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EXTENSIONS OF MULTILINEAR MAPPINGS TO POWERS OF LINEAR SPACES

We consider the question of the possibility to recover a multilinear mapping from the restriction to the diagonal of its extension to a Cartesian power of a space.

Key words and phrases: multilinear mapping, polarization formula.

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INTRODUCTION

Let X and Y be linear spaces over the same field \mathbb{K} . It is well-known (see e. g. [1, Theorem 1.10]) that every *symmetric* n -linear mapping $A : X^n \rightarrow Y$ can be recovered from its restriction to the diagonal $\widehat{A} : X \rightarrow Y$, $\widehat{A}(x) = A(x, \dots, x)$, by means of the so-called Polarization Formula:

$$A(x_1, \dots, x_n) = \frac{1}{n!2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n = \pm 1} \varepsilon_1 \dots \varepsilon_n \widehat{A}(\varepsilon_1 x_1 + \dots + \varepsilon_n x_n).$$

But in general if A is non-symmetric, it cannot be recovered from \widehat{A} . For example, if A is alternating, then \widehat{A} is equal to zero. Let us recall that A is called *alternating* if $A(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = (-1)^\sigma A(x_1, \dots, x_n)$ for every $x_1, \dots, x_n \in X$ and $\sigma \in S_n$, where S_n is the group of all permutations of n elements and $(-1)^\sigma$ is the sign of the permutation σ .

In [1, p. 8] it has been introduced mappings between complex linear spaces, which are linear with respect to some arguments and antilinear with respect to other arguments. If such a mapping is symmetric with respect to “linear” and “antilinear” arguments separately, then it can be recovered from its restriction to the diagonal by means of polarization formulas, proved in [2] and [3]. Note that in this case there are no any requirements of symmetry between “linear” and “antilinear” arguments. In some cases for multilinear mappings there is a similar situation. For example, if $A : X^n \rightarrow Y$ is an n -linear mapping, then a mapping $\widetilde{A} : (X^n)^n \rightarrow Y$, defined by

$$\widetilde{A}(x_1, \dots, x_n) = A(x_1^{(1)}, \dots, x_n^{(n)}),$$

where $x_j = (x_j^{(1)}, \dots, x_j^{(n)}) \in X^n$, $j \in \{1, \dots, n\}$, is an n -linear mapping too (in general, non-symmetric) and its restriction to the diagonal $\widetilde{A}(x)$ is equal to $A(x^{(1)}, \dots, x^{(n)})$ for $x = (x^{(1)}, \dots, x^{(n)}) \in X^n$. Therefore, A and, consequently, \widetilde{A} , can be recovered from the restriction of \widetilde{A} to the diagonal.

We consider the question of the possibility of recovering of a multilinear mapping from the restriction to the diagonal of its extension to a power of a space.

УДК 517.98

2010 *Mathematics Subject Classification:* 46J20, 46E15.

1 THE MAIN RESULT

Let $M = (m_{ij})_{i,j=1}^n$ be a matrix of scalars from \mathbb{K} . Then for every n -linear mapping $A : X^n \rightarrow Y$ a mapping $E_M(A) : (X^n)^n \rightarrow Y$, defined by

$$E_M(A)(x_1, \dots, x_n) = A(m_{11}x_1^{(1)} + \dots + m_{1n}x_1^{(n)}, \dots, m_{n1}x_n^{(1)} + \dots + m_{nn}x_n^{(n)}),$$

where $x_1, \dots, x_n \in X^n$, is an n -linear mapping. Its restriction to the diagonal is equal to

$$\widehat{E_M(A)}(x) = \sum_{k_1=1}^n \dots \sum_{k_n=1}^n m_{1k_1} \dots m_{nk_n} A(x^{(k_1)}, \dots, x^{(k_n)}). \quad (1)$$

Note that if $m_{ij} = 1, i = 1, \dots, n$, for the fixed $j \in \{1, \dots, n\}$, then $E_M(A)$ is an extension of A .

Proposition 1.1. For every n -linear alternating mapping $A : X^n \rightarrow Y$,

$$\widehat{E_M(A)}(x) = \det(M)A(x^{(1)}, \dots, x^{(n)}),$$

where $x = (x^{(1)}, \dots, x^{(n)}) \in X^n$.

Proof. Since A is alternating, $A(x^{(k_1)}, \dots, x^{(k_n)}) = 0$ if $k_l = k_s$ for some $l \neq s$. Therefore, by (1),

$$\widehat{E_M(A)}(x) = \sum_{\sigma \in S_n} m_{1\sigma(k_1)} \dots m_{n\sigma(k_n)} A(x^{(\sigma(1))}, \dots, x^{(\sigma(n))}).$$

Since $A(x^{(\sigma(1))}, \dots, x^{(\sigma(n))}) = (-1)^\sigma A(x^{(1)}, \dots, x^{(n)})$, therefore

$$\widehat{E_M(A)}(x) = \sum_{\sigma \in S_n} (-1)^\sigma m_{1\sigma(k_1)} \dots m_{n\sigma(k_n)} A(x^{(1)}, \dots, x^{(n)}) = \det(M)A(x^{(1)}, \dots, x^{(n)}).$$

□

Let us consider recovering of multilinear mappings, which in general are neither symmetric nor alternating. It can be easily seen that if M is a diagonal matrix, then

$$\widehat{E_M(A)}(x) = m_{11} \dots m_{nn} A(x^{(1)}, \dots, x^{(n)})$$

for every n -linear mapping A . Let us construct a non-diagonal matrix M' such that every n -linear mapping A can be recovered from $\widehat{E_{M'}(A)}$. Let

$$M' = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & -1 & 1 & \dots & 1 \\ 1 & 1 & -1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & -1 \end{pmatrix}.$$

For $k \in \{1, \dots, n\}$ let $i_k : X \rightarrow X^n$, $i_k(x) = (\underbrace{0, \dots, 0}_{k-1}, x, 0, \dots, 0)$.

Theorem 1. *The n -linear mapping A can be recovered from $\widehat{E_{M'}(A)}$ by means of the formula:*

$$A(x_1, \dots, x_n) = \frac{1}{2^{2n-1}} \sum_{j_2, \dots, j_n=0}^1 (-1)^{j_2+\dots+j_n} \sum_{\varepsilon_1, \dots, \varepsilon_n=\pm 1} \varepsilon_1 \dots \varepsilon_n \widehat{E_{M'}(A)} \\ \times \left(\varepsilon_1 i_1(x_1) + \varepsilon_2 p_{j_2}^{(2)}(x_2) + \dots + \varepsilon_n p_{j_n}^{(n)}(x_n) \right), \quad (2)$$

where

$$p_{j_k}^{(k)}(x) = \begin{cases} i_1(x), & \text{if } j_k = 0, \\ i_k(x), & \text{if } j_k = 1 \end{cases}$$

for $k \in \{2, \dots, n\}$.

Proof. Let $y_1 = i_1(x_1), y_2 = p_{j_2}^{(2)}(x_2), \dots, y_n = p_{j_n}^{(n)}(x_n)$. Notice that

$$\sum_{\varepsilon_1, \dots, \varepsilon_n=\pm 1} \varepsilon_1 \dots \varepsilon_n \widehat{E_{M'}(A)}(\varepsilon_1 y_1 + \dots + \varepsilon_n y_n) \\ = \sum_{k_1, \dots, k_n=1}^n E_{M'}(A)(y_{k_1}, \dots, y_{k_n}) \sum_{\varepsilon_1, \dots, \varepsilon_n=\pm 1} \varepsilon_1 \dots \varepsilon_n \varepsilon_{k_1} \dots \varepsilon_{k_n}$$

and

$$\sum_{\varepsilon_1, \dots, \varepsilon_n=\pm 1} \varepsilon_1 \dots \varepsilon_n \varepsilon_{k_1} \dots \varepsilon_{k_n} = \begin{cases} 2^n, & \text{if } k_1 \neq \dots \neq k_n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$\frac{1}{2^n} \sum_{\varepsilon_1, \dots, \varepsilon_n=\pm 1} \varepsilon_1 \dots \varepsilon_n \widehat{E_{M'}(A)}(\varepsilon_1 y_1 + \dots + \varepsilon_n y_n) = \sum_{\sigma \in S_n} E_{M'}(A)(y_{\sigma(1)}, \dots, y_{\sigma(n)}).$$

For $\sigma \in S_n$ such that $\sigma(n) = n$ we have

$$\sum_{j_n=0}^1 (-1)^{j_n} E_{M'}(A)(y_{\sigma(1)}, \dots, y_{\sigma(n-1)}, y_{\sigma(n)}) = \sum_{j_n=0}^1 (-1)^{j_n} E_{M'}(A)(y_{\sigma(1)}, \dots, y_{\sigma(n-1)}, p_{j_n}^{(n)}(x_n)) \\ = E_{M'}(A)(y_{\sigma(1)}, \dots, y_{\sigma(n-1)}, i_1(x_n)) - E_{M'}(A)(y_{\sigma(1)}, \dots, y_{\sigma(n-1)}, i_n(x_n)) \\ = 2E_{M'}(A)(y_{\sigma(1)}, \dots, y_{\sigma(n-1)}, i_1(x_n)).$$

For $\sigma \in S_n$ such that $\sigma(n) \neq n$ we have

$$\sum_{j_n=0}^1 (-1)^{j_n} E_{M'}(A)(y_{\sigma(1)}, \dots, y_{\sigma(n)}) = \sum_{j_n=0}^1 (-1)^{j_n} (y_{\sigma(1)}, \dots, p_{j_n}^{(n)}(x_n), \dots, y_{\sigma(n)}) \\ = E_{M'}(A)(y_{\sigma(1)}, \dots, i_1(x_n), \dots, y_{\sigma(n)}) - E_{M'}(A)(y_{\sigma(1)}, \dots, i_n(x_n), \dots, y_{\sigma(n)}) = 0.$$

Therefore, the right-hand side of (2) is equal to

$$\frac{1}{2^{n-2}} \sum_{j_2, \dots, j_{n-1}=0}^1 (-1)^{j_2+\dots+j_{n-1}} \sum_{\sigma \in S_n, \sigma(n)=n} E_{M'}(A)(y_{\sigma(1)}, \dots, y_{\sigma(n-1)}, i_1(x_n)).$$

After applying this method $n-1$ times we obtain that the right-hand side of (2) is equal to $E_{M'}(A)(i_1(x_1), i_1(x_2), \dots, i_1(x_n))$, which is equal to $A(x_1, x_2, \dots, x_n)$. \square

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Received 30.10.2016

Василишин Т.В. *Продовження мультилінійних відображень на степені лінійних просторів // Карпатські матем. публ.* — 2016. — Т.8, №2. — С. 211–214.

Розглянуто питання можливості відновлення мультилінійного відображення зі звуження на діагональ продовження цього відображення на деякий декартів степінь простору.

Ключові слова і фрази: мультилінійне відображення, поляризаційна формула.