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ON THE DIMENSION OF VERTEX LABELING OF k-UNIFORM DCSL OF k-UNIFORM CATERPILLAR

A distance compatible set labeling (dcsl) of a connected graph G is an injective set assignment $f : V(G) \to 2^X$, X being a nonempty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \to 2^X \setminus \{\varnothing\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $\mid f^{\oplus}(uv) \mid = k^{f}_{(u,v)} d_G(u,v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u,v)$ denotes the path distance between u and v and $k^{f}_{(u,v)}$ is a constant, not necessarily an integer. A dcsl f of G is k-uniform if all the constant of proportionality with respect to f are equal to k, and if G admits such a dcsl then G is called a k-uniform dcsl graph. The *k-uniform dcsl index* of a graph G, denoted by $\delta_k(G)$ is the minimum of the cardinalities of X, as X varies over all k-uniform dcsl-sets of G. A *linear extension* $\mathbf{L}$ of a partial order $\mathbf{P} = (P, \preceq)$ is a linear order on the elements of P, such that $x \preceq y$ in $\mathbf{P}$ implies $x \preceq y$ in $\mathbf{L}$, for all $x, y \in P$. The dimension of a poset $\mathbf{P}$, denoted by $dim(\mathbf{P})$, is the minimum number of linear extensions on $\mathbf{P}$ whose intersection is '$\preceq$'. In this paper we prove that $dim(\mathcal{F}) \leq \delta_k(P_n^{+k})$, where $\mathcal{F}$ is the range of a k-uniform dcsl of the k-uniform caterpillar, denoted by P_n^{+k} $(n \geq 1, k \geq 1)$ on '$n(k+1)$' vertices.

Key words and phrases: k-uniform dcsl index, dimension of a poset, lattice.

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INTRODUCTION

Acharya [1] introduced the notion of vertex *set-valuation* as a set-analogue of number valuation. For a graph $G = (V, E)$ and a nonempty set X, Acharya defined a *set-valuation* of G as an injective *set-valued* function $f : V(G) \to 2^X$, and defined a *set-indexer* $f^{\oplus} : E(G) \to 2^X \setminus \{\varnothing\}$ as a *set-valuation* such that the function given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all subsets of X and '$\oplus$' is the binary operation of taking the symmetric difference of subsets of X.

Acharya and Germina [2], introduced the particular kind of set-valuation for which a metric, especially the cardinality of the symmetric difference, associated with each pair of vertices is k (where k be a constant) times that of the distance between them in the graph [2]. In other words, determine those graphs $G = (V, E)$ that admit an injective set-valued function $f : V(G) \to 2^X$, where 2^X is the power set of a nonempty set X, such that, for each pair of distinct vertices u and v in G, the cardinality of the symmetric difference $f(u) \oplus f(v)$ is k times

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that of the usual path distance $d_G(u,v)$ between u and v in G, where k is a non-negative constant. They in [2] called such a *set-valuation* f of G a *k-uniform distance-compatible set-labeling* (*k-uniform dcsl*) of G, and the graph G which admits k-uniform dcsl, a *k-uniform distance-compatible set-labeled graph* (*k-uniform dcsl graph*) and the non empty set X corresponding to f, a *k-uniform dcsl-set*. The *k-uniform dcsl index* [4] of a graph G, denoted by $\delta_k(G)$ is the minimum of the cardinalities of X, as X varies over all k-uniform dcsl-sets of G.

Consider a *partially ordered set* or a *poset* $\mathbf{P}$ as a structure $(P, \preceq)$ where P is a nonempty set and '$\preceq$' is a partial order relation on P. We denote $(x,y) \in \mathbf{P}$ by $x \preceq y$, and identify the ground set of a poset with the whole poset. Two elements of $\mathbf{P}$ standing in the relation of $\mathbf{P}$ are called *comparable*, otherwise they are *incomparable*. We denote the incomparable elements x and y of $\mathbf{P}$ by $x \parallel y$. A poset is a *chain* if it contains no incomparable pair of elements, and in this case, the partial order is a *linear order*. A poset is an *antichain* if all of its pairs are incomparable. The length of a chain is one less than the number of elements in the chain. An element $p \in \mathbf{P}$ of a finite poset is on *level k*, if there exists a sequence of elements $p_0, p_1, \ldots, p_k = p$ in $\mathbf{P}$ such that $p_0 \preceq p_1 \preceq \ ldots \preceq p_k = p$ and any other such sequences in $\mathbf{P}$ has length less than or equal to k. The size of a largest chain in a poset $\mathbf{P}$ is called the *height* of the poset, denoted by $height(\mathbf{P})$ or $h(\mathbf{P})$, and that of a largest antichain is called its *width*, denoted by $width(\mathbf{P})$ or $w(\mathbf{P})$. A *Hasse diagram* of a poset $(P, \preceq)$ is a drawing in which the points of P are placed so that if y *covers* x (we say, z *covers* y if and only if $y \prec z$ and $y \preceq x \preceq z$ implies either $x = y$ or $x = z$), then y is placed at a higher level than x and joined to x by a line segment. A poset $\mathbf{P}$ is *connected*, if its Hasse diagram is connected as a graph. A *Cover graph or Hasse graph* of a poset $(P, \preceq)$ is the graph with vertex set P such that $x, y \in P$ are adjacent if and only if one of them covers the other.

Let $\mathbf{P} = (P, \preceq_P)$ and $\mathbf{Q} = (Q, \preceq_Q)$ be two partially ordered sets. A mapping f from the poset $\mathbf{P}$ to the poset $\mathbf{Q}$ is called *order preserving* if for every two elements x and y of P, $x \preceq_P y$ implies $f(x) \preceq_Q f(y)$. A poset $\mathbf{Q}$ is a subposet of $\mathbf{P}$ if $Q \subseteq P$, and $\preceq_Q$ is the restriction of $\preceq_P$ to $Q \times Q$. i.e., for $a, b \in Q$, $a \preceq_Q b$ if and only if $a \preceq_P b$. Two posets $\mathbf{P}$ and $\mathbf{Q}$ are called *isomorphic* if there is a one to one order preserving mapping Φ from the poset $\mathbf{P}$ onto the poset $\mathbf{Q}$ such that for every two elements x and y of P, $x \preceq_P y$ in $\mathbf{P}$ if and only if $\Phi(x) \preceq_Q \Phi(y)$ in $\mathbf{Q}$. The poset $\mathbf{Q}$ is said to be *embedded* or *contained* in $\mathbf{P}$, denoted by $\mathbf{Q} \sqsubseteq \mathbf{P}$, if $\mathbf{Q}$ is isomorphic to a subposet of $\mathbf{P}$. Let $\mathbf{R}$ and $\mathbf{S}$ are two partial orders (with respect to $\preceq$) on the same set X, we call $\mathbf{S}$ an *extension* of $\mathbf{R}$ if $\mathbf{R} \subseteq \mathbf{S}$, i.e., $x \preceq y$ in $\mathbf{R}$ implies $x \preceq y$ in $\mathbf{S}$ for all $x, y \in X$. In particular if $\mathbf{S}$ is a chain, then we call it as a *linear extension* of $\mathbf{R}$. For convenience, let $\mathbf{L} : [x_1, x_2, \ldots, x_n]$ denote linear order on $\{x_1, x_2, \ldots, x_n\}$ in which $x_1 \preceq x_2 \preceq \cdots \preceq x_n$.

Definition 1 ([8]). *A set $\mathcal{R} = \{\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_k\}$ of linear extensions of $\mathbf{P}$ is a **realizer** of $\mathbf{P}$ if for every incomparable pair $x, y \in \mathbf{P}$, there are $\mathbf{L}_i, \mathbf{L}_j \in \mathcal{R}$ with $x \preceq y$ in $\mathbf{L}_i$ and $x \succeq y$ in $\mathbf{L}_j$ for $1 \leq i \neq j \leq k$. The **dimension** of $\mathbf{P}$ (denoted by $dim(\mathbf{P})$) is the minimum cardinality of a realizer.*

There are equivalent definitions for $dim(\mathbf{P})$. It is defined as the minimum k for which there are linear extensions $\mathbf{L}_1, \ldots, \mathbf{L}_k$ such that $\mathbf{P} = \mathbf{L}_1 \cap \mathbf{L}_2 \cap \cdots \cap \mathbf{L}_k$, where the intersection is taken over the sets of relations of $\mathbf{L}_i$, for $1 \leq i \leq k$. Another characterization of dimension, in terms of coordinates, is obtained by using an embedding of $\mathbf{P}$ into R^t (called t-dimensional poset) [11]. Let R^t denotes the poset of all t-tuples of real numbers, partially ordered by inequality in each coordinate: $(a_1, a_2, \ldots, a_t) \leq (b_1, b_2, \ldots, b_t)$ if and only if $a_i \leq b_i$, for $i = 1, 2, \ldots, t$. Then the dimension of a poset $\mathbf{P}$ is the minimum number t such that $\mathbf{P}$ is embedded in $\mathbf{R^t}$, denoted as $\mathbf{P} \sqsubseteq \mathbf{R^t}$. For more results on dimension of poset one may see [7, 9, 12, 13].

A poset $(L, \preceq)$ is a *lattice* if every pair of elements $x, y \in L$, has a *least upper bound* (*lub*), denoted by $x \vee y$ (called join), and a *greatest lower bound* (*glb*), denoted by $x \wedge y$ (called meet). In general, a lattice is denoted by $(L, \preceq)$. Throughout this paper lattice (and poset) means lattice (and poset) under set inclusion $\subseteq$. Unless otherwise mentioned, for all the terminology in graph theory and lattice theory, the reader is asked to refer, respectively [5, 6].

This paper initiates a study on the dimension of vertex labeling of k-uniform dcsl of k-uniform caterpillar, and prove that $dim(\mathcal{F}) \leq \delta_k(P_n^{+k})$, where $\mathcal{F}$ is the range of a k-uniform dcsl of the k-uniform caterpillar, denoted by P_n^{+k} $(n \geq 1, k \geq 1)$ on '$n(k+1)$' vertices that forms a poset under set inclusion $\subseteq$.

Following are the definitions and results used in this paper.

Definition 2 ([10]). *The height-2 poset H_n on $2n$ elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ is the poset of height two consisting of two antichains $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that $b_i \preceq a_j$ in H_n exactly if $i = j$, and $j = i - 1$.*

Proposition 1 ([10]). *For $n \geq 2$, $dim(H_n) = 2$.*

Proposition 2 ([10]). *Let $\mathcal{F}$ be the range of a vertex labeling of 1-uniform dcsl path $P_n (n > 2)$, which is embedded in H_n, then $dim(\mathcal{F}) = 2$.*

Definition 3 ([10]). *A width-2 poset W_n is the poset $(\{a_1, \ldots, a_n, b_1, \ldots, b_n\}, \preceq)$ of width two consisting of two chains $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that $a_{i-1} \preceq a_i$ for $2 \leq i \leq n$, $b_i \preceq b_{i+1}$ for $1 \leq i \leq n-1$, $a_1 \preceq b_i$ for $1 \leq i \leq n$, and for $2 \leq i \leq n$ and $1 \leq j \leq n$, $a_i \mid\mid b_j$.*

Proposition 3 ([10]). *For $n \geq 2$, $dim(W_n) = 2$.*

Proposition 4 ([10]). *Let $\mathcal{F}$ be the range of a vertex labeling of 1-uniform dcsl path $P_n (n > 2)$, which is embedded in W_n, then $dim(\mathcal{F}) = 2$.*

Lemma 1 ([3]). *$\delta_d(P_n) = n - 1$, for $n > 2$.*

Lemma 2 ([10]). *$\delta_k(P_n) = k(n-1)$, for $n > 2$.*

1 MAIN RESULTS

Since the existence of vertex labeling of 1-uniform dcsl graph is not unique, the problem of determining posets which embeds the vertex labeling of 1-uniform dcsl of k-uniform caterpillar is same as determining the existence of different vertex labels f of 1-uniform dcsl of k-uniform caterpillar whose corresponding range, say $\mathcal{F} = Range(f)$ forms a poset under set inclusion $\subseteq$. Thus, there is a one to one correspondence between the vertex labeling f of 1-uniform dcsl of k-uniform caterpillar and its corresponding poset $\mathcal{F}$. Thus, it is always possible to find a 1-uniform dcsl f of a graph G so that $\mathcal{F} = Range(f)$ forms a poset under set inclusion $\subseteq$. Hence, $\mathcal{F}$ contains the vertex labeling f of 1-uniform dcsl graph G as an embedding of itself. Hence, the problem of determining the 1-uniform dcsl vertex labeling f of a graph G is equivalent in determining the poset $\mathcal{F}$ which embeds the 1-uniform dcsl vertex labeling f of the same graph G.

Definition 4. *Let $\mathbf{P} = (\{a_1, \ldots, a_n\}, \preceq)$ be a poset. We define k-uniform extended poset or, simply, k-extended poset of $\mathbf{P}$, denoted by $\mathbf{P}^k$ as*

$$(\{a_1, a_1^1, a_1^2, \ldots, a_1^k, a_2, a_2^1, a_2^2, a_2^k, \ldots, a_n, a_n^1, a_n^2, \ldots, a_n^k\}, \preceq),$$

which is an extension of $\mathbf{P}$, and for $1 \leq i \leq n$, each $k(\geq 1)$ elements a_i^1, a_i^2, $\ldots$, a_i^k of $\mathbf{P}^k$ covers only a_i. We call $\mathbf{P}$ as an underline poset of $\mathbf{P}^k$.

Remark 1. *It is interesting to note the following in a k-extended posets.*

(i) If there exist any two distinct elements which belong to the same level in $\mathbf{P}^k$, then they are incomparable.

(ii) For each $k(\geq 1)$ elements a_i^1, a_i^2, $\ldots$, a_i^k of $\mathbf{P}^k$ covers only a_i, where $1 \leq i \leq n$. This implies that there exist no element in $\mathbf{P}^k$ that covers any one of the k elements a_i^1, a_i^2, $\ldots$, a_i^k. Hence, the k elements a_i^1, a_i^2, $\ldots$, a_i^k are maximal elements of $\mathbf{P}^k$. Thus, they are the nk maximal elements, namely, a_i^j in $\mathbf{P}^k$, $1 \leq i \leq n$ and $1 \leq j \leq k$.

Proposition 5. *For any poset $\mathbf{P}$ (finite and connected) of size greater than 1, the k-extended poset $\mathbf{P}^k(k \geq 1)$ of $\mathbf{P}$, does not form a lattice.*

Proof. If possible let, $\mathbf{P}^k$ forms a lattice, then $\mathbf{P}^k$ has unique glb and unique lub, say g and l respectively. Since l is the lub of $\mathbf{P}^k$, $x \preceq l$, for every $x \in \mathbf{P}^k$, which in turn implies one of the element from the maximal elements a_n^1, a_n^2, $\ldots$, a_n^k of $\mathbf{P}^k$ should be equal to l, say, a_n^1. Hence for $2 \leq i \leq n$, we have $a_n^i \preceq l$ which is a contradiction as remarked in Remark 1. □

Proposition 6. *Let $\mathbf{P}$ be a linear order as of the form: $a_{i-1} \preceq a_i$, for $2 \leq i \leq n$, then the dimension of k-extended poset $\mathbf{P}^k(k \geq 1)$ of $\mathbf{P}$ is 2.*

Proof. Let $\mathcal{R} = \{\mathbf{L}_1, \mathbf{L}_2\}$ be linear extensions of $\mathbf{P}^k$, where
$\mathbf{L}_1 : [a_1, a_1^1, \ldots, a_1^k, a_2, a_2^1, \ldots, a_2^k, \ldots, a_n, a_n^1, \ldots, a_n^k]$ and
$\mathbf{L}_2$: $[a_1, \ldots, a_n, a_n^k, \ldots, a_n^1, a_{n-1}^k, \ldots, a_{n-1}^1, \ldots, a_1^k, \ldots, a_1^1]$.

Then $\mathcal{R}$ is a realizer of $\mathbf{P}^k$, and hence $dim(\mathbf{P}^k) \leq 2$. We prove that there is no proper subset $\mathcal{S}$ of $\mathcal{R}$ which realizes $\mathbf{P}^k$. For, if there is a proper subset $\mathcal{S}$ of $\mathcal{R}$ which realizes $\mathbf{P}^k$, then, the only one member in $\mathcal{S}$ give rise to the poset $\mathbf{P}^k$, and hence, all the elements of $\mathbf{P}^k$ are comparable, which is a contradiction. Hence $dim(\mathbf{P}^k) = 2$. □

Since the graph P_n^{+k} is the extension of P_n, the k-extended poset can embed the vertex labeling of a 1-uniform dcsl k-uniform caterpillar only when its corresponding underline poset embed the vertex labeling of a 1-uniform dcsl path.

Now, we aim to determine the dimension of k-extended posets which embeds the vertex labeling of a 1-uniform dcsl of a k-uniform caterpillar.

Proposition 7. *Let $\mathbf{P}$ be a linear order as $a_{i-1} \preceq a_i$, for $2 \leq i \leq n$, then the k-extended poset $\mathbf{P}^k$ embeds the vertex labeling of a 1-uniform dcsl of the k-uniform caterpillar.*

Proof. Let $G = P_n^{+k}$ be the k-uniform caterpillar with $n(k+1)$ vertices, where $n \geq 2$ and $k \geq 1$. Let $V(G) = \{v_i, v_i^j \mid 1 \leq i \leq n, 1 \leq j \leq k\}$, where v_i are the internal vertices and v_i^j are the pendant vertices which are adjacent to v_i.

First we claim that there exist a vertex labeling f of a 1-uniform dcsl of the k-uniform caterpillar, whose range is suitable for the embedding of k-extended poset $\mathbf{P}^k$. Let $X = \{1, 2, \ldots, n(k+1) - 1\}$. Define $f : V(G) \to 2^X$ such that $f(v_1) = \varnothing$ and $f(v_j) = \{1, 2, \ldots, j-1\}, 2 \le j \le n$. For, $1 \le i \le n$ and $1 \le j \le k$,

$$f(v_i^j) = f(v_i) \cup \{(n-1) + (i-1)k + j\} = \{1, 2, \ldots, i-1, (n-1) + (i-1)k + j\}.$$

Case 1: When $u = v_l$ and $v = v_m$, $l = 1$ and $2 \le m \le n$. Then,

$$|f(v_l) \oplus f(v_m)| = |\varnothing \oplus \{1, 2, \ldots, m-1\}| = |\{1, 2, \ldots, m-1\}| = m - l = d(v_l, v_m).$$

Case 2: When $u = v_l$ and $v = v_m$, $l \ne m, 2 \le l,\ m \le n$. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m)| &= |\{1, 2, \ldots, l-1\} \oplus \{1, 2, \ldots, m-1\}| \\ &= |\{l, l+1, \ldots, m-1\}| = m - l = d(v_l, v_m), \quad 2 \le l < m \le n. \end{aligned}$$

Case 3: When $u = v_l$ and $v = v_m^j$, $l = 1, 2 \le m \le n$ and $1 \le j \le k$. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m^j)| &= |\varnothing \oplus \{1, 2, \ldots, m-1, (n-1) + (m-1)k + j\}| \\ &= |\{1, 2, \ldots, m-1, (n-1) + (m-1)k + j\}| = m = d(v_l, v_m^j). \end{aligned}$$

Case 4: When $u = v_l$ and $v = v_m^j$, $l \ne m, 2 \le l,\ m \le n$ and $1 \le j \le k$. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m^j)| &= |\{1, 2, \ldots, l-1\} \oplus \{1, 2, \ldots, m-1, (n-1) + (m-1)k + j\}| \\ &= |\{l, l+1, \ldots, m-1, (n-1) + (m-1)k + j\}| \\ &= m - l + 1 = d(v_l, v_m^j), \quad 2 \le l < m \le n \text{ and } 1 \le j \le k. \end{aligned}$$

Case 5: When $u = v_l^i$ and $v = v_m^j$, $l = 1, 2 \le m \le n$ and $1 \le i, j \le k$. Then,

$$\begin{aligned} |f(v_l^i) \oplus f(v_m^j)| &= |\{(n-1) + (l-1)k + i\} \\ &\quad \oplus \{1, \ldots, m-1, (n-1) + (m-1)k + j\}| \\ &= |\{1, \ldots, m-1, (n-1) + (m-1)k + j, (n-1) + (l-1)k + i\}| = m + 1 = d(v_l^i, v_m^j). \end{aligned}$$

Case 6: When $u = v_l^i$ and $v = v_m^j$, $l \ne m, 2 \le l,\ m \le n$ and $1 \le i, j \le k$. Then,

$$\begin{aligned} f(v_l^i) \oplus f(v_m^j)| &= |\{1, \ldots, l-1, (n-1) + (l-1)k + i\} \\ &\quad \oplus \{1, \ldots, m-1, (n-1) + (m-1)k + j\}| \\ &= |\{(n-1) + (l-1)k + i, l, l+1, ., m-1, (n-1) + (m-1)k + j\}| \\ &= m - l + 2 = d(v_l^i, v_m^j), \quad 2 \le l < m \le n \text{ and } 1 \le i \le j \le k. \end{aligned}$$

Hence, for any distinct $u, v \in V(G)$, $|f(u) \oplus f(v)| = \mathrm{d}(u, v)$. Thus, f is a 1-uniform dcsl of G.

Now, to prove, $\mathcal{F} \sqsubseteq \mathbf{P}^k$, where $\mathcal{F}$ is the range of f which forms a poset under '$\subseteq$' and $\mathbf{P}$ a linear order as $a_{i-1} \preceq a_i, 2 \le i \le n$. Define $\Phi : \mathcal{F} \to \mathbf{P}^k$ as follows.

Case 1. On the internal vertices v_i of $V(G)$, define $\Phi(f(v_i)) = a_i$.
Case 2. On the pendant vertices v_i^j of $V(G)$, define $\Phi(f(v_i^j)) = a_i^j$.

In Case 1, the corresponding vertex labels of a pair of internal vertices are comparable where as in Case 2, for any pair of pendant vertices the corresponding vertex labels are incomparable. Hence, $f(v_i) \subseteq f(v_j)$ in $\mathcal{F}$ if and only if $a_i \preceq a_j$ in $\mathbf{P}^k$ and $f(v_i^r) \parallel f(v_i^s)$ in $\mathcal{F}$ if and only if $a_i^r \parallel a_i^s$ in $\mathbf{P}^k$. Also, $f(v_i) \subseteq f(v_i^j)$ in $\mathcal{F}$ if and only if $a_i \preceq a_i^j$ in $\mathbf{P}^k$ and $f(v_i) \parallel f(v_{i-1}^s)$ in $\mathcal{F}$ if and only if $a_i \parallel a_{i-1}^s$ in $\mathbf{P}^k$. Therefore, $\mathcal{F} \sqsubseteq \mathbf{P}^k$. □

Using Proposition 6 and Proposition 7, we have the following result.

Proposition 8. *Let $\mathcal{F}$ be the range of a 1-uniform dcsl of the k-uniform caterpillar such that $\mathcal{F} \sqsubseteq \mathbf{P}^k$, where $\mathbf{P}$ is a linear order of finite length. Then $dim(\mathcal{F}) = 2$.*

Remark 2. *From Proposition 2 and Proposition 4, we have seen that the height-2 poset, H_n and width-2 poset, W_n on '$2n$' elements embeds the vertex labeling of a 1-uniform dcsl path. Choosing these posets as underline posets defined on 'n' elements, the corresponding k-extended posets embedding, restricted to height-2 poset and width-2 poset on n elements, give two subposets, namely min height poset (denoted by Min_n) and avg height poset(denoted by Avg_n), respectively. Further, the poset Min_n end up with $b_{\lceil \frac{n}{2} \rceil}$, when n is odd; $a_{\frac{n}{2}}$ if n is even. Hence, $Min_n \sqsubseteq H_n$. For the poset Avg_n, $Avg_n \sqsubseteq W_n$. For, without loss of generality, consider the poset as $(\{a_1, \ldots, a_{\lceil \frac{n}{2} \rceil = h}, b_1, \ldots, b_{n-h}\}, \preceq)$ of width two consisting of two chains $A = \{a_1, \ldots, a_h\}$ and $B = \{b_1, \ldots, b_{n-h}\}$ such that a_{i-1} preceqa_i for $2 \leq i \leq h$, $b_i \preceq b_{i+1}$ for $1 \leq i \leq n-h-1$, $a_1 \preceq b_i$ for $1 \leq i \leq n-h$, and for $2 \leq i \leq h$ and $1 \leq j \leq n-h$, $a_i \parallel b_j$. In particular, if the underline poset is of linear order, then it posses maximum height and by Proposition 6, the k-extended poset of it has dimension 2.*

Proposition 9. *For a k-extended poset Min_n, $dim(Min_n^k) = 2$.*

Proof. We define the linear extensions $\mathbf{L}_1$ and $\mathbf{L}_2$ of Min_n^k, in two cases.
Case 1: When n is even. Consider,

$$\mathbf{L}_1 : [b_1, b_1^1, \ldots, b_1^k, b_2, b_2^1, \ldots, b_2^k, \ldots, b_{\frac{n}{2}}, b_{\frac{n}{2}}^1, \ldots, b_{\frac{n}{2}}^k, a_1, a_1^1, \ldots, a_1^k, a_2, a_2^1, \ldots, a_2^k, \ldots, a_{\frac{n}{2}}, a_{\frac{n}{2}}^1, \ldots, a_{\frac{n}{2}}^k] \text{ and}$$

$$\mathbf{L}_2 : [b_{\frac{n}{2}}, a_{\frac{n}{2}}, b_{\frac{n}{2}-1}, a_{\frac{n}{2}-1}, \ldots, b_1, a_1, a_{\frac{n}{2}}^k, \ldots, a_{\frac{n}{2}}^1, a_{\frac{n}{2}-1}^k, \ldots, a_{\frac{n}{2}-1}^1, \ldots, a_1^k, \ldots, a_1^1, b_{\frac{n}{2}}^k, \ldots, b_{\frac{n}{2}}^1, b_{\frac{n}{2}-1}^k, \ldots, b_{\frac{n}{2}-1}^1, \ldots, b_1^k, \ldots, b_1^1].$$

Since, these extensions intersect to yield the partial order on Min_n^k, $dim(Min_n^k) \leq 2$.
Case 2: When n is odd. Consider,

$$\mathbf{L}_1 : [b_{\lceil \frac{n}{2} \rceil}, b_{\lceil \frac{n}{2} \rceil}^1, \ldots, b_{\lceil \frac{n}{2} \rceil}^k, b_{\lceil \frac{n}{2} \rceil-1}, b_{\lceil \frac{n}{2} \rceil-1}^1, \ldots, b_{\lceil \frac{n}{2} \rceil-1}^k, \ldots, b_1, b_1^1, \ldots, b_1^k, a_{\lceil \frac{n}{2} \rceil-1}, a_{\lceil \frac{n}{2} \rceil-1}^1, \ldots, a_{\lceil \frac{n}{2} \rceil-1}^k, \ldots, a_1, a_1^1, \ldots, a_1^k] \text{ and}$$

$$\mathbf{L}_2 : [b_1, a_1, b_2, a_2, \ldots, b_{\lceil \frac{n}{2} \rceil-1}, a_{\lceil \frac{n}{2} \rceil-1}, b_{\lceil \frac{n}{2} \rceil}, a_1^k, \ldots, a_1^1, a_2^k, \ldots, a_2^1, \ldots, a_{\lceil \frac{n}{2} \rceil-1}^k, \ldots, a_{\lceil \frac{n}{2} \rceil-1}^1, b_1^k, \ldots, b_1^1, b_2^k, \ldots, b_2^1, \ldots, b_{\lceil \frac{n}{2} \rceil}^k, \ldots, b_{\lceil \frac{n}{2} \rceil}^1].$$

Clearly, these extensions produces a realizer of Min_n^k, hence $dim(Min_n^k) \leq 2$. Following as in the proof of Proposition 6, the dimension cannot be less than 2. Therefore, $dim(Min_n^k) = 2$. $\square$

Proposition 10. *The k-extended poset Min_n^k embeds the vertex labeling of a 1-uniform dcsl of the k-uniform caterpillar.*

Proof. Let $V(P_n^k) = \{v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k \ldots, v_n, v_n^1, \ldots, v_n^k\}$, where v_i are the internal vertices and v_i^j are the pendant vertices which are adjacent to v_i.

Let $X = \{1, 2, \ldots, w, \ldots, n, \ldots, m = n(k+1) - 1\}$, where $w = \lceil \frac{|V(P_n)|}{2} \rceil$.

We claim that there exists a poset $\mathcal{F}$ which can be obtained from a vertex labeling of 1-uniform dcsl caterpillar, that suits for the embedding of Min_n^k.

Define $f : V(P_n^k) \to 2^X$, on internal vertices, by

$$f(v_1) = \{1, 2, \ldots, w-1\}, f(v_2) = \{1, 2, \ldots, w-1, w\}, f(v_3) = \{2, \ldots, w-1, w\},$$
$$f(v_4) = \{2, \ldots, w-1, w, w+1\}, f(v_5) = \{3, \ldots, w, w+1\}, \ldots, f(v_n) = \{w, w+1, \ldots, n-1\},$$

when n is odd; otherwise, $f(v_n) = \{w, w+1, \ldots, n\}$. In general, for $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} \{ \frac{i+1}{2}, \frac{i+1}{2}+1, \ldots, \frac{i+1}{2}+w-2 \}, & \text{if } i \text{ is odd} \\ \{ \frac{i}{2}, \frac{i}{2}+1, \ldots, \frac{i}{2}+w-1 \}, & \text{otherwise,} \end{cases}$$

and on pendant vertices, vertex labeling is same, as in Proposition 7.

Case 1: When $u = v_i$ and $v = v_{i+1}$, where i is odd. Then,

$$\begin{aligned} |f(v_i) \oplus f(v_{i+1})| &= | \{\frac{i+1}{2}, \ldots, \frac{i+1}{2}+w-2\} \oplus \{\frac{i+1}{2}, \ldots, \frac{i+1}{2}+w-1\} | \\ &= | \{\frac{i+1}{2}+w-1\} | = 1 = d(v_i, v_{i+1}). \end{aligned}$$

Case 2: When $u = v_{i+1}$ and $v = v_i$, where i is even. Then,

$$\begin{aligned} |f(v_{i+1}) \oplus f(v_i)| &= | \{\frac{i+2}{2}, \ldots, \frac{i+2}{2}+w-2\} \oplus \{\frac{i}{2}, \ldots, \frac{i}{2}+w-1\} | \\ &= | \{\frac{i}{2}\} | = 1 = d(v_{i+1}, v_i). \end{aligned}$$

Case 3: When $u = v_l$ and $v = v_m$, $l \neq m$, $1 \leq l,\ m \leq n$ and both l and m are odd. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m)| &= | \{\frac{l+1}{2}, \ldots, \frac{l+1}{2}+w-2\} \oplus \{\frac{m+1}{2}, \ldots, \frac{m+1}{2}+w-2\} | \\ &= | \{\frac{l+1}{2}, \ldots, \frac{m+1}{2}+w-2\} | = m-l = d(v_l, v_m), \quad 1 \leq l < m \leq n. \end{aligned}$$

Case 4: When $u = v_l$ and $v = v_m$, $l \neq m$, $1 \leq l,\ m \leq n$ and both l and m are even. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m)| &= | \{\frac{l}{2}, \ldots, \frac{l}{2}+w-1\} \oplus \{\frac{m}{2}, \ldots, \frac{m}{2}+w-1\} | \\ &= | \{\frac{l}{2}, \ldots, \frac{m}{2}+w-1\} | = m-l = d(v_l, v_m), \quad 1 \leq l < m \leq n. \end{aligned}$$

Case 5: When $u = v_i$ and $v = v_i^j$, $1 \leq i \leq n$ and $1 \leq j \leq k$. Then,

$$|f(v_i) \oplus f(v_i^j)| = | \{n + (i-1)k + (j-1)\} | = 1 = d(v_i, v_i^j).$$

Case 6: When $u = v_i$ and $v = v_{i+1}^j$, $1 \le j \le k$ and i is odd. Then,

$$\begin{aligned}|f(v_i) \oplus f(v_{i+1}^j)| =& | \{\frac{i+1}{2}, \dots, \frac{i+1}{2} + w - 2\} \\ & \oplus \{\frac{i+1}{2}, \dots, \frac{i+1}{2} + w - 1, n + (i)k + (j-1)\} | \\ & =| \{\frac{i+1}{2} + w - 1, n + (i)k + (j-1)\} |= 2 = d(v_i, v_{i+1}^j).\end{aligned}$$

Case 7: $u = v_{i+1}$ and $v = v_i^j$, $1 \le j \le k$ and i is even. Then,

$$\begin{aligned}|f(v_{i+1}) \oplus f(v_i)| =& | \{\frac{i+2}{2}, \frac{i+2}{2} + 1, \dots, \frac{i+2}{2} + w - 2\} \\ & \oplus \{\frac{i}{2}, \frac{i}{2} + 1, \dots, \frac{i}{2} + w - 1, n + (i-1)k + (j-1)\} | \\ & =| \{\frac{i}{2}, n + (i-1)k + (j-1)\} |= 2 = d(v_{i+1}, v_i^j).\end{aligned}$$

Case 8: When $u = v_l$ and $v = v_m^j$, $l \ne m$, $1 \le l,\ m \le n$, $1 \le j \le k$ and both l and m are odd. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| =& | \{\frac{l+1}{2}, \frac{l+1}{2} + 1, \dots, \frac{l+1}{2} + w - 2\} \\ & \oplus \{\frac{m+1}{2}, \frac{m+1}{2} + 1, \dots, \frac{m+1}{2} + w - 2, n + (m-1)k + (j-1)\} | \\ & =| \{\frac{l+1}{2}, \dots, \frac{m+1}{2} + w - 2, n + (m-1)k + (j-1)\} |= m - l + 1 = d(v_l, v_m^j), \\ & 1 \le l < m \le n \text{ and } 1 \le j \le k.\end{aligned}$$

Case 9: When $u = v_l$ and $v = v_m^j$, $l \ne m$, $1 \le l,\ m \le n$, $1 \le j \le k$ and both l and m are even. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| =& | \{\frac{l}{2}, \frac{l}{2} + 1, \dots, \frac{l}{2} + w - 1\} \\ & \oplus \{\frac{m}{2}, \frac{m}{2} + 1, \dots, \frac{m}{2} + w - 1, n + (m-1)k + (j-1)\} | \\ & =| \{\{\frac{l}{2}, \dots, \frac{m}{2} + w - 1, n + (m-1)k + (j-1)\} |= m - l + 1 = d(v_l, v_m^j), \\ & 1 \le l < m \le n \text{ and } 1 \le j \le k.\end{aligned}$$

Case 10: When $u = v_i^r$ and $v = v_{i+1}^s$, $1 \le r,\ s \le k$ and i is odd. Then,

$$\begin{aligned}|f(v_i^r) \oplus f(v_{i+1}^s)| =& | \{\frac{i+1}{2}, \dots, \frac{i+1}{2} + w - 2, n + (i-1)k + (r-1)\} \\ & \oplus \{\frac{i+1}{2}, \dots, \frac{i+1}{2} + w - 1, n + (i)k + (s-1)\} | \\ & =| \{n + (i-1)k + (r-1), \frac{i+1}{2} + w - 1, n + (i)k + (s-1)\} |= 3 = d(v_i^r, v_{i+1}^s).\end{aligned}$$

Case 11: $u = v^r_{i+1}$ and $v = v^s_i$, $1 \leq r,\ s \leq k$ and i is even. Then,

$$\begin{aligned}|f(v^r_{i+1}) \oplus f(v^s_i)| =& |\ \{\frac{i+2}{2}, \dots, \frac{i+2}{2} + w - 2, n + (i)k + (r-1)\} \\ &\oplus \{\frac{i}{2}, \dots, \frac{i}{2} + w - 1, n + (i-1)k + (j-1)\}\ | \\ &= |\ \{\frac{i}{2}, n + (i)k + (r-1), n + (i-1)k + (s-1)\}\ | = 3 = d(v^r_{i+1}, v^s_i).\end{aligned}$$

Case 12: When $u = v^i_l$ and $v = v^j_m$, $l \neq m$, $1 \leq l,\ m \leq n$, $1 \leq i,\ j \leq k$ and both l and m are odd. Then,

$$\begin{aligned}|f(v^i_l) \oplus f(v^j_m)| =& |\ \{\frac{l+1}{2}, \dots, \frac{l+1}{2} + w - 2, n + (l-1)k + (i-1)\} \\ &\oplus \{\frac{m+1}{2}, \dots, \frac{m+1}{2} + w - 2, n + (m-1)k + (j-1)\}\ | \\ &= |\ \{\{\frac{l+1}{2}, \dots, \frac{m+1}{2} + w - 2, n + (l-1)k + (i-1), n + (m-1)k + (j-1)\}\ | \\ &= m - l + 2 = d(v^i_l, v^j_m), \quad 1 \leq l < m \leq n \text{ and } 1 \leq i, j \leq k.\end{aligned}$$

Case 13: When $u = v^i_l$ and $v = v^j_m$, $l \neq m$, $1 \leq l,\ m \leq n$, $1 \leq i,\ j \leq k$ and both l and m are even. Then,

$$\begin{aligned}|f(v^i_l) \oplus f(v^j_m)| =& |\ \{\frac{l}{2}, \dots, \frac{l}{2} + w - 1, n + (l-1)k + (i-1)\} \\ &\oplus \{\frac{m}{2}, \dots, \frac{m}{2} + w - 1, n + (m-1)k + (j-1)\}\ | \\ &= |\ \{\{\frac{l}{2}, \dots, \frac{m}{2} + w - 1, n + (l-1)k + (i-1), n + (m-1)k + (j-1)\}\ | \\ &= m - l + 2 = d(v^i_l, v^j_m), \quad 1 \leq l < m \leq n \text{ and } 1 \leq i, j \leq k.\end{aligned}$$

Thus, for any distinct $u, v \in V(P^k_n)$, $|f(u) \oplus f(v)| = d(u, v)$ and hence f admits 1-uniform dcsl. Also, to prove $\mathcal{F} \sqsubseteq Min^k_n$, where $\mathcal{F}$ is the range of f, which forms a poset, we define $\Phi : \mathcal{F} \to Min^k_n$ as follows in two different cases.

Case 1. On the internal vertices v_i of $V(P^k_n)$. $\Phi(f(v_i)) = \begin{cases} a_{\frac{i}{2}}, & \text{if } i \text{ is even,} \\ b_{\lceil \frac{i}{2} \rceil}, & \text{otherwise.} \end{cases}$

Case 2. On the pendant vertices v^j_i of $V(P^k_n)$. $\Phi(f(v^j_i)) = \begin{cases} a^j_{\frac{i}{2}}, & \text{if } i \text{ is even,} \\ b^j_{\lceil \frac{i}{2} \rceil}, & \text{otherwise.} \end{cases}$

In Case 1, the internal vertex labeling of $V(P^k_n)$, exhibits the embedding of $\mathcal{F}$ into the underline poset of Min^k_n; and in Case 2, the pendent vertex labeling of $V(P^k_n)$, exhibits the embedding of $\mathcal{F}$ into the outermost labeling of an underline set of Min^k_n. Thus, all together, we get $\mathcal{F} \sqsubseteq Min^k_n$. □

Analogously, from Proposition 9 and Proposition 10, we have.

Proposition 11. *Let $\mathcal{F}$ be the range of a 1-uniform dcsl of the k-uniform caterpillar such that $\mathcal{F} \sqsubseteq Min_n^k$. Then $dim(\mathcal{F}) = 2$.*

Proposition 12. *For the k-extended poset Avg_n^k, $dim(Avg_n^k) = 2$.*

Proof. Let us take the linear extensions of Avg_n^k as

$$\begin{aligned}\mathbf{L}_1 :& [a_1, a_1^1, \ldots, a_1^k, a_2, a_2^1, \ldots, a_2^k, \ldots, a_h, a_h^1, \ldots, a_h^k, b_1, b_1^1, \ldots, b_1^k, b_2, b_2^1, \ldots, b_2^k, \ldots, b_{n-h}, \\ & b_{n-h}^1, \ldots, b_{n-h}^k] \text{ and} \\ \mathbf{L}_2 :& [a_1, b_1, b_2, \ldots, b_{n-h}, a_2, \ldots, a_h, b_{n-h}^k, \ldots, b_{n-h}^1, b_{n-h-1}^k, \ldots, b_{n-h-1}^1, \ldots, b_1^k, \ldots, b_1^1, \\ & a_h^k, \ldots, a_h^1, a_{h-1}^k, \ldots, a_{h-1}^1, \ldots, a_1^k, \ldots, a_1^1].\end{aligned}$$

Then dimension of Avg_n^k is at most 2. Again, as in Proposition 6 the dimension cannot be less than 2. Hence $dim(Avg_n^k) = 2$. □

Proposition 13. *The k-extended poset Avg_n^k embeds the vertex labeling of a 1-uniform dcsl of the k-uniform caterpillar.*

Proof. Let $v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k, \ldots, v_n, v_n^1, \ldots,$ and v_n^k be the vertices of $V(P_n^k)$.

Let $X = \{1, 2, \ldots, h, \ldots, n, \ldots, m = n(k+1) - 1\}$, where $h = \lceil \frac{|V(P_n)|}{2} \rceil$. To prove the existence of a poset $\mathcal{F}$ from a vertex labeling of 1-uniform dcsl of the k-uniform caterpillar, that suits for the embedding of Avg_n^k, define $f : V(P_n^k) \to 2^X$, on internal vertices, by

$$\begin{aligned} &f(v_j) = \{1, \ldots, n-h-(j-1)\},\ 1 \le j \le n-h, \quad f(v_{n-h+1}) = \varnothing, \\ &f(v_{n-h+i}) = \{n-h+1, \ldots, n-h+(i-1)\},\ 2 \le i \le h \end{aligned}$$

and we consider the vertex labeling on pendant vertices which is same as mentioned in Proposition 7.

Case 1: When $u = v_l$ and $v = v_m$, $l \neq m$, $1 \le l \le n-h$ and $m = n-h+1$. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m)| &= | \{1, \ldots, n-h-(l-1)\} \oplus \varnothing | \\ &= | \{1, \ldots, n-h-(l-1)\} | = n-h-(l-1) = d(v_l, v_m). \end{aligned}$$

Case 2: When $u = v_l$ and $v = v_m$, $l \neq m$, $n-h+2 \le l \le n$ and $m = n-h+1$. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m)| &= | \{n-h+1, \ldots, l-1\} \oplus \varnothing | \\ &= | \{n-h+1, \ldots, l-1 = n-h+(l-m)\} | = l-m = d(v_l, v_m). \end{aligned}$$

Case 3: When $u = v_l$ and $v = v_m$, $l \neq m$, $1 \le l \le n-h$ and $n-h+2 \le m \le n$. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m)| &= | \{1, \ldots, n-h-(l-1)\} \oplus \{n-h+1, \ldots, m-1\} | \\ &= | \{1, \ldots, n-h-(l-1), n-h+1, \ldots, m-1\} | = m-l = d(v_l, v_m). \end{aligned}$$

Case 4: When $u = v_l$ and $v = v_m^j$, $l \neq m$, $1 \le l \le n-h$, $m = n-h+1$ and $1 \le j \le k$. Then,

$$\begin{aligned} |f(v_l) \oplus f(v_m^j)| &= | \{1, \ldots, n-h-(l-1)\} \oplus \{n-1+(m-1)k+j\} | \\ &= | \{1, \ldots, n-h-(l-1), n-1+(m-1)k+j\} | = m-l+1 = d(v_l, v_m^j). \end{aligned}$$

Case 5: When $u = v_l$ and $v = v_m^j$, $l \neq m$, $n-h+2 \leq l \leq n$, $m = n-h+1$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| &=| \{n-h+1,\ldots,l-1\} \oplus \{n-1+(m-1)k+j\} | \\ &=| \{n-h+1,\ldots,l-1,n-1+(m-1)k+j\} |= l-m+1 = d(v_l, v_m^j).\end{aligned}$$

Case 6: When $u = v_l$ and $v = v_m^j$, $l \neq m$, $1 \leq l \leq n-h$, $n-h+2 \leq m \leq n$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}|f(v_l) \oplus f(v_m^j)| &=| \{1,\ldots,n-h-(l-1)\} \oplus \{n-h+1,\ldots,m-1,n-1+(m-1)k+j\} | \\ &=| \{1,\ldots,n-h-(l-1),n-h+1,\ldots,m-1,n-1+(m-1)k+j\} | \\ &= m-l+1 = d(v_l, v_m^j).\end{aligned}$$

Case 7: When $u = v_l^i$ and $v = v_m^j$, $l \neq m$, $1 \leq l \leq n-h$, $m = n-h+1$ and $1 \leq i,\ j \leq k$. Then,

$$\begin{aligned}|f(v_l^i) \oplus f(v_m^j)| &=| \{1,\ldots,n-h-(l-1),n-1+(l-1)k+i\} \oplus \{n-1+(m-1)k+j\} | \\ &=| \{1,\ldots,n-h-(l-1),n-1+(l-1)k+i,n-1+(m-1)k+j\} | \\ &= m-l+2 = d(v_l^i, v_m^j).\end{aligned}$$

Case 8: When $u = v_l^i$ and $v = v_m^j$, $l \neq m$, $n-h+2 \leq l \leq n$, $m = n-h+1$ and $1 \leq i,\ j \leq k$. Then,

$$\begin{aligned}&|f(v_l^i) \oplus f(v_m^j)| =| \{n-h+1,\ldots,l-1,n-1+(l-1)k+i\} \oplus \{n-1+(m-1)k+j\} | \\ &=| \{n-h+1,\ldots,l-1,n-1+(l-1)k+i,n-1+(m-1)k+j\} |= l-m+2 = d(v_l^i, v_m^j).\end{aligned}$$

Case 9: When $u = v_l^i$ and $v = v_m^j$, $l \neq m$, $1 \leq l \leq n-h$, $n-h+2 \leq m \leq n$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}&|f(v_l^i) \oplus f(v_m^j)| =| \{1,\ldots,n-h-(l-1),n-1+(l-1)k+i\} \\ &\oplus \{n-h+1,\ldots,m-1,n-1+(m-1)k+j\} | \\ &=| \{1,\ldots,n-h-(l-1),n-1+(l-1)k+i,n-h+1,\ldots,m-1,n-1+(m-1)k+j\} | \\ &= m-l+2 = d(v_l^i, v_m^j).\end{aligned}$$

Thus, for any distinct vertices $u, v \in V(P_n^k)$, $|f(u) \oplus f(v)|$ = d(u, v), and hence f admits 1-uniform dcsl.

Finally, to prove $\mathcal{F} \sqsubseteq Avg_n^k$, where $\mathcal{F}$ is the range of f, which forms a poset, define $\Psi : \mathcal{F} \to Avg_n^k$ as follows.

Case 1. On the internal vertices v_i of $V(P_n^k)$. $\Psi(f(v_i)) = \begin{cases} b_i, & \text{when } 1 \leq i \leq n-h, \\ a_{i-(n-h)}, & \text{otherwise.} \end{cases}$

Case 2. On the pendant vertices v_i^j of $V(P_n^k)$. $\Phi(f(v_i^j)) = \begin{cases} b_i^j, & \text{when } 1 \leq i \leq n-h, \\ a_{i-(n-h)}^j, & \text{otherwise.} \end{cases}$

In Case 1, we can identify the internal vertex labeling of $V(P_n^k)$, as the embedding of $\mathcal{F}$ into the underline poset of Avg_n^k. In Case 2, the pendent vertex labeling of $V(P_n^k)$, list the embedding of $\mathcal{F}$ into the outermost labeling of an underline set of Avg_n^k. Thus, from Case 1 and Case 2, we get $\mathcal{F} \sqsubseteq Avg_n^k$. □

The following result follows from Proposition 12 and Proposition 13.

Proposition 14. *Let $\mathcal{F}$ be the range of vertex labeling of a 1-uniform dcsl k-uniform caterpillar such that $\mathcal{F} \sqsubseteq Avg_n^k$. Then $dim(\mathcal{F}) = 2$.*

Theorem 1 ([7])**.** *If $\mathbf{T}$ is a tree*[1]*, then $dim(\mathbf{T}) \leq 2$ unless $\mathbf{T}$ contains one or more of the trees J_1 and J_2 or their duals as subposets.*

Theorem 2. *Let $\mathcal{F}$ be the poset. Then there exists a 1-uniform dcsl f (the vertex labeling of a k-uniform caterpillar) such that $\mathcal{F} = Range(f) = \{f(v) \mid v \in V(P_n^k)\}$, where $n > 2$ and $k \geq 1$, and $dim(\mathcal{F}) = 2$.*

Proof. Let f be a vertex labeling of 1-uniform dcsl k-uniform caterpillar on '$n(k+1)$' vertices, where $n > 2$ and $k \geq 1$, other than the labeling which is mentioned in Proposition 7, Proposition 10 and Proposition 13, respectively, and let $\mathcal{F}$ be the range of f. Hence, $\mathcal{F} = Range(f) = \{f(v) \mid v \in V(P_n^k)\}$, is a poset.

We prove that $dim(\mathcal{F}) = 2$.

Since the Hasse diagram of $\mathcal{F}$ is a tree, from Theorem 1, we have $dim(\mathcal{F}) \leq 2$. But, $dim(\mathcal{F})$ is never less than 2. For, if it is of dimension 1, then the Hasse diagram of it resembles a path, which is not possible. Hence, $dim(\mathcal{F}) = 2$. ☐

Recall that [3] the minimum cardinality of the underlying set X such that G admits a 1-uniform dcsl is called the 1-uniform dcsl index $\delta_d(G)$ of G. Following discussion is an attempt to establish the relationship between the 1-uniform dscl index of a k-uniform caterpillar and the dimension of the poset $\mathcal{F} = Range(f) = \{f(v) \mid v \in V(P_n^k)\}$, where $n \geq 1$ and $k \geq 1$.

Lemma 3. *The 1-uniform dcsl index of P_n^k $(n \geq 1,\ k \geq 1)$ is $n(k+1) - 1$.*

Proof. Let $V(P_n^k) = \{v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k, \ldots, v_n, v_n^1, \ldots, v_n^k\}$, and let f be the dcsl labeling of P_n^k with the underlying set as X. First, we claim that $\mid X \mid \geq n(k+1) - 1$. By Lemma 1, the 1-uniform dcsl index of P_n is $n-1$, and hence for the internal vertices of P_n^k, the dcsl index is $n-1$. For the remaining 'nk' vertices (pendant vertices), we need to have atleast 'nk' subsets of X other than the subsets which has already been labeled for the internal vertices. Hence, the cardinality of X is atleast $nk + n - 1$. By Proposition 7, the vertex labeling of 1-uniform dcsl of P_n^k with underlying set X is of cardinality $n(k+1) - 1$. Hence, $\delta_d(P_n^k) = n(k+1) - 1$. ☐

In Propositions 7, 10 and 13, the existence of different vertex labeling of 1-uniform dcsl of k-uniform caterpillar and their embedding in respective posets have been established.

In the following theorem we determine the bounds of the poset $\mathcal{F}$, where $\mathcal{F} = Range(f) = \{f(v) \mid v \in V(P_n^k)\}$.

Theorem 3. *Let $\mathcal{F}$ be the poset which is the range of a 1-uniform dcsl of the k-uniform caterpillar, with respect to set inclusion '$\subseteq$'. Then, $dim(\mathcal{F}) \leq \delta_d(P_n^k)$.*

Proof. Let f be a 1-uniform dcsl of $P_n^k (n \geq 1,\ k \geq 1)$, such that $\mathcal{F} = \{f(v) \mid v \in V(P_n^k)\}$ forms a poset with respect to set inclusion '$\subseteq$'. Depending on the number of vertices of $V(P_n^k)$, we prove the theorem for the following four cases.

[1] we call a poset is a tree if its Hasse diagram is a tree in the graph theoretic sense.

Case 1: When $n = 1$ and $k = 1$. In this case, the poset $\mathcal{F}$ is isomorphic to a poset which is a chain of length 1, and hence $dim(\mathcal{F}) = 1$. But by Lemma 3, $\delta_d(P_1^1) = 1$. Thus, we have $dim(\mathcal{F}) = \delta_d(P_n^k)$.

Case 2: When $n = 2$ and $k = 1$. By Lemma 3, we have $\delta_d(P_2^1) = 3$. Also $\mathcal{F}$ is isomorphic to any of the four posets namely, a poset which is a chain of length 3, poset Avg_4, poset $\hat{Avg_4}$ or poset $\mathbf{P}^1$, where $\mathbf{P}$ is a chain of length 1. If $\mathcal{F}$ is isomorphic to chain of length 3, then $dim(\mathcal{F}) = 1$, and hence $dim(\mathcal{F}) < \delta_d(P_n^k)$. If $\mathcal{F} \cong Avg_4$, then by Proposition 14, $dim(\mathcal{F}) = 2$, and hence $dim(\mathcal{F}) < \delta_d(P_n^k)$. Since, for a poset $\mathbf{P}$, $dim(\mathbf{P}) = dim(\hat{\mathbf{P}})$ (see [7]), so if $\mathcal{F} \cong \hat{Avg_4}$, then $dim(\mathcal{F}) = dim(\hat{\mathcal{F}}) = dim(Avg_4) = 2$. Thus, $dim(\mathcal{F}) < \delta_d(P_n^k)$. If $\mathcal{F} \cong \mathbf{P}^1$, where $\mathbf{P}$ is a chain of length 1, then by Proposition 8, $dim(\mathcal{F}) = 2$, and hence, $dim(\mathcal{F}) < \delta_d(P_n^k)$.

Case 3: When $n \geq 3$ and $k \geq 1$. In this case, we prefer k-extended posets that embeds $\mathcal{F}$, as it is not easy to predict all the variations of the poset $\mathcal{F}$. Thus, based on the underline posets of the k-extended posets, since by Lemma 3, $\delta_d(P_n^k) = n(k+1) - 1$, it is enough to consider the following subcases under Case 3.

Case 3.1: If the underline poset is a linear order of finite length, say $\mathbf{L} : a_{i-1} \preceq a_i$, for $2 \leq i \leq n$, then by Proposition 8, $dim(\mathcal{F}) = 2$. Hence $\delta_d(P_n^k) > dim(\mathcal{F})$.

Case 3.2: If the underline poset is isomorphic to Min_n, then by Proposition 11, $dim(\mathcal{F}) = 2$. Hence $dim(\mathcal{F}) < \delta_d(P_n^k)$.

Case 3.3: If the underline poset is isomorphic to Avg_n, then by Proposition 14, $dim(\mathcal{F}) = 2$. Hence $dim(\mathcal{F}) < \delta_d(P_n^k)$.

Case 4: When the poset $\mathcal{F}$ is not isomorphic to either $\mathbf{P}^k$, Min_n^k or Avg_n^k. We have from Theorem 2, $dim(\mathcal{F}) = 2$ and, by Lemma 3, $\delta_d(P_n^k) = n(k+1) - 1$, hence $dim(\mathcal{F}) < \delta_d(P_n^k)$. Thus in all the cases we get $dim(\mathcal{F}) \leq \delta_d(P_n^k)$. □

Theorem 4. *The k-uniform caterpillar P_n^k admits a k-uniform dcsl.*

Proof. Consider $G = P_n^k$ with $n(k+1)$ vertices, say $v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k, \ldots, v_n, v_n^1, \ldots,$ and v_n^k. Let $X = \{1, 2, \ldots, h, \ldots, n, \ldots, n(k+1) - 1, \ldots, k(n(k+1) - 1)\}$.

Define $f : V(G) \to 2^X$ by $f(v_1) = \varnothing$, $f(v_i) = \{1, 2, \ldots, (i-1)k\}$ for $2 \leq i \leq n$, and for $1 \leq i \leq k$,

$$f(v_1^i) = f(v_1) \cup \{(n-1)k + (i-1)k + 1, \ldots, (n-1)k + (i-1)k + k\},$$
$$f(v_2^i) = f(v_2) \cup \{(n-1)k + k^2 + (i-1)k + 1, \ldots, (n-1)k + k^2 + (i-1)k + k\} \text{ and}$$
$$f(v_n^i) = f(v_n) \cup \{(n-1)k + (n-1)k^2 + (i-1)k + 1, \ldots, (n-1)k + (n-1)k^2 + (i-1)k + k\}.$$

In general, for $1 \leq i \leq n$ and $1 \leq j \leq k$,
$f(v_i^j) = f(v_i) \cup \{(n-1)k + (i-1)k^2 + (j-1)k + 1, \ldots, (n-1)k + (i-1)k^2 + (j-1)k + k\}$.
Case 1: When $u = v_l$ and $v = v_m$, $l = 1$ and $2 \leq m \leq n$. Then,

$$|f(v_l) \oplus f(v_m)| = |\varnothing \oplus \{1, 2, \ldots, (m-1)k\}| = |\{1, 2, \ldots, (m-1)k\}| = (m-1)k = kd(v_l, v_m).$$

Case 2: When $u = v_l$ and $v = v_m$, $l \neq m$, $2 \leq l,\ m \leq n$. Then,

$$|f(v_l) \oplus f(v_m)| = |\{1, 2, \ldots, (l-1)k\} \oplus \{1, 2, \ldots, (m-1)k\}| = |\{(l-1)k + 1, \ldots, (m-1)k\}| = (m-l)k = kd(v_l, v_m), \quad 2 \leq l < m \leq n.$$

Case 3: When $u = v_l$ and $v = v_m^j$, $l = 1, 2 \leq m \leq n$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}
&|f(v_l) \oplus f(v_m^j)| \\
&=| \varnothing \oplus \{1,2,\ldots,(m-1)k,(n+j-2)k+(m-1)k^2+1,\ldots,(n+j-2)k+(m-1)k^2+k\} | \\
&=| \{1,2,\ldots,(m-1)k,(n+j-2)k+(m-1)k^2+1,\ldots,(n+j-2)k+(m-1)k^2+k\} | \\
&= (m-l+1)k = kd(v_l, v_m^j).
\end{aligned}$$

Case 4: When $u = v_l$ and $v = v_m^j$, $l \neq m, 2 \leq l,\ m \leq n$ and $1 \leq j \leq k$. Then,

$$\begin{aligned}
&|f(v_l) \oplus f(v_m^j)| \\
&=| \{1,2,\ldots,(l-1)k\} \oplus \{1,2,\ldots,(m-1)k,(n+j-2)k+(m-1)k^2+1,\ldots, \\
&(n+j-2)k+(m-1)k^2+k\} | \\
&=| \{(l-1)k+1,\ldots,(m-1)k,(n+j-2)k+(m-1)k^2+1,\ldots, \\
&(n+j-2)k+(m-1)k^2+k\} | \\
&= (m-l+1)k = kd(v_l, v_m^j), \quad 2 \leq l < m \leq n \text{ and } 1 \leq j \leq k.
\end{aligned}$$

Case 5: When $u = v_l^i$ and $v = v_m^j$, $l = 1, 2 \leq m \leq n$ and $1 \leq i, j \leq k$. Then,

$$\begin{aligned}
&|f(v_l^i) \oplus f(v_m^j)| \\
&=| \{(n-1)k+(i-1)k+1,\ldots,(n-1)k+(i-1)k+k\} \oplus \{1,\ldots,(m-1)k, \\
&(n-1)k+(m-1)k^2+(j-1)k+1,\ldots,(n-1)k+(m-1)k^2+(j-1)k+k\} | \\
&=| \{1,\ldots,(m-1)k,(n-1)k+(m-1)k^2+(j-1)k+1,\ldots, \\
&(n-1)k+(m-1)k^2+(j-1)k+k,(n-1)k+(i-1)k+1,\ldots,(n-1)k+(i-1)k+k\} | \\
&= (m-l+2)k = kd(v_l^i, v_m^j).
\end{aligned}$$

Case 6: When $u = v_l^i$ and $v = v_m^j$, $l \neq m, 2 \leq l,\ m \leq n$ and $1 \leq i, j \leq k$. Then,

$$\begin{aligned}
&|f(v_l^i) \oplus f(v_m^j)| \\
&=| \{1,\ldots,(l-1)k,(n-1)k+(l-1)k^2+(i-1)k+1,\ldots, \\
&(n-1)k+(l-1)k^2+(i-1)k+k\} \oplus \{1,\ldots,(m-1)k,(n-1)k+(m-1)k^2+ \\
&(j-1)k+1,\ldots,(n-1)k+(m-1)k^2+(j-1)k+k\} | \\
&=| \{(n-1)k+(l-1)k^2+(i-1)k+1,\ldots,(n-1)k+(l-1)k^2+(i-1)k+k, \\
&(l-1)k+1,\ldots,(m-1)k,(n-1)k+(m-1)k^2+(j-1)k+1,\ldots, \\
&(n-1)k+(m-1)k^2+(j-1)k+k\} | \\
&= (m-l+2)k = kd(v_l^i, v_m^j), \quad 2 \leq l < m \leq n \text{ and } 1 \leq i \leq j \leq k.
\end{aligned}$$

Hence, for any distinct $u, v \in V(G)$, $|f(u) \oplus f(v)| = kd(u, v)$. Which shows that f admits k-uniform dcsl. □

Lemma 4. *For* $n \geq 1, k \geq 1$, $\delta_k(P_n^k) = k(n(k+1)-1)$.

Proof. Let $V(P_n^k) = \{v_1, v_1^1, \ldots, v_1^k, v_2, v_2^1, \ldots, v_2^k, \ldots, v_n, v_n^1, \ldots, v_n^k\}$, and let f be the dcsl labeling of P_n^k with the underlying set as X. By Lemma 2, the 1-uniform dcsl index of P_n is $k(n-1)$, which implies that for internal vertices of P_n^k, the required dcsl index is $k(n-1)$, where as for remaining 'nk' vertices (pendant vertices), we need at least 'k^2n' subsets of X other than the subsets which has already been labeled. Hence the cardinality of X is atleast $k^2n + k(n-1)$. Since by Theorem 4, P_n^k is a k-uniform dcsl with underlying set X of cardinality $k(n(k+1)-1)$, thus we have, $\delta_k(P_n^k) = k(n(k+1)-1)$. □

Theorem 5 ([4]). *If G is k-uniform dcsl, and m is a positive integer, then G is mk-uniform dcsl.*

It has been already established in [4] that path admits arbitrary k-uniform dcsl labeling and k-uniform dcsl index, $\delta_k(P_n)$ is k times that of 1-uniform dcsl index. In this paper, this result is extended to a k-uniform caterpillar, and we prove that the k-uniform dcsl index, $\delta_k(P_n^k)$ is k times that of the 1-uniform dcsl index of k-uniform caterpillar. It is interesting to note that the range of any arbitrary k-uniform dcsl of a k-uniform caterpillar, P_n^k need not form a connected poset. However, there always exists a k-uniform dcsl of P_n^k, whose range is a connected poset. Hence, the Hasse diagram (or poset) which embeds the vertex labeling of 1-uniform dcsl P_n^k, can also embed the vertex labeling of k-uniform dcsl P_n^k. Hence, for such postes the dimension corresponding to 1-uniform dcsl P_n^k and the dimension corresponding to k-uniform dcsl P_n^k are same. Thus, we have the following theorem.

Theorem 6. *If $\mathcal{F}$ is the range of a k-uniform dcsl of the k-uniform caterpillar P_n^k $(n \geq 1, k \geq 1)$, that forms a poset with respect to set inclusion '$\subseteq$', then, $dim(\mathcal{F}) \leq \delta_k(P_n^k)$.*

Proof. Proof is immediate from Theorem 5, Lemma 4 and Theorem 3. □

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Наґесвара Рао К., Ґерміна К.А., Шаіні П. *Про розмірність маркування вершин k-однорідного dcsl k-однорідного графа* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 134–149.

Сумісне з відстанню множинне маркування (dcsl) зв'язного графа G є ін'єктивним відображенням $f : V(G) \to 2^X$, де X є непорожною базовою множиною такою, що відповідна індукована функція $f^{\oplus} : E(G) \to 2^X \setminus \{\varnothing\}$, задана рівністю $f^{\oplus}(uv) = f(u) \oplus f(v)$, задовольняє $| f^{\oplus}(uv) |= k^f_{(u,v)} d_G(u,v)$ для довільної пари різних вершин $u, v \in V(G)$, де $d_G(u,v)$ позначає відстань між u і v та $k^f_{(u,v)}$ є числом, не обов'язково цілим. Сумісне з відстанню множинне маркування f графа G є k-однорідним, якщо всі коефіцієнти пропорційності відносно f рівні k, і якщо G допускає таке маркування, то G називають k-однорідним dcsl графом. *k-однорідний dcsl індекс* графа G, що позначається $\delta_k(G)$, є мінімальним серед потужностей X, де X пробігає всі k-однорідні dcsl-множини графа G. *Лінійне розширення* $\mathbf{L}$ часткового порядку $\mathbf{P} = (P, \preceq)$ є лінійним порядком на елементах із P таким, що з $x \preceq y$ в $\mathbf{P}$ слідує, що $x \preceq y$ в $\mathbf{L}$ для всіх $x, y \in P$. Розмірність множини $\mathbf{P}$, яка позначається $dim(\mathbf{P})$, є мінімальним числом лінійних розширень на $\mathbf{P}$, перетин яких є '$\preceq$'. У цій статті ми доводимо, що $dim(\mathcal{F}) \leq \delta_k(P_n^{+k})$, де $\mathcal{F}$ є образом k-однорідного dcsl k-однорідного графа, позначеного P_n^{+k} $(n \geq 1, k \geq 1)$ на '$n(k+1)$' вершинах.

Ключові слова і фрази: k-однорідний dcsl індекс, розмірність множини з частковим порядком, решітка.