1(t)a_1(t)\Delta u_3(0, y_0, t) - v_2(t)a_1(t)\Delta u_3(h, y_0, t).
\end{aligned} \tag{36}$$

Denote by $\hat{G}_{22}(x, y, t, \xi, \eta, \tau)$ a Green function of the problem (31)–(34). Since $a_1(t)$ is a known function, the solution to the problem (31)–(34) is unique and can be calculated by the formula:

$$u_3(x, y, t) = \int_0^t \int_0^l \int_0^h \hat{G}_{22}(x, y, t, \xi, \eta, \tau)a_3(\tau)\Delta u_2(\xi, \eta, \tau)d\xi d\eta d\tau. \tag{37}$$

By differentiating (37) with respect to y and applying to (37) the Laplacian , we obtain

$$u_{3y}(x, y, t) = \int_0^t \int_0^l \int_0^h \hat{G}_{22y}(x, y, t, \xi, \eta, \tau)a_3(\tau)\Delta u_2(\xi, \eta, \tau)d\xi d\eta d\tau, \tag{38}$$

$$\Delta u_3(x, y, t) = \int_0^t d\tau \int_0^l \int_0^h \Delta \hat{G}_{22}(x, y, t, \xi, \eta, \tau)a_3(\tau)\Delta u_2(\xi, \eta, \tau)d\xi d\eta. \tag{39}$$

Therefore, by applying (37)–(39) to (36), we obtain an equation with respect to $a_3(t)$

$$\begin{aligned}
a_3(t) = \frac{-1}{v_1(t)\Delta u_2(0, y_0, t) + v_2(t)\Delta u_2(h, y_0, t)} \int_0^t d\tau \int_0^l \int_0^h \Bigg((v_1'(t) + v_1(t)c(0, y_0, t)) \\
\times \hat{G}_{22}(0, y_0, t, \xi, \eta, \tau) + (v_2'(t) + v_2(t)c(h, y_0, t))\hat{G}_{22}(h, y_0, t, \xi, \eta, \tau) \\
+ v_1(t)b_2(0, y_0, t)\hat{G}_{22y}(0, y_0, t, \xi, \eta, \tau) + v_2(t)b_2(h, y_0, t)\hat{G}_{22y}(h, y_0, t, \xi, \eta, \tau) \\
+ v_1(t)a_1(t)\Delta \hat{G}_{22}(0, y_0, t, \xi, \eta, \tau) + v_2(t)a_1(t)\Delta \hat{G}_{22}(h, y_0, t, \xi, \eta, \tau) \Bigg) \\
\times a_3(\tau)\Delta u_2(\xi, \eta, \tau)d\xi d\eta.
\end{aligned} \tag{40}$$

It is still necessary to ensure that for

$$\nu_1(t)\Delta u_2(0, y_0, t) + \nu_2(t)\Delta u_2(h, y_0, t) > 0. \tag{41}$$

Since (a_2, u_2) is a solution of (1)–(5) it follows from (9) that $t \in [0, T]$

$$
\begin{aligned}
\nu_1(t)\Delta u_2(0, y_0, t) + \nu_2(t)\Delta u_2(h, y_0, t) &= \frac{1}{a_2(t)}(\mu_3'(t) - \nu_1(t)f(0, y_0, t) - \nu_2(t) \\
&\times f(h, y_0, t) - (\nu_1'(t) + \nu_1(t)c(0, y_0, t))u_2(0, y_0, t) - (\nu_2'(t) + \nu_2(t)c(h, y_0, t)) \\
&\times u_2(h, y_0, t) - \nu_1(t)b_2(0, y_0, t)u_{2y}(0, y_0, t) - \nu_2(t)b_2(h, y_0, t)u_{2y}(h, y_0, t)).
\end{aligned}
\tag{42}
$$

Thus, it follows from (42), (20) and (25), ensured by **(A2)**, that

$$\nu_1(t)\Delta u_2(0, y_0, t) + \nu_2(t)\Delta u_2(h, y_0, t) \geqslant \frac{m_1}{2a_2(t)} > 0, \quad t \in [0, t_1]. \tag{43}$$

Hence, (40) is a homogeneous Volterra integral equation of the second kind on $[0, t_1]$. Since $\Delta u_2 \in C^{\alpha, 0}(\overline{Q}_{t_1})$, according to [4] the kernel of (40) is integrable. Therefore, (40) has a unique solution $a_3(t) = 0$, $t \in [0, t_1]$, and from the equality (37) it follows that $u_3(x, y, t) = 0$, $(x, y, t) \in \overline{Q}_{t_1}$. The proof of the theorem is complete. $\qquad\square$

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Кінаш Н.Є. *Обернена задача для двовимірного параболічного рівняння із нелокальними умовами перевизначення* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 107–117.

Розглядаємо обернену задачу визначення залежного від часу коефіцієнта $a(t)$ у двовимірному параболічному рівнянні:

$$u_t = a(t)\Delta u + b_1(x,y,t)u_x + b_2(x,y,t)u_y + c(x,y,t)u + f(x,y,t),\ (x,y,t) \in Q_T,$$

із початковою умовою, крайовими умовами Неймана та нелокальною умовою перевизначення

$$\nu_1(t)u(0,y_0,t) + \nu_2(t)u(h,y_0,t) = \mu_3(t), \quad t \in [0,T],$$

де y_0 фіксоване значення із $[0,l]$.

Встановлено умови існування та єдиності класичного розв'язку задачі. З цією метою застосовано метод функції Гріна, теорему Шаудера про нерухому точку та теорію інтегральних рівнянь Вольтерра.

Ключові слова і фрази: обернена задача, визначення коефіцієнтів, параболічне рівняння, нелокальна умова перевизначення, прямокутна область.