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FAMILY OF WAVELET FUNCTIONS ON THE GALOIS FUNCTION BASE

We construct a family of wavelet systems on the Galois function base. We research and prove properties of systems of the built family.

Key words and phrases: wavelet, Galois function, scaling function, wavelet function.

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INTRODUCTION

The main methods of solving problems of a digital signal processing are spectral analysis, synthesis, filtering, coding and compressing based on discrete orthogonal transforms and wavelet transforms [1–5]. The signal is presented in the form of a function of time. Wavelet transforms may be considered as time-frequency representations or decompositions of a signal. A signal decomposition can be done by the basis built from a single wavelet function using wavelet scale changes and shifts [1–4, 6]. Each function of the basis describes some frequency of the signal and its location in the time domain.

An important step of a wavelet analysis is the choice of transform basis which depends on the processing tasks and on the signal. The problem of choice of a basis and the wavelet transform based on it is rather relevant and it is being researched subject.

The paper [4] systematizes basises of wavelet functions and wavelet transforms, but the problem of choice of a wavelet is solved only partially [1–4, 6]. For descrete analysis the wavelets of Daubechies, Haar, Meyer, Coifman, symlets, biorthogonal wavelets and wavelet-packet Walsh functions are used [1–4, 6].

To solve practical problems orthogonal or symmetric wavelets with compact carrier that ensure efficient transform algorithm can be chosen. But wavelets that simultaneously satisfy all of this properties are unknown. The only symmetric orthogonal wavelets with compact support are Haar wavelets but they do not satisfy the given processing qualities in many problems. To ensure symmetry multivalued biorthogonal wavelets are used. Daubechies wavelets are much smoother than Haar wavelets but they are multivalued and do not have analytical expression that complicates the process of their forming and calculation transforming.

From the recursively ordered Walsh system the Galois functions are generated [5], the latter take only two values (± 1) and the sequence of values is in full correlation. These features can provide simple algorithms for information processing in the basis based on Galois functions [5], but the researches of the Galois functions properties in various spaces and the possibility of its application for wavelet transform have not been done yet.

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Thus performing of a time-frequency analysis and processing of a broad class of one-dimensional signals with finite and limited energy, mathematical models of which are functions in space $L_2([0,T))$, necessitated the construction of wavelet basis on the base of Galois functions in this space and research their properties.

The goal of this article is the construction of a family of wavelet systems based on mother or generating Galois functions and proving properties of the constructed systems to create basises for discrete wavelet transform in the space $L_2([0,T))$.

The article provides the results of building of wavelet systems based on Galois functions, of synthesed scaling functions for built wavelets systems and proves the required properties of wavelets basises in the space $L_2([0,T))$.

1 DEFINING WAVELET SYSTEM ON GALOIS FUNCTIONS BASE

For the purpose of constructing of a system of wavelet functions for discrete transforms of signals presented by functions $f \in L_2([0,T))$ as a mother wavelet the first function $Gal_{n,0}(\theta)$ of a recursively ordered Galois system, which is defined in [5] is used.

The Galois functions system [5, p. 46] with the recursive ordering [5, p. 36] $\{Gal_{n,i}(\theta)\}$, $\theta \in [0,M)$ is defined according to the generating vector of Galois field $GF(2^n)$ from a recursive sequence or a recursive orderly system of Walsh functions [5, p. 36], where $M = 2^n$, $M \leq T$, $n = 1,2,3,\ldots$ is a degree of irreducible polynomial Galois fields $GF(2^n)$; $i = 0,1,\ldots 2^n - 1$. Examples of creating recursive sequences are shown in the following text.

Example 1. *Vector of coefficients* $(p_0,p_1,p_2) = (1,1,1)$ *corresponds to irreducible polynomial* x^2+x+1*, which generates Galois field* $GF(2^2)$*. Non-zero elements of vector determine the rule* $p_{i+2} = p_i \oplus p_{i+1}$ *for the formation of a recursive sequence. Initial vector with unitary elements* $(v_0,v_1) = (1,1)$ *is chosen as a primary vector. From the primary vector according to this rule* $v_{i+2} = v_i \oplus v_{i+1}$ *there are defined the elements of a recursive sequence wich are repeated with period* 2^n-1*. Fragment of* $n-1$ *zero elements of the sequence is supplemented by one zero. Elements of supplemented sequence are denoted as* g_i*:*

$$\{0, v_{i+2}, v_i, v_{i+1}\} = \{g_0, g_1, g_2, g_3\} = \{0,0,1,1\},$$

where $\oplus$ *denotes the addition modulo two.*

Example 2. *Vector of coefficients* $(p_0,p_1,p_2,p_3) = (1,1,0,1)$ *corresponds to irreducible polynomial* x^3+x^2+1*, which generates Galois field* $GF(2^3)$*. This vector also determines the rule* $p_{i+3} = p_i \oplus p_{i+1}$ *for the formation of a recursive sequence. From the initial vector* $(v_0,v_1,v_2) = (1,1,1)$ *according to the rule* $v_{i+3} = v_i \oplus v_{i+1}$ *there are defined the elements of a recursive sequence, supplemented by zero and submitted the following fragment:*

$$\{0, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6}, v_i, v_{i+1}, v_{i+2}\} = \{g_0, g_1, g_2, \ldots, g_7\} = \{0,0,0,1,0,1,1,1\}.$$

Example 3. *Vector of coefficients* $(p_0,p_1) = (1,1)$ *corresponds to irreducible polynomial* $x+1$*, which generates Galois field* $GF(2^1)$*. This vector also determines the rule* $p_{i+1} = p_i$ *for the formation of a recursive sequence. From the initial vector* $(v_0) = (1)$ *according to this rule* $v_{i+1} = v_i \oplus 1$ *there are defined the elements of a recursive sequence, supplemented by zero and submitted the following fragment:* $\{0, v_i\} = \{g_0, g_1\} = \{0,1\}$.

Elements of the fragment of a recursive sequence supplemented by zero are signed as $\{g_0, g_1, g_2, \ldots, g_{2^n-1}\}$.

The value of the first function $Gal_{n,0}(\theta)$ of a recursively ordered Galois system $\{Gal_{n,i}(\theta)\}$ of order n at the points $\theta = \theta_j = j$ in the interval $\theta \in [0, M)$ is obtained from an element of a recursive sequences fragment via transform

$$Gal_{n,0}(\theta_j) = 1 - 2g_j, \tag{1}$$

where $j = 0, 1, \ldots 2^n - 1$, g_j — elements of a fragment of a recursive sequence.

In the intervals $\theta \in [j, j+1)$ functions $Gal_{n,0}(\theta)$ are continuous constants and take values

$$Gal_{n,0}(\theta) = Gal_{n,0}(\theta_j). \tag{2}$$

Since $g_j = 1$ or $g_j = 0$, therefore according to (1) and (2) functions $Gal_{n,0}(\theta) = \pm 1$.

Each next function of Galois system $\{Gal_{n,i}(\theta)\}$ is received from the previous unit cyclic shift either left or right by $\theta = 1$ [5], so the first function can create two different systems. For each irreducible polynomial of Galois field $GF(2^n)$ or generating vector several systems Galois functions can be built.

These functions $Gal_{n,0}(\theta)$ are defined as mother wavelets for systems of order n

$$Gal_n(\theta) = Gal_{n,0}(\theta).$$

Mother wavelet $Gal_n(\theta)$ is defined in the interval $[0, M)$, outside this interval the function $Gal_n(\theta) = 0$.

The norm of function $Gal_n(\theta)$ equals $\|Gal_n(\theta)\| = (\int\limits_0^M Gal_n^2(\theta)\, d\theta)^{\frac{1}{2}}$. Wavelet-functions must have unitary norm $\|Gal_n(\theta)\| = 1$, that is why function values are $Gal_n(\theta) = \pm\sqrt{\frac{1}{2^n}}$.

The graphics mother Galois wavelets $Gal_1(\theta)$, $Gal_2(\theta)$, $Gal_3(\theta)$, $Gal_4(\theta)$ are shown in fig. 1 — fig. 4 accordingly.

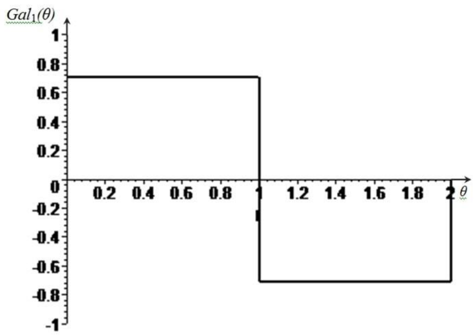

Figure 1: Galois wavelet, $n = 1$.

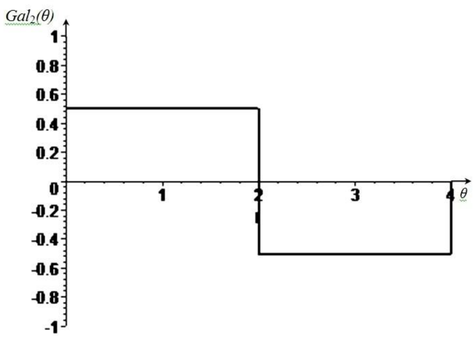

Figure 2: Galois wavelet, $n = 2$.

On the basis of each mother function $Gal_n(\theta)$ with the help of scale and parralel shift a system of wavelet-function is formed and defined as

$$Gal_{n,m,k}(t) = 2^{\frac{m-1}{2}} Gal_n(2^{m-1}t - Nk), \tag{3}$$

where $t = \frac{N}{M}\theta$; $N = 2^p$ is the quantity of functions in the system; $p = 1, 2, 3, \ldots$; $m = 0, 1, \ldots, \log_2 N + 1$; $k = 0, 1, \ldots, N \cdot 2^{-m}$.

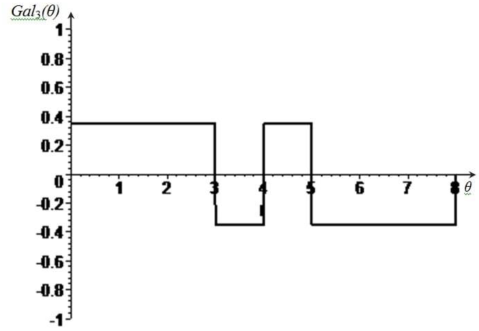

Figure 3: Galois wavelet, $n = 3$.

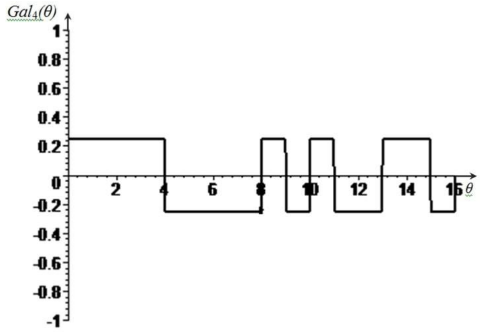

Figure 4: Galois wavelet, $n = 4$.

Non-normalized functions are $Gal_{n,m,k}(t) = \pm 1$ for $t \in [0, T)$, $T = N$ and $Gal_{n,m,k}(t) = 0$ for other t.

Normalized functions $Gal_{n,m,k}(t) = \pm\sqrt{\frac{2^{m-1}}{N}}$ are piecewise constants in intervals $t \in \left[\frac{q}{l}, \frac{q+1}{l}\right)$, where $q = 0, 1, \ldots, lN - 1$; $l = 2^{n-1}$.

The graphics of eight wavelet functions $\{Gal_{2,m,k}(t)\}$ built by the formula (3) from mother Galois wavelet $Gal_2(\theta)$ are shown in fig. 5.

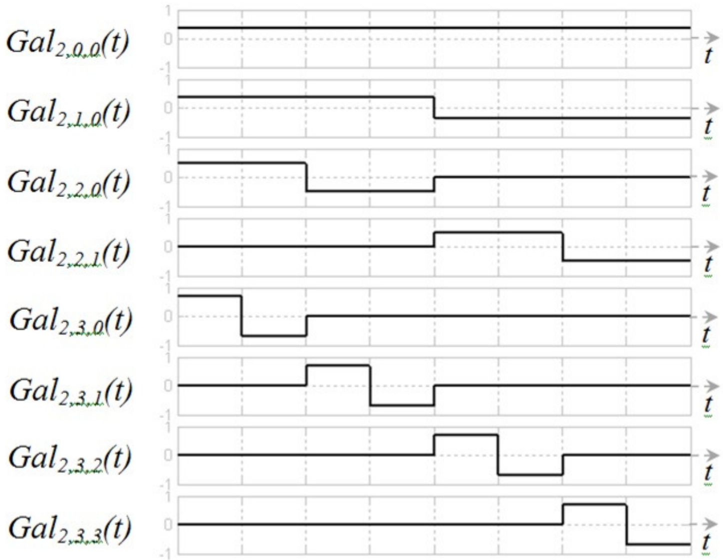

Figure 5: Graphics of wavelet functions of two-order system with mother Galois wavelet.

The graphics of eight wavelet functions $\{Gal_{3,m,k}(t)\}$, built by the formula (3) from mother Galois wavelet $Gal_3(\theta)$ are shown in fig. 6.

The set $\{Gal_{n,m,k}(t)\}$ of systems, based on mother wavelets for different values of $n = 1, 2, 3, \ldots$ forms a family of wavelet functions on the Galois functions basis.

From the result of construction of wavelet functions according (1) — (2) and fig. 1 — fig. 2 we can conclude that mother wavelets $Gal_1(\theta)$ i $Gal_2(\theta)$ of systems by orders $n = 1$ and $n = 2$ are Haar wavelets and the system wavelet functions built on their basis (fig. 5) is an orthogonal Haar system.

It is known that Haar system or Haar wavelet functions is the orthonormal basis [1–6] in the space $L_2\left([0, T)\right)$, that is why in this paper proving properties and synthesis of scaling functions will be done for cases $n \geq 3$.

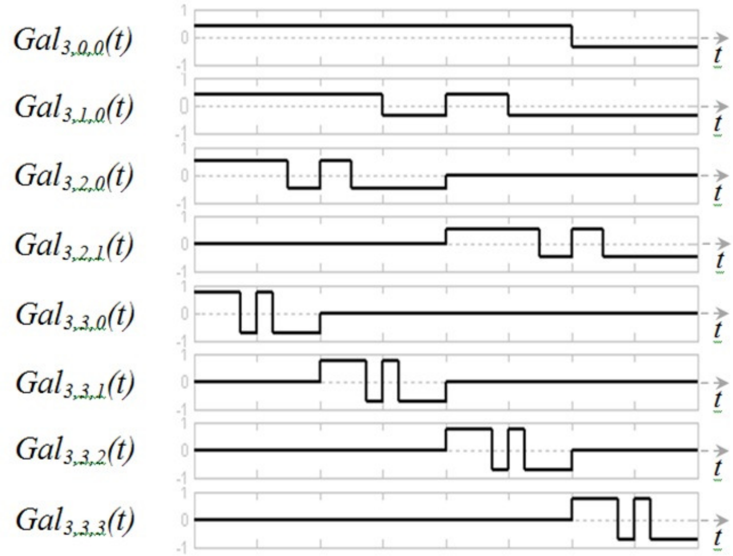

Figure 6: Graphics of wavelet functions of third-order system with mother Galois wavelet, $n = 3$.

2 SYNTHESIS OF SCALING FUNCTION

To execute the multiresolution decomposition [6, p. 86] or multiresolution analysis and to record wavelet transform in the filter form the scaling functions are used.

Scaling functions must form the basis, in which mother wavelet decomposes [1,3,4,6].

To build scaling functions for Galois wavelets a well known method of construction of scaling functions for Haar systems [3,6] is used.

For mother wavelet $Gal_n(\theta)$ the scaling function $\varphi(\theta)$ is defined as

$$\varphi(\theta) = \begin{cases} 1, \theta \in [0,1), \\ 0, \theta \in [1, M). \end{cases}$$

In the space $L_2(R)$ there is build the system of functions $\varphi_{0,b}(\theta)$, $b \in Z$, received from $\varphi(\theta)$ by shifts on integer number b

$$\varphi_{0,b}(\theta) = \varphi(\theta - b).$$

Space in $L_2(R)$, being generated by linear combinations of shift functions, is a closure of linear span of system $\varphi_{0,b}(\theta)$, signed V_0. Obviously, the system $\varphi_{0,b}(\theta)$ forms an orthonormal basis of space V_0.

On the next step a system of functions $\varphi_{1,b}(\theta)$ is created by scaling and shifting of function $\varphi_{0,b}(\theta)$

$$\varphi_{1,b}(\theta) = \sqrt{2}\varphi(2\theta - b).$$

System $\varphi_{1,b}(\theta)$ creates an orthonormal basis in space V_1, which is the closure of the linear span of the system $\varphi_{1,b}(\theta)$.

Function $\varphi(\theta) \in V_0$ is a linear combination of elements of space V_1

$$\begin{aligned} \varphi(\theta) &= \varphi(2\theta) + \varphi(2\theta - 1), \\ \varphi(\theta) &= \frac{1}{\sqrt{2}}\varphi_{1,0}(2\theta) + \frac{1}{\sqrt{2}}\varphi_{1,1}(2\theta - 1). \end{aligned} \tag{4}$$

On the next step there is built a space V_2, generated by functions

$$\varphi_{2,b}(\theta) = 2\varphi(2^2\theta - b).$$

For constructed spaces V_0, V_1, V_0 insertion $V_0 \subset V_1 \subset V_2$ is right. The procedure of construction of functions system is extended for any $k \in Z$. It results in a constructed orthonormal functions system

$$\varphi_{k,b}(\theta) = \sqrt{2^k}\varphi(2^k\theta - b).$$

There are the following inclusion of spaces $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_k$.

According to the definition [6, p. 76] the function $\varphi(\theta) \in L_2(R)$ is called a scaling function if it can be presented in the following form

$$\varphi(\theta) = \sqrt{2}\sum_{s\in Z} h_s\varphi(2\theta - s),$$

where numbers h_s satisfy the condition $\sum\limits_{s\in Z} |h_s|^2 < \infty$.

Decomposition (4) proves performing of the scaling function definition for $\varphi(\theta)$.

Mother wavelet $Gal_n(\theta)$ is decomposed into the functions system $\{\varphi(2\theta)\}$

$$Gal_n(\theta) = \sqrt{2}\sum_{s=0}^{2^{n+1}-1} h_s\varphi(2\theta - s),$$

where the coefficients h_s are called filters.

Example 4. *Non-normalized wavelet function $Gal_3(\theta)$ is decomposed in the system of scaling functions $\varphi(2\theta)$ by the following way*

$$\begin{aligned}Gal_3(\theta) = {} & 1\cdot\varphi(2\theta) + 1\cdot\varphi(2\theta-1) + 1\cdot\varphi(2\theta-2) + 1\cdot\varphi(2\theta-3) + 1\cdot\varphi(2\theta-4)\\ & + 1\cdot\varphi(2\theta-5) + (-1)\cdot\varphi(2\theta-6) + (-1)\cdot\varphi(2\theta-7) + 1\cdot\varphi(2\theta-8) + 1\cdot\varphi(2\theta-9)\\ & + (-1)\cdot\varphi(2\theta-10) + (-1)\cdot\varphi(2\theta-11) + (-1)\cdot\varphi(2\theta-12) + (-1)\cdot\varphi(2\theta-13)\\ & + (-1)\cdot\varphi(2\theta-14) + (-1)\cdot\varphi(2\theta-15).\end{aligned}$$

The corresponding filters are $h_0 = 1, h_1 = 1, h_2 = 1, h_3 = 1, h_4 = 1, h_5 = 1, h_6 = -1, h_7 = -1, h_8 = 1, h_9 = 1, h_{10} = -1, h_{11} = -1, h_{12} = -1, h_{13} = -1, h_{14} = -1, h_{15} = -1.$

3 PROPERTIES OF WAVELET SYSTEMS BASED ON GALOIS FUNCTIONS IN $L_2([0,T])$

The wavelets system (3) based on Galois functions may be used as a basis for wavelet transforms if the following properties of wavelet basises are performed [3,4,6]:

1) it has a compact carrier (a finite time interval);

2) it has at least one zero moment;

3) a basis is orthogonal or it is a Riesz basis.

These properties for systems of wavelets with mother Galois functions are proved by the following propositions.

1) The existence of a compact carrier of a wavelet.

It is known that the function $f(t)$ has a compact carrier if $f(t) = 0$ for $t < a$ or $t > b$, where $-\infty < a < b < \infty$ [3, p. 15]. Wavelets with a compact carrier have a finite number of nonzero coefficients of expansion.

Proposition 1. *Mother wavelet $Gal_n(\theta)$ of Galois wavelet system has a compact carrier.*

Proof. According to the definition (1)—(2) function $Gal_n(\theta)$ in interval $[0, M)$ is piecewise constant, it has non-zero values $Gal_n(\theta) = \pm\sqrt{\frac{1}{2^n}}$ and outside the interval its value equals zero, therefore it has a compact carrier. □

2) The existence of one zero moment.

According to the definition [6, p. 129], function $f(t) \in L_2(R)$ has L zero moment if equality is satisfied

$$\int_{-\infty}^{\infty} t^r f(t)\, dt = 0 \tag{5}$$

for all integers $r = 0, 1, \ldots, L-1$. If the mother-wavelet has successive moments equal to zero the wavelet coefficients decrease quickly.

Proposition 2. *The mother wavelet $Gal_n(\theta)$ of the Galois wavelet system has one zero moment*

$$\int_{-\infty}^{\infty} Gal_n(\theta)\, d\theta = 0.$$

Proof. According to the property of Galois function [5] it is

$$\int_{0}^{M} Gal_n(\theta)\, d\theta = 0,$$

and outside the interval $[0, M)$ value of function is zero.

Sums of lengths the intervals where $Gal_n(\theta) = \sqrt{\frac{1}{2^n}}$ and $Gal_n(\theta) = -\sqrt{\frac{1}{2^n}}$ are equal. Therefore, according to the definition (5) functions $Gal_n(\theta)$ have a zero moment and satisfy the basic requirements for wavelet functions. However, there is only one zero moment because the direct checking shows that

$$\int_{-\infty}^{\infty} \theta Gal_n(\theta)\, d\theta \neq 0.$$

□

3) Orthogonality of system or Riesz basis. Built systems $\{Gal_{1,m,k}(t)\}$ and $\{Gal_{2,m,k}(t)\}$ coincide with the orthogonal Haar system. Built systems $\{Gal_{n,m,k}(t)\}$ for $n = 3, 4, \ldots$ are nonorthogonal. We know that the demand for orthogonality of wavelets system may be weakened, but it is necessary for the system to form the Riesz basis [2–4, 6].

According to the definition [6, p. 111] system $\varphi_v(t)$ in Hilbert space H called Riesz basis if there are such positive constants A i B that for any element $f(t) \in H$ the following inequality is performed

$$A\|f(t)\|^2 \leqslant \sum_{v=1}^{\infty} |\langle f(t), \varphi_v(t)\rangle|^2 \leqslant B\|f(t)\|^2. \tag{6}$$

Proposition 3. *System $\{Gal_{n,m,k}(t)\}$ is the Riesz basis in space $L_2\left([0,T)\right)$.*

Proof. To prove that the properties (6) of wavelet systems with mother Galois functions form Riesz basis, it must be established that there are such constants A i B, $0 < A \le B < \infty$ for which the inequality is performed

$$A\|f(t)\|^2 \leqslant \sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 \leqslant B\|f(t)\|^2, \tag{7}$$

where $\|f(t)\|^2 = \int\limits_0^N f^2(t)\,dt$, $v = 1, 2, \ldots, N$ is serial number of the wavelet in the system $\{Gal_{n,m,k}(t)\}$.

Numbers m and k in the system $\{Gal_{n,m,k}(t)\}$ with the triple numeration are connected with the serial number v of the wavelet by the formula $v = 2^{m-1} + k + 1$.

Since the number of functions in the proposed system is finite and equals N, the sum in the middle of inequality (7) contains a finite number of components

$$\sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 = \sum_{v=1}^{N} \left| \int\limits_0^T f(t) \cdot Gal_v(t)\,dt \right|^2.$$

With Bunyakovsky inequality $\left(\int\limits_a^b x(t) \cdot y(t)\,dt\right)^2 \le \int\limits_a^b x^2(t)\,dt \int\limits_a^b y^2(t)\,dt$ for any $x(t)$, $y(t)$ an assessment of the latter expression and following transforms there are performed

$$\begin{aligned}\sum_{v=1}^{N} \left(\int\limits_0^T f(t) \cdot Gal_v(t)\,dt\right)^2 &\le \sum_{v=1}^{N} \left(\int\limits_0^T f^2(t)\,dt \cdot \int\limits_0^T Gal_v^2(t)\,dt\right) \\ &= \|f(t)\|^2 \cdot \sum_{v=1}^{N} \|Gal_v(t)\|^2.\end{aligned}$$

Functions $\{Gal_v(t)\}$ are normalized and the norm is $\|Gal_v(t)\| = 1$. Selection of the first and the last expressions in latest inequality sets the ratio

$$\sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 \le \|f(t)\|^2 \cdot \sum_{v=1}^{N} \|Gal_v(t)\|^2 = \|f(t)\|^2 \cdot N.$$

So there exists the constant $N > 0$ and the right side of inequality (7) is proved. On the other hand, we must prove that there exists a constant $A > 0$ and there performs the inequality

$$A\|f(t)\|^2 \le \sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 \text{ or} \tag{8}$$

$$A \le \frac{\sum\limits_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2}{\|f(t)\|^2}. \tag{9}$$

Since function $f(t)$ is bounded and it is designated as $p \le f(t) \le P$, then the following inequalities are executed $\int\limits_{q}^{q+1} f(t)\,dt \ge \int\limits_{q}^{q+1} p\,dt$ and $\int\limits_{q}^{q+1} (-f(t))\,dt \ge \int\limits_{q}^{q+1} (-P)\,dt$.

Since normalized functions $Gal_v(t) = \pm\sqrt{\frac{2^{m-1}}{N}}$ are piecewise constants in intervals $t \in \left[\frac{q}{l}, \frac{q+1}{l}\right)$, $q = 0, 1, \dots, lN-1$ and each function $Gal_v(t) = Gal_{n,m,k}(t) \ne 0$ is not zero-value in the interval $t \in \left[\frac{k}{2^{m-log_2 N-1}}, \frac{k+1}{2^{m-log_2 N-1}}\right)$, then

$$\sum_{v=1}^{N} |\langle f(t), Gal_v(t)\rangle|^2 = \sum_{v=1}^{N} \left| \int\limits_{0}^{T} f(t) Gal_v(t)\,dt \right|^2 = \sum_{v=1}^{N} \left| \int\limits_{\frac{k}{2^{m-log_2 N-1}}}^{\frac{k+1}{2^{m-log_2 N-1}}} f(t) Gal_{n,m,k}(t)\,dt \right|^2 .$$

Assume designation $I_1 = \cup \left[\frac{q_s}{l}, \frac{q_s+1}{l}\right)$ — for combining intervals, where values of functions are $Gal_v(t) = \sqrt{\frac{2^{m-1}}{N}}$, and $I_2 = \cup \left[\frac{q_r}{l}, \frac{q_r+1}{l}\right)$ — for combining intervals, where values of functions are $Gal_v(t) = -\sqrt{\frac{2^{m-1}}{N}}$, $s = 0, 1, \dots, lN-1$, $r = 0, 1, \dots, lN-1$.

$$\begin{aligned}
&\sum_{v=1}^{N} \left| \int\limits_{\frac{k}{2^{m-log_2 N-1}}}^{\frac{k+1}{2^{m-log_2 N-1}}} f(t) Gal_{n,m,k}(t)\,dt \right|^2 = \sum_{v=1}^{N} \left| \int\limits_{I_1} \sqrt{\frac{2^{m-1}}{N}} f(t)\,dt + \int\limits_{I_2} \sqrt{\frac{2^{m-1}}{N}} (-f(t))\,dt \right|^2 \\
&= \sum_{v=1}^{N} \frac{2^{m-1}}{N} \left| \left(\int\limits_{I_1} f(t)\,dt + \int\limits_{I_2} (-f(t))\,dt \right) \right|^2 \ge \sum_{v=1}^{N} \frac{2^{m-1}}{N} \left| \left(\int\limits_{I_1} p\,dt + \int\limits_{I_2} (-P)\,dt \right) \right|^2 \\
&= \sum_{v=1}^{N} \frac{2^{m-1}}{N} \left| \left(p \int\limits_{I_1} dt + (-P) \int\limits_{I_2} dt \right) \right|^2 = \sum_{v=1}^{N} \frac{2^{m-1}}{N} \left| \left(p \frac{N}{2^m} + (-P) \frac{N}{2^m} \right) \right|^2 \\
&= \sum_{v=1}^{N} \frac{2^{m-1} N^2}{N 2^{2m}} (p-P)^2 = 2^{-m-1} N^2 (p-P)^2.
\end{aligned}$$

The function $f^2(t)$ is bounded. It is assumed that $f^2(t) \le S$, $S \in R$, then the inequality is executed

$$\int\limits_{0}^{T} f^2(t)\,dt \le \int\limits_{0}^{T} S\,dt = S \cdot N.$$

Substituting the last result in (9) allows to reach the following conclusion: when choosing $A \le \frac{N(p-P)^2}{2^{m+1} S}$ inequalities (8) i (7) are performed. The statement is proved. □

According to proven propositions 1 — 3 mother Galois wavelet functions have a compact carrier, one vanishing zero moment, wavelet function systems $Gal_{n,m,k}(t)$ for different n form Riesz basises, that satisfy the necessary conditions for wavelet basises in space $L_2([0,T])$.

4 Conclusions

Thus, it was proved that the first functions of recursively ordered Galois systems are mother or generating wavelets. There was synthesized the orthogonal scaling functions system in which mother wavelets decompose.

On the basis of mother wavelets of different orders n there were built wavelet functions systems. The set of built systems is a family of wavelet functions that are generated by Galois functions.

The article also proves necessary conditions (properties) of wavelet system for the built system. It is proved that each system of family is the Riesz basis. The proved conditions enable using wavelet systems with generating functions Galois as basises of discrete wavelet transforms in the space $L_2\left([0,T)\right)$. A significant advantage of implementation of these transforms compared to others is that all the basic functions are piecewise constant and take only two values.

Transforms in built basises can be used for analysis and processing of one-dimensional signals with finite energy.

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Ключові слова і фрази: вейвлет, функція Ґалуа, масштабуюча функція, система вейвлет-функцій.