ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2018, **10** (1), 14–30
doi:10.15330/cmp.10.1.14-30

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2018, Т.10, №1, С.14–30

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THE NONLOCAL PROBLEM FOR THE $2n$ DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATOR COEFFICIENTS AND THE INVOLUTION

We study a problem with periodic boundary conditions for a $2n$-order differential equation whose coefficients are non-self-adjoint operators. It is established that the operator of the problem has two invariant subspaces generated by the involution operator and two subsystems of the system of eigenfunctions which are Riesz bases in each of the subspaces. For a differential-operator equation of even order, we study a problem with non-self-adjoint boundary conditions which are perturbations of periodic conditions. We study cases when the perturbed conditions are Birkhoff regular but not strongly Birkhoff regular or nonregular. We found the eigenvalues and elements of the system V of root functions of the operator which is complete and contains an infinite number of associated functions. Some sufficient conditions for which this system V is a Riesz basis are obtained. Some conditions for the existence and uniqueness of the solution of the problem with homogeneous boundary conditions are obtained.

Key words and phrases: operator of involution, differential-operator equation, eigenfunctions, Riesz basis.

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1 INTRODUCTION

The theory of differential equations with an unbounded operator coefficient was initiated by Hill and Yosida where the first theorems on the existence of the Cauchy problem solution for a linear homogeneous differential equation with respect to a function with values in a Banach space were obtained. Among works on this subject should be noted works of Kato T. , Krein S.G., Mizohata S., Phillips R.S.

The boundary value problems for linear differential-operator equations are used in the simulation of boundary value problems for differential equations with partial derivatives, in particular, in the study of nonlocal problems. Significant results concerning the theory of boundary value problems for differential-operator equations were obtained in the papers of Vishik M.I., Boehner M., Gorbachuk V.I. and Gorbachuk M.L., Dezin O.O., Dubinsky Yu.V., Kochubei A.N., Lions J.-L., Mamedov K.S., Romanko V.K., Shakhmurov Veli B., Triebel Kh., Yakubov S., Yurchuk N.Yu.

During recent years the number of publications with the use of an involution operator in various sections of the theory of ordinary differential equations (see [2, 8–10, 12, 13, 15, 16]), partial differential equations (see [1, 7, 11, 14, 17, 18]) and differential equations with operator coefficients (see [3–6]) increased significantly.

In our article we will use the following notations. Let H be a separable Hilbert space and $A : D(A) \subset H \to H$ be the closed unbounded linear operator with the discrete spectrum

УДК 517.927.5, 517.984.5
2010 *Mathematics Subject Classification:* 34G10,34K10,34K30,34L10.

$\sigma(A) \equiv \{z_k \in \mathbb{R},\ z_k = \alpha(k)^\gamma,\ \alpha, \gamma > 0,\ k = 1,2,\dots\}$. We denote by $V(A) \equiv \{v_k \in H : k = 1,2,\dots\}$ the system of the eigenfunctions of A which forms a Riesz basis in H, by $W(A) \equiv \{w_m \in H : m = 1,2,\dots\}$ the biorthogonal system of the functions in the sense of equalities $(v_k, w_m; H) = 0, k \neq m,\ (v_k, w_k; H) = 1,\ k,\ m = 1,2,\dots;\ H(A^s) \equiv \{h \in H : A^s h \in H\}$, $s \geq 0$. Let $H_1 \equiv L_2((0,1), H)$ and $D_x : H_1 \to H_1$ is a strong derivative in the space H_1; $\left\| \frac{u(x+\triangle x)-u(x)}{\triangle x} - D_x u; H_1 \right\| \to 0, \triangle x \to 0$. Denote by $H_2 \equiv \{u \in H_1 : D_x^{2n} u \in H_1,\ A^{2n} u \in H_1\}$; by $[H]$ the algebra of the bounded linear operators $B : H \to H$. Denote by $H_0 \equiv L_2(0,1)$; let I be the operator of the involution in the space H_0, $Iy(x) \equiv y(1-x)$, and let E be the identity transformation in H_0, $p_j \equiv \frac{1}{2}(E + (-1)^j I)$ are the orthoprojectors in the space H_0, $H_{0,j} \equiv \{y \in H_0 : y \equiv p_j y\}$, $j = 0,1$. Let us denote by $W^{2n}(0,1) \equiv \{y \in H_0 : y^{(m)} \in C[0,1]$, $m = \overline{0, 2n-1},\ y^{(2n)} \in H_0\}$, by W^* the space of the continuous linear functionals on the space $W^{2n}(0,1)$ and by $W_j^* \equiv \{l \in W^* : ly = 0, y \in H_{2,1-j} \bigcap W^{2n}(0,1)\};\ j = 0,1$.

We consider the following boundary problem

$$\begin{aligned} Lw \equiv & (-1)^n D_x^{2n} w(x) + A^{2n} w(x) \\ & + \sum_{s=1}^{n} a_s \left(D_x^{2s-1} w(x) + D_x^{2s-1} w(1-x) \right) = f(x), \quad x \in (0,1), \end{aligned} \tag{1}$$

$$\ell_j w \equiv D_x^{2j-1} w(0) - D_x^{2j-1} w(1) + l_j^1 w = \varphi_j, \tag{2}$$

$$\ell_{n+j} w \equiv D_x^{2j-2} w(0) - D_x^{2j-2} w(1) = \varphi_{n+j}, \tag{3}$$

where

$$\ell_j^1 w \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)), \quad j = 1,2,\dots,n. \tag{4}$$

The function w is called the solution of the problem (1)–(4) if

$$\|Lw - f; H_1\| = 0, \quad \|l_j w - \varphi_j; H\left(A^{\beta_j}\right)\| = 0,$$

$$\beta_{n+j} = 2n - 2j + \frac{3}{2}, \quad \beta_j = 2n - \max(m_j,\ 2j-1) - \frac{1}{2},$$

$$a_j, b_{j,r,s} \in \mathbb{R}, \quad r = 0,1,\dots, \quad m_j \leq 2n-1, \quad s = 0,1, \quad j = 1,2,\dots,n.$$

The paper is arranged as follows. In Section 2 we investigate the properties of the operator of problem with periodic conditions for the equation $(-1)^n y^{(2n)} = \lambda y$. In Section 3 we study the spectral properties of the operator of a problem with boundary conditions that are periodic perturbations. In Sections 4 we construct a commutative group of operators that map the root functions of the operators of perturbed boundary-value problems. In Section 5 using these operators, systems of root functions of boundary-value problem operators are constructed and conditions for the completeness and basis property of these systems are established. In Section 6 some analogous results are obtained for the operators of boundary problems generated by differential equations with an involution.

2 A SPECTRAL PROBLEM WITH PERIODIC BOUNDARY CONDITIONS FOR A DIFFERENTIAL-OPERATOR EQUATION

Consider the partial case of the problem (1)–(4) with $a_j = 0,\ b_{j,r,s} = 0,\ r = 0,1,\dots,m_j$, $s = 0,1,\ j = 1,2,\dots,n$, namely

$$(-1)^n D_x^{2n} u(x) + A^{2n} u(x) = f(x), \quad x \in (0,1), \tag{5}$$

$$\ell_{0,j} u \equiv D_x^{2j-1} u(0) - D_x^{2j-1} u(1) = 0, \tag{6}$$

$$\ell_{0,n+j} u \equiv D_x^{2j-2} u(0) - D_x^{2j-2} u(1) = 0, \quad j = 1,2,\dots,n. \tag{7}$$

Let L_0 be the operator of the problem (5)–(7),

$$L_0 u \equiv (-1)^n \ D_x^{2n} u + \ A^{2n} u, \quad u \in D(L_0), \ D(L_0) \equiv \{u \in H_2 : l_{0,j} u = 0, \ j = 1, 2, \ldots, 2n\}.$$

Consider the spectral problem

$$(-1)^n \ D_x^{2n} u(x) + \ A^{2n} u(x) = \lambda u(x), \quad \lambda \in \mathbb{C}, \tag{8}$$

$$\ell_{0,j} u \equiv \ u^{(2j-1)}(0) - \ u^{(2j-1)}(1) = 0, \quad j = 1, 2, \ldots, 2n. \tag{9}$$

We find the solution of the spectral problem (8), (9) as the product $u(x) = y(x) v_k$, $v_k \in V(A)$, $k = 1, 2, \ldots$.

To determine the unknown function $y \in W^{2n}(0,1)$ we obtain the spectral problem

$$(-1)^n \ y^{(2n)}(x) + \ z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{10}$$

$$\ell_{0,j} y \equiv \ y^{(2j-1)}(0) - \ y^{(2j-1)}(1) = 0, \quad j = 1, 2, \ldots, n, \tag{11}$$

$$\ell_{0,n+j} y \equiv \ y^{(2j-2)} y(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \ldots, n. \tag{12}$$

Let $L_{0,k}$ be the operator of the problem (10)–(12),

$$L_{0,k} y \equiv (-1)^n y^{(2n)} + z_k^{2n} y, \quad y \in D(L_{0,k}), \ D(L_{0,k}) \equiv \{y \in W^{2n}(0,1) : l_{0,j} y = 0, j = \overline{1, 2n}\}.$$

The roots ρ_j of the characteristic equation $(-1)^n \rho^{2n} = \lambda - z_k^{2n}$ of the differential equation

$$(-1)^n \ y^{(2n)}(x) + \ z_k^{2n} y(x) = \lambda y(x), \tag{13}$$

are defined by the relations

$$\rho_j = \omega_j \rho, \ \omega_1 = i, \omega_j = i \exp\left(i \frac{\pi(j-1)}{2n}\right), \quad j = 2, 3, \ldots, n.$$

The fundamental system of the solutions of the differential equation (13) is defined by the formulas

$$Y_j(x, \rho) \equiv \frac{1}{2}(\exp \omega_j \rho x + \ \exp \omega_j \rho (1-x)) \in H_{2,0}, \quad j = 1, 2, \ldots, n, \tag{14}$$

$$Y_{n+j}(x, \rho) \equiv \frac{1}{2}(\exp \omega_j \rho x - \ \exp \omega_j \rho (1-x)) \in H_{2,1}, \quad j = 1, 2, \ldots, n. \tag{15}$$

Substituting the general solution

$$y(x, \rho) = \sum_{s=1}^{2n} C_s Y_s(x, \rho)$$

of the differential equation (13) into the boundary conditions (11), (12) we obtain the equation for determination the eigenvalues of the operator $L_{0,k}$

$$\Delta(\rho) = \det(l_r Y_j)_{r,j=1}^{2n} = 0. \tag{16}$$

By substituting the functions (14), (15) in the boundary conditions (11), (12), we obtain $l_{0,r} Y_{n+j} = 0$, $l_{0,n+r} Y_j = 0$, j, $r = 1, 2, \ldots, n$. Therefore $\Delta(\rho) = \Delta_0(\rho) \Delta_1(\rho) = 0$, where $\Delta_s(\rho) = \det(l_{sn+r} Y_{sn+j})_{r,j=1}^{n}$, $s = 0, 1$.

The operator $L_{0,k}$ is self-adjoint (see [15]). Therefore the solutions of the equation (16) are $\rho_q = 2q\pi i$, $q = 0, 1, 2, \ldots,$ which are numbered in ascending order and lie on the half-line $Im \rho = 0$, $Re \rho \geq 0$.

Thus, the operator $L_{0,k}$ has eigenvalues $\lambda_{q,k} = (\rho_q)^{2n} + z_k^{2n}$, $q = 0, 1, \ldots$. We obtain the following result.

Lemma 2.1. *The self-adjoint operator $L_{0,k}$ has a point spectrum*

$$\sigma\left(L_{0,k}\right)=\left\{\lambda_q \in \mathbb{R}: \lambda_q=(2\pi q)^{2n}+z_k^{2n},\ q=0,1,\dots\right\}$$

and a system of eigenfunctions

$$V\left(L_{0,k}\right) \equiv \Big\{v_q(x) \in L_2(0,1): v_0(x)=1,\ v_{2q}(x) \equiv \sqrt{2}\cos 2\pi q x,$$
$$v_{2q-1}(x) \equiv \sqrt{2}\sin 2\pi q x,\ q=1,2,\dots\Big\},$$

which is an orthonormal basis of the space H_0.

Remark 2.1. *The systems*

$$V_0\left(L_{0,k}\right) \equiv \left\{v_{2q}(x):\ q=0,1,\dots\right\},\ V_1\left(L_{0,k}\right) \equiv \left\{v_{2q-1}(x):\ q=1,2,\dots\right\}$$

form an orthonormal basis in spaces $H_{0,0}$ and $H_{0,1}$, respectively.

Therefore, the operator L_0 has the following eigenfunctions in the space H_1

$$V(L_0) \equiv \left\{v_{q,k}(x,L_0) \in H_1:\ v_{q,k}(x,L_0) \equiv v_q(x)v_k,\ q=\overline{0,\infty},\ k=\overline{1,\infty}\right\}.$$

A system of functions $\{h_s\}_{s=1}^{\infty} \subset H$ is called a Riesz basis in a Hilbert space H , if $\{h_s\}_{s=1}^{\infty}$ is complete in the space H, and for any orthonormal basis $\{e_s\}_{s=1}^{\infty} \subset H$ there exists an isomorphism $B: H \to H,\ Be_s=h_s,\ s=1,2,\dots$.

The product of a system $V(A)$ and an orthonormal system $V(L_{0,k})$ is the Riesz basis (see [9]) in the space H_1. Thus, the following theorem is true.

Theorem 1. *The operator L_0 has a discrete spectrum*

$$\sigma(L_0) \equiv \left\{\lambda_{q,k} \in \mathbb{R}: \lambda_{q,k} \equiv \rho_q^{2n}+\ z_k^{2n},\ k=\ \overline{1,\infty},\ q=\overline{0,\infty}\right\},$$

and the system of the eigenfunctions $V(L_0)$ forms the Riesz basis in the space H_1.

Let us consider the functions

$$\begin{aligned} y_r\left(x,\rho_q\right) &\equiv \tfrac{1}{2}\left(1+\ e^{\omega_r\rho_q}\right)^{-1}\left(e^{\omega_r\rho_q x}+\ e^{\omega_r\rho_q(1-x)}\right), \\ y_1\left(x,\rho_q\right) &\equiv \tfrac{1}{2}(1-2x)\sin\rho_q x, \quad r=2,3,\dots,n,\ q=1,2,\dots, \end{aligned} \tag{17}$$

and determine the square matrix

$$\mathrm{B}_{0,p}\left(x,\rho_q\right) \equiv (\beta_{p,s}^0)_{p,s=1}^n$$

of the order n according to the following: the row with number p is determined by the elements of the system (17) $\beta_{p,s}^0\left(x,\rho_q\right) \equiv y_s\left(x,\rho_q\right)$ and the other lines by the formulas $\beta_{j,s}^0\left(x,\rho_q\right) \equiv (\omega_s)^{2j-1}$ with $j \neq p,\ j,s=1,2,\dots,n$.

We denote the determinant of the matrix $\mathrm{B}_{0,p}\left(x,\rho_q\right)$ by $y_{1,p}\left(x,\rho_q\right)$.

Substituting the determinant into conditions (11), (12), we obtain

$$l_{0,r}y_{1,p}=0,\ r \neq p,\ l_{0,p}y_{1,p}=(\rho_q)^{2p-1}h(i)W^n, \tag{18}$$

where the Vandermonde determinant W^n is constructed by the numbers

$$1, (\omega_2)^2, \ldots, (\omega_n)^2, h(i) = (-i)^{n-1} i,\ i = \sqrt{-1},\ r = 1, \ldots, 2n,\ m = 1, 2, \ldots, n.$$

Consider the functions $y_{2,p}(x, \rho_q) \equiv (h(i)W^n)^{-1} y_{1,p}(x, \rho_q),\ p = 1, 2, \ldots, n.$
From the relation (18) we obtain

$$l_{0,r} y_{2,p} = 0,\ r \neq p,\ l_{0,p} y_{2,p} = (\rho_q)^{2p-1},\ p, r = 1, 2, \ldots, n. \tag{19}$$

Similarly, let us consider the system of functions

$$\begin{aligned} y_{n+r}(x, \rho_q) &\equiv \tfrac{1}{2}\left(1 - e^{w_r \rho_q}\right)^{-1}\left(e^{w_r \rho_q x} - e^{w_r \rho_q (1-x)}\right), \\ y_{n+1}(x, \rho_q) &\equiv \tfrac{1}{2}(1 - 2x)\cos \rho_q x, \quad r = 2, 3, \ldots, n,\ q = 1, 2, \ldots, \end{aligned} \tag{20}$$

and a square matrix

$$\mathrm{B}_{1,r}(x, \rho_q) \equiv (\beta^1_{p,s})^n_{p,s=1}$$

of the order n which rows are determine by following: the row with number r is determined by the elements of the system (20) $\beta^1_{r,s}(x, \rho_q) \equiv y_{n+s}(x, \rho_q)$ and the other lines by the equalities

$$\beta^1_{j,s}(x, \rho_q) \equiv (\omega_s)^{2j-2},\ j \neq r,\ r, s = 1, 3, \ldots, n.$$

We denote the determinant of the matrix by $y_{1,n+r}(x, \rho_q)$.
Substituting it into conditions (11), (12), we get

$$l_{0,j} y_{1,n+r} = 0,\ j \neq n + r,\ l_{0,n+r} y_{1,n+r} = W^n (\rho_q)^{2r-2}. \tag{21}$$

Let us define the functions $y_{2,n+r}(x, \rho_q) \equiv (W^n)^{-1} y_{1,n+r}(x, \rho_q),\ r = 1, 2, \ldots, n.$
Taking the relation (21) for the functions $y_{2,n+r}(x, p_q)$ into account, we obtain

$$l_{0,j} y_{2,n+r} = 0,\ j \neq n + r,\ l_{0,n+r} y_{2,n+r} = (\rho_q)^{2r-2},\ j = 1, \ldots, 2n,\ r = 1, 2, \ldots, n.$$

Remark 2.2. *There exist positive numbers K_0, K_1 such that*

$$K_1 \leq \|y_{2,j}(x, \rho_q); H_0\| \leq K_2 < \infty, \quad j = 1, 2, \ldots, 2n,\ q = 1, 2, \ldots. \tag{22}$$

Here $K_s,\ s \in \mathbb{N}$, are positive constants.

3 NONLOCAL BOUNDARY VALUE PROBLEM

For the differential-operator equation (5) and an arbitrary fixed $p \in \{1, 2, \ldots, n\}$ and $b \in \mathbb{R}$ we consider the boundary value problem

$$\ell_{1,j} u \equiv D_x^{2j-1} u(0) - D_x^{2j-1} u(1) = 0, \quad j \neq p,\ j = 1, 2, \ldots, n, \tag{23}$$

$$\ell_{1,p} u \equiv D_x^{2p-1} u(0) - D_x^{2p-1} u(1) + l_p^2 u = 0, \tag{24}$$

$$\ell_{1,n+j} u \equiv D_x^{2j-2} u(0) - D_x^{2j-2} u(1) = 0, \quad j = 1, 2, \ldots, n, \tag{25}$$

with

$$\ell_p^2 u \equiv b(D_x^{2p-1} u(0) + D_x^{2p-1} u(1)) = 0, \quad b \in \mathbb{R}. \tag{26}$$

We will use following notations. Let L_1 be the operator of the problem (5), (23)–(26) and $L_1 u \equiv (-1)^n D_x^{2n} u(x) + A^{2n} u(x),\ u \in D(L_1),\ D(L_1) \equiv \{u \in H_2 : l_{1,r} u = 0,\ r = \overline{1,2n}\}$.

We find the solution of the spectral problem (8), (23)–(26) as the product $u(x) = y(x) v_k$, $v_k \in V(A),\ k = 1, 2, \dots$.

To determine the unknown function $y \in W^{2n}(0,1)$ we consider the spectral problem

$$(-1)^n\, y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{27}$$

$$\ell_{1,j} y \equiv y^{(2j-1)}(0) - y^{(2j-1)}(1) = 0, \quad j \neq p,\ j = 1, 2, \dots, n, \tag{28}$$

$$\ell_{1,p} y \equiv y^{(2p-1)}(0) - y^{(2p-1)}(1) + l_p^2 y = 0, \tag{29}$$

$$l_{1,n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \dots, n, \tag{30}$$

with

$$l_p^2 y \equiv b\left(y^{(2p-1)}(0) + y^{(2p-1)}(1)\right). \tag{31}$$

Let $L_{1,k} \equiv L_{1,k,p}$ be the operator of the problem (27)–(31) and

$$L_{1,k} y \equiv (-1)^n y^{(2n)}(x) + z_k^{2n} y(x), \quad y \in D(L_{1,k}), D(L_{1,k}) \equiv \{y \in W^{2n}(0,1) : l_{1,j} y = 0, j = \overline{1,2n}\}.$$

Let $V(L_{1,k})$ be the system of root functions of the operator $L_{1,k}$, let $R(L_{1,k}) \equiv E + S(L_{1,k})$ be the operator which maps the system $V(L_{0,k})$ into the system $V(L_{1,k})$.

Theorem 2. *For any $b \in \mathbb{R},\ p \in \{1, 2, \dots, n\}$, the operator $L_{1,k}$ has the point spectrum $\sigma(L_{0,k})$ and the system of root functions $V(L_{1,k})$ forms a Riesz basis in H_0.*

Proof. We will show that eigenvalues of the operators $L_{0,k}$ and $L_{1,k}$ coincide.

We substitute the fundamental system (14), (15) for the solutions of the differential equation (27) into the boundary conditions (28)–(31). Using $l_p^2 y_{n+j}(x, \rho) = 0,\ j, p = \overline{1,n}$, we obtain the same equations for determination the spectrum

$$\det(l_{1,j} y_r(x,\rho))_{j,r=1}^{2n} = \det(l_{1,j} y_r(x,\rho))_{j,r=1}^{n} \det(l_{1,n+j} y_{n+r}(x,\rho))_{j,r=1}^{n}.$$

Let us define elements of the system $V(L_{1,k})$.

It is easy to see that $v_{2q}(x) \in D(L_{1,k}), L_{1,k} v_{2q}(x) = \lambda_q v_{2q}(x), q = 0, 1, 2, \dots$. Hence

$$v_{2q}(x, L_{1,k}) \equiv v_{2q}(x), \quad q = 0, 1, \dots . \tag{32}$$

We define the root functions of the operator $L_{1,k}$ as

$$v_{2q-1}(x, L_{1,k}) \equiv v_{2q-1}(x) + c_{b,p} y_{2,p}(x, \rho_q). \tag{33}$$

Substituting (33) into the boundary condition (29) and taking the equality (19) into account, we obtain $c_{b,p} = -\sqrt{2} b$,

$$v_{2q-1}(x, L_{1,k}) \equiv v_{2q-1}(x) - \sqrt{2} b y_{2,p}(x, \rho_q), \quad q = 1, 2, \dots . \tag{34}$$

Thus, the operator $L_{1,k}$ has a system of root functions (32)–(34) in the sense of equalities

$$L_{1,k} v_{2q-1}(x, L_{1,k}) = \lambda_{q,k} v_{2q-1}(x, L_{1,k}) + \xi_{b,q} v_{2q}(x, L_{1,k}),$$
$$\xi_{b,q} = -4\sqrt{2} bn (\rho_q)^{2n-1},\ L_{1,k} v_{2q}(x, L_{1,k}) = \lambda_{q,k} v_{2q}(x, L_{1,k}), \quad q = 1, 2, \dots .$$

Given the regularity according to Birkhoff (see [15]) of boundary conditions (28)–(31) we obtain that the system $V(L_{1,k})$ is complete and minimal in the space H_0. □

Let W^{n-1} be the Vandermonde determinant constructed by elements $\omega_2^2,\ \omega_3^2,\ \ldots,\omega_n^2$; let $R_1 \equiv E + S_1$ be the operator which maps the system $V(L_0)$ into the system V_1 and elements of this system are

$$\begin{aligned} &v_{2q-1,1}(x) \equiv (1 - \sqrt{2}^{-1} W^n (h(i)W^{n-1})^{-1} b(1-2x)) v_{2q-1}(x), \\ &v_{0,1}(x) \equiv v_0(x),\ v_{2q,1}(x) \equiv v_{2q}(x), \quad q = 1,2,\ldots . \end{aligned} \tag{35}$$

Lemma 3.1. *The system V_1 forms a Riesz basis in H_0.*

Proof. For an arbitrary function $\varphi \in H_0$ we have

$$\begin{aligned} &\varphi = \varphi_0 v_0(x) + \sum_{q=0}^{\infty} \left(\varphi_{2q} v_{2q}(x) + \varphi_{2q-1} v_{2q-1}(x)\right) \in H_0, \\ &\|\varphi; H_0\|^2 = |\varphi_0|^2 + \sum_{q=1}^{\infty} \left(|\varphi_{2q}|^2 + |\varphi_{2q-1}|^2\right) < \infty, \end{aligned}$$

consider the function

$$\begin{aligned} &\varphi_1 = R_1 \varphi = \varphi_0 v_{0,1}(x) + \sum_{q=1}^{\infty} \left(\varphi_{2q} v_{2q,1}(x) + \varphi_{2q-1} v_{2q-1,1}(x)\right), \\ &\|R_1 \varphi; H_0\|^2 \le K_3 \|\varphi; H_0\|^2, \quad K_3 = 2\left(1 + |W^n (W^{n-1})^{-1} b|^2\right). \end{aligned}$$

Therefore $\|R_1; [H_0]\|^2 \le K_3 < \infty,\ R_1 \equiv E + S_1 \in [H_0],\ R_1^{-1} \equiv E - S_1 \in [H_0]$. Taking into account the Bari Theorem (see [9]), we obtain the following statement: the system V_1 forms the Riesz basis in H_0. □

Therefore, the operator L_1 has the following system of root functions in the space H_1

$$V(L_1) \equiv \left\{ v_{q,k}(x, L_1) \equiv v_q(x, L_{1,k})\ v_k \in H_1 :\ q = \overline{0,\infty},\ k = \overline{1,\infty} \right\}.$$

Remark 3.1. *The operator L_1 has a system of root functions in the means of equalities*

$$\begin{aligned} &L_1 v_{2q-1,k}(x, L_1) = \lambda_{q,k} v_{2q-1,k}(x, L_1) + \xi_{b,q} v_{2q,k}(x, L_1), \\ &\xi_{b,q} = -4\sqrt{2} bn \left(\rho_q\right)^{2n-1}, \quad q,k = 1,2,\ldots, \\ &L_1 v_{2q,k}(x, L_1) = \lambda_{q,k} v_{2q,k}(x, L_1), \quad q = 0,1,\ldots,\ k = 1,2,\ldots . \end{aligned}$$

Theorem 3. *For any fixed numbers $p \in \{1,2,\ldots,n\}$, $b \in \mathbb{R}$, the system $V(L_1)$ is the Riesz basis of the space H_1.*

Proof. Let $R(L_{1,k}) \equiv E + S(L_{1,k}) : V(L_{0,k}) \to V(L_{1,k})$, let p_k be a projection in H, $p_k y \equiv (y, w_k(A); H) v_k$, $R(L_1) \equiv \sum_{k=1}^{\infty} R(L_{1,k}) p_k$.

From the definition of the operator $R(L_1) = E + S(L_1)$ it follows that $R^{-1}(L_1) = E - S(L_1)$.

Therefore the system $V(L_1)$ is complete and minimal in the space H_1. Taking into account the representations of the elements of the system $V(L_{1,k})$ and Theorem 2, we obtain $\|R(L_1); [H_1]\| \le K_4 \|R(L_{1,k}); [H_0]\| < \infty$.

Taking into account the Bari Theorem (see [9]), we obtain the following statement: the system V_1 forms the Riesz basis in H_1.

□

4 TRANSFORMATION OPERATORS

For any $k \in \mathbb{N}$, $p \in \{1,2,\dots,n\}$, we define the operator $B_p : H_0 \to H_0$ as the operator whose eigenvalues coincide with eigenvalues of the operator $L_{0,k}$, and the root functions are defined by

$$v_{2s}(x,B_p) \equiv v_{2s}(x),\ v_{2q-1}(x,B_p) \equiv v_{2q-1}(x) + c_q(B_p)y_{2,p}(x,\rho_q), \tag{36}$$

where $c_q(B_p) \in \mathbb{R}$, $s = 0,1,\dots,$ $q = 1,2,\dots$.

The operator which maps the system $V(L_{0,k})$ into the system $V(B_p)$ of the root functions of the operator B_p is denoted by $R(B_p) \equiv E + S(B_p)$, where $S(B_p) : H_{0,0} \to H_{0,1}$, $S(B_p) : H_{0,1} \to 0$.

We denote by $G_{0,p}(L_{0,k}) \equiv \{R(B_p)\}$ such that the root functions of the operator B_p are defined by the equalities (36), and $G_{0,p,c}(L_{0,k}) \equiv G_{0,p}(L_{0,k}) \cap [H_0]$.

Remark 4.1. *Using that $S(B_p) : H_{0,0} \to H_{0,1}$, $S(B_p) : H_{0,1} \to 0$ we obtain $S^2(B_p) = 0$, $R^{-1}(B_p) \equiv E - S(B_p)$.*

Consequently, the operator $R(B_p)$ has a dense domain in the space H_0 and the system of root functions is complete and minimal in H_0.

Similarly, using the root functions of an adjoint operator $L^*_{1,k}$, we define the functions

$$w_0(x,B_p) \equiv v_0(x) + c_0(1-2x),$$
$$w_{2q}(x,B_p) \equiv v_{2q}(x) + c_q(B_p)y_{2,2n-p-1}(x,\rho_q),\ w_{2q-1}(x,B_p) \equiv v_{2q-1}(x), \quad q = 1,2,\dots,$$

and the set of operators $G_{1,p}(L_{0,k}) \equiv \left\{R\left(B_p^*\right) = E + S\left(B_p^*\right),\ R(B_p) \in G_{0,p}(L_{0,k})\right\}$.

Theorem 4. *For any $b \in \mathbb{R}$, $p \in \{1,2,\dots,n\}$, the operator B_p has the point spectrum $\sigma(L_{0,k})$ and the system of root functions $V(L_{1,k})$ forms the Riesz basis in H_0 if and only if the sequence $c_q(B_p)$ is bounded, i.e. $|c_q(B_p)| \le K_5 < \infty$, $q = 1,2,\dots$.*

Proof. The necessity. Let the system $V(B_p)$ be the Riesz basis in H_0, i.e. $R(B_p) \in [H_0]$, then $S(B_p) = E - R_p(B) \in [H_0]$. From the definition of the operator B_p we have

$$S(B_p)v_{2q-1}(x) = c_q(B_p)y_{2,p}(x,\rho_q), \quad q = 1,2,\dots.$$

Therefore, taking the estimate (22) into account, we obtain

$$\begin{aligned}&|c_q(B_p)| \le \|S(B_p);[H_0]\|\|y_{2,p}(x,\rho_q);H_0\|^{-1} \le K_6 < \infty,\\&K_6 = K_5^{-1}\|S(B_p);[H_0]\|, \quad q = 1,2,\dots.\end{aligned}$$

The sufficiency. The completeness of the system $V(B_p)$ in the space H_0 follows from Remark 4.1.

Let $\varphi \in H_0$, $\varphi = \varphi_0 + \varphi_1$, $\varphi_s \in H_{0,s}$, $s = 0,1$. Then we have

$$\varphi = \varphi_0 v_0(x) + \sum_{q=1}^{\infty}\left(\varphi_{2q}v_{2q}(x) + \varphi_{2q-1}v_{2q-1}(x)\right) \in H_0,$$

$$\|\varphi;H_0\|^2 = |\varphi_0|^2 + \sum_{q=1}^{\infty}\left(\left|\varphi_{2q}\right|^2 + \left|\varphi_{2q-1}\right|^2\right) < \infty,$$

$$R(B_p)\varphi = \varphi_0 v_0(x,B_p) + \sum_{q=1}^{\infty}\left(\varphi_{2q}v_{2q}(x,B_p) + \varphi_{2q-1}v_{2q-1}(x,B_p)\right) \in H_0,$$

$$R(B_p)\varphi = \varphi_0 v_0(x,B_p) + \sum_{q=1}^{\infty}\left(\varphi_{2q}v_{2q}(x) + \varphi_{2q-1}c_q(B_p)\left(v_{2q-1}(x,L_{1,k}) - v_{2q-1}(x)\right)\right),$$

$$\|R(B_p)\varphi;H_0\|^2 \le K_7\|\varphi;H_0\|^2,\ K_7 = 3\left(1 + K_6^2 + K_6^2\|R(L_{1,k});[H_0]\|^2\right).$$

Therefore, $\|R(B_p);[H_0]\|^2 \leq K_7 < \infty$.

Consider equalities $R\left(B_p\right) = E + S\left(B_p\right),\ R^{-1}\left(B_p\right) = E - S\left(B_p\right)$. We have

$$R^{-1}\left(B_p\right) = 2E - R\left(B_p\right).$$

Therefore, $\left\|R^{-1}\left(B_p\right);[H_0]\right\|^2 \leq K_8 < \infty,\ K_8 = 8 + 2K_7$. Taking into account the Bari Theorem (see [9]), we obtain that the system V_1 forms the Riesz basis in H_0. □

Suppose that $Q_0(I)$ is a set of operators $R = E + S,$ such that $S : H_{0,0} \to H_{0,1},\ S : H_{0,1} \to 0,$ $Q_{0,c}(I) \equiv [H_0] \bigcap Q_0(I)$. Using that $S^2(B_p) = 0,\ R(B_p) \in G_{0,p}(L_{0,k}) \subset Q_0(I)$ on the set $Q_0(I)$, we can define the operation of multiplication

$$R_1R_2 \equiv (E + S_1)(E + S_2) = E + S_1 + S_2,\ R_1,\ R_2 \in Q_0(I).$$

In particular, $(E + S)(E - S) = E - S^2 = E,\ R = E + S \in Q_0(I)$.

Therefore, for each operator $R = E + S \in Q_0(I)$ there exists a unique inverse operator $R^{-1} = E - S$.

According to the definition of the operator B_p and the set $G_{0,p}(L_{0,k})$ we have the inclusions

$$G_{0,p}(L_{0,k}) \subset Q_0(I),\ G_{c,0,p}(L_{0,k}) \subset Q_{0,c}(I), \quad p \in \{1, 2, \ldots, n\}.$$

Thus, the set $Q_0(I)$ is an Abelian group which contains the Abelian subgroups $Q_{c,0}(I)$, $G_{0,p}(L_{0,k}),\ G_{0,c,p}(L_{0,k}),\ p \in \{1, 2, \ldots, n\}$. Therefore, for all operators $R_j = E + S_j \in Q_0(I)$, $j = 1, 2 \ldots, d,\ d \in \mathbb{N}$, the following equality

$$\prod_{j=1}^{d} R_j \equiv \prod_{j=1}^{d}\left(E + S_j\right) = E + \sum_{j=1}^{d} S_j, \quad d \in \mathbb{N}, \tag{37}$$

holds.

5 NONLOCAL BOUNDARY VALUE PROBLEMS FOR A DIFFERENTIAL-OPERATOR EQUATION

5.1. For the differential-operator equation (5) and arbitrary fixed indices $b_{p,r,s} \in \mathbb{R},\ p \in \{1, 2, \ldots, n\},\ r = 0, 1, \ldots, k_j,\ s = 0, 1,\ j = 1, 2, \ldots, n$, we consider the boundary problem generated by nonlocal conditions

$$\ell_{2,j}w \equiv D_x^{2j-1}w(0) - D_x^{2j-1}w(1) = 0, \quad j \neq p, \tag{38}$$

$$\ell_{2,p}w \equiv D_x^{2p-1}w(0) - D_x^{2p-1}w(1) + l_p^1 w = 0, \tag{39}$$

$$\ell_{2,n+j}w \equiv D_x^{2j-2}w(0) - D_x^{2j-2}w(1) = 0, \quad j = 1, 2, \ldots, n, \tag{40}$$

where

$$\ell_p^1 w \equiv \sum_{r=0}^{m_p}(b_{p,r,0}D_x^r w(0) + b_{j,r,1}D_x^r w(1)). \tag{41}$$

Assumption P_1: $b_{p,r,0} = (-1)^{r+1}b_{p,r,1},\ r = 0, 1, \ldots, m_p,\ j,\ p = 1, 2, \ldots, n$.

Assumption P_2: $m_p \leq 2p - 1,\ p = 1, 2, \ldots, n$.

Remark 5.1. *Assumption* P_1 *implies that* $l_p^1 \in W_1^*,\ p = 1, 2, \ldots, n$.

Let $L_2 \equiv L_{2,p}$ be the operator of the problem (5), (38)–(41) and

$$L_2 u \equiv (-1)^n D_x^{2n} u(x) + A^{2n} u(x), \quad u \in D(L_2),\ D(L_2) \equiv \{u \in H_2 : l_{2,j} u = 0,\ j = 1,2,\ldots,2n\}.$$

The solution of the spectral problem (5), (38)–(41) is defined as the product $w(x) = y(x) v_k$, $v_k \in V(A),\ k = 1,2,\ldots$.

To determine the unknown function $y \in W^{2n}(0,1)$ we consider the spectral problem

$$(-1)^n\, y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \tag{42}$$

$$\ell_{2,j} y \equiv y^{(2j-1)}(0) - y^{(2j-1)}(1) = 0, \quad j \neq p, \tag{43}$$

$$\ell_{2,p} y \equiv y^{(2p-1)}(0) - y^{(2p-1)}(1) + l_p^1 y = 0, \tag{44}$$

$$\ell_{2,n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2}(1) = 0, \quad j = 1,2,\ldots,n. \tag{45}$$

Let $L_{2,k} \equiv L_{2,k,p}$ be the operator of the problem (42)–(45) and

$$L_{2,k} y \equiv (-1)^n\, y^{(2n)}(x) + z_k^{2n} y(x),$$

$$y \in D(L_{2,k}),\ D(L_{2,k}) \equiv \{y \in W^{2n}(0,1) : l_{2,j} y = 0,\ j = 1,2,\ldots,2n\}.$$

Theorem 5. *Suppose that the Assumption P_1 holds. Then for arbitrary numbers $b_{p,r,s} \in \mathbb{R}$, $s = 0,1,\ r = 0,1,\ldots,m_p,\ p \in \{1,2,\ldots,n\}$, the following statements hold*

1) the eigenvalues of the operators $L_{0,k}$ and $L_{2,k}$ coincide;

2) the system $V(L_{2,k})$ is complete and minimal in the space H_0 ;

3) if in addition the Assumption P_2 holds, then the system $V(L_{2,k})$ is the Riesz basis of the space H_0.

Proof. The proof of part 1 of the theorem can be made in the same way as in Theorem 2.

Let us define the elements of the system $V(L_{2,k})$. A direct substitution gives that the function $v_{2q}(x),\ q = 0,1,\ldots,$ satisfies the conditions (43)–(45) . Therefore, the root function of the operator $L_{2,k}$ with respect to the eigenvalue $\lambda_{q,k}$ is defined by

$$v_{2q}(x, L_{2,k}) = v_{2q}(x, L_{0,k}), \quad q = 0,1,\ldots,$$
$$v_{2q-1}(x, L_{2,k}) = v_{2q-1}(x) + c_{q,p} y_{2,p}(x, \rho_q),$$
$$c_{q,p} = -l_p^1(v_{2q-1}(x))(l_{2,p} y_{2,p}(x, \rho_q))^{-1}, \quad q = 1,2,\ldots .$$

Consequently $L_{2,k} \in Q_0(I)$. If the Assumption P_2 holds, then from the inequality $|l_{p,b}^1 v_{2q-1}| \leq K_9(\rho_q)^{2p-2}$ we obtain the inequality

$$|l_p^1(v_{2q-1}(x))(l_{2,n+p} y_{2,p}(x, \rho_{0,q}))^{-1}| \leq K_{10} < \infty. \tag{46}$$

Taking Theorem 4 into account, we obtain the third statement of the theorem. □

Therefore, the operator L_2 has the following system of root functions in the space H_1

$$V(L_2) \equiv \left\{ v_{q,k}(x, L_2) \equiv v_q(x, L_{2,k}) v_k : q = \overline{0,\infty},\ k = \overline{1,\infty} \right\}.$$

Remark 5.2. *The operator L_2 has a system of root functions in the means of equalities*

$$L_2 v_{2q-1,k}(x, L_2) = \lambda_{q,k} v_{2q-1,k}(x, L_2) + \xi_{q,p} v_{2q,k}(x, L_2), \tag{47}$$

$$\xi_{q,p} = -4\sqrt{2}(\rho_q)^{2n-1} c_{q,p}, \quad q,k = 1,2,\ldots, \tag{48}$$

$$L_2 v_{2q,k}(x, L_2) = \lambda_{q,k} v_{2q,k}(x, L_2), \quad q = 0,1,\ldots, k = 1,2,\ldots . \tag{49}$$

Theorem 6. *Suppose that the Assumption P_1 holds. Then, for arbitrary numbers $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \ldots, m_p$, $p \in \{1, 2, \ldots, n\}$, the following statements hold*

1) the eigenvalues of the operators L_0 and L_2 coincide;

2) the system $V(L_2)$ is complete and minimal in the space H_1;

3) if in addition the Assumption P_2 holds, then the system $V(L_2)$ forms the Riesz basis of the space H_1.

Proof. Taking Theorem 5 into account, it is possible to determine the elements of a system $W(L_{2,k})$ which is biorthogonal to the system $V(L_{2,k})$ in the space H_0.

Therefore, there exists $W(L_2) \equiv \{w_{q,k}(x, L_2)w_k : \ q = 0, 1, \ldots, \ k = 1, 2, \ldots .\}$ which is the biorthogonal system of functions to the system $V(L_2)$ in the space H_1.

Thus the second statement of the theorem is proved.

Suppose that the Assumption P_2 holds. Taking the inequalities (46) into account, we obtain the estimate

$$\|R(L_2); [H_1]\| \leq K_{11} < \infty.$$

From the Bari Theorem (see [9]) we obtain the statement: the system V_1 forms the Riesz basis in H_1. ☐

5.2. Consider the spectral problem

$$(-1)^n D_x^{2n} w(x) + \ A^{2n} w(x) = \ \lambda w(x), \tag{50}$$

$$\ell_j w \equiv \ D_x^{2j-1} w(0) - D_x^{2j-1} w(1) + l_j^1 w = 0, \tag{51}$$

$$\ell_{n+j} w \equiv \ D_x^{2j-2} w(0) - D_x^{2j-2} w(1) = 0, \tag{52}$$

where

$$\ell_j^1 w \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)), \quad j = 1, 2, \ldots, n. \tag{53}$$

Let L_3 be the operator of the problem (50)–(53) and

$$L_3 u \equiv (-1)^n \ D_x^{2n} u + \ A^{2n} u, \quad u \in D(L_3), \ D(L_3) \equiv \{u \in H_2 : l_j u = 0, \ j = 1, 2, \ldots , 2n\}.$$

We find the solution of the spectral problem (50)–(53) as the product $w(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1, 2, \ldots .$

To determine the unknown function $y \in W^{2n}(0, 1)$ we obtain the spectral problem

$$(-1)^n y^{(2n)} + \ z_k^{2n} y \ = \ \lambda y, \quad \lambda \in \mathbb{C}, \tag{54}$$

$$\ell_j y \equiv \ y^{(2j-1)}(0) + \ y^{(2j-1)}(1) + \ l_j^1 y = 0, \tag{55}$$

$$\ell_{n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \ldots, n. \tag{56}$$

Let $L_{3,k}$ be the operator of the problem (54)–(56);

$$L_{3,k} y\,(x) \equiv (-1)^n \, y^{(2n)}\,(x) + \, z_k^{2n};$$

$$y \in D\,(L_{3,k})\,; \ D\,(L_{3,k}) \equiv \left\{ y \in W^{2n}\,(0, 1) : l_j y = 0, \ j = 1, 2, \ldots, 2n \right\};$$

let $V\,(L_{3,k})$ be the system of root functions of the operator $L_{3,k}$.

We can prove that

$$v_{2q}\,(x) \in D\,(L_{3,k})\,, \ L_{3,k} v_{2q}\,(x) = \lambda_{q,k} v_{2q}\,(x)\,, \quad q = 0, 1, \ldots .$$

Therefore,

$$v_{2q}\left(x, L_{3,k}\right) \equiv v_{2q}\left(x\right), \quad q = 0, 1, \ldots . \tag{57}$$

The root functions of the operator $L_{3,k}$ are determined by the equalities

$$v_{2q-1}\left(x, L_{3,k}\right) \equiv v_{2q-1}\left(x\right) + \sum_{p=1}^{n} c_{1,q,p} y_{2,p}\left(x, \rho_q\right), \quad q = 1, 2, \ldots . \tag{58}$$

Substituting the expression (58) into the boundary conditions (55), (56), we obtain

$$c_{1,q,p} = -\sqrt{2} \sum_{r=0}^{m_p} \left(-1\right)^{r-2p+1} b_{p,r,0} \left(\rho_q\right)^{2+r-2p}, \quad p = 1, 2, \ldots, n, \ q = 1, 2, \ldots . \tag{59}$$

Thus, the operator $L_{3,k}$ has the system of root functions (57)–(59) in the means of equalities

$$\begin{aligned}
&L_{3,k} v_{2q-1}\left(x, L_{3,k}\right) = \lambda_{q,k} v_{2q-1}\left(x, L_{3,k}\right) + \xi_q^0 v_{2q}\left(x, L_{3,k}\right), \\
&\xi_q^0 = 2\sqrt{2} n \left(\rho_q\right)^{2n-1} \sum_{p=1}^{n} c_{p,q}, \quad q = 1, 2, \ldots, \\
&L_{3,k} v_{2k}\left(x, L_{3,k}\right) = \lambda_{q,k} v_{2k}\left(x, L_{3,k}\right), \quad q = 0, 1, \ldots .
\end{aligned}$$

Let $R\left(L_{3,k}\right)$ be the operator which acts as $V\left(L_{0,k}\right) \to V\left(L_{3,k}\right)$. From the formulas (37), (58), we obtain the relation

$$R(L_{3,k}) = \prod_{p=1}^{n} R(L_{2,k,p}) = E + \sum_{p=1}^{n} S(L_{2,k,p}). \tag{60}$$

Therefore, we have the inclusion $R\left(L_{3,k}\right) \in G_0\left(L_{0,k}\right) \subset Q_0\left(I\right)$. Thus, we obtain the following statement.

Lemma 5.1. *Suppose that the Assumption P_1 holds. Then, for the any fixed $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \ldots, m_p$, $p = 1, 2, \ldots, n$, the system $V(L_{3,k})$ is complete and minimal in the space H_0.*

Consider the system V_2 of functions

$$\begin{aligned}
&v_{0,2}\left(x\right) \equiv v_0\left(x\right), \ v_{2q,2}\left(x\right) \equiv v_{2q}\left(x, \right), \quad q = 1, 2, \ldots, \\
&v_{2q-1,2}(x) \equiv (1 + \tau_2)(1 - 2x)) v_{2q-1}(x), \\
&\tau_2 \equiv W^n (W^{n-1})^{-1} c_b, \ c_b \equiv \sum_{p=1}^{n} b_{p,2p-1,0}.
\end{aligned}$$

Let $R_2 = E + S_2$ be the operator which acts as $V\left(L_{0,k}\right) \to V_2$.

Using that $S_2 : H_{0,1} \longrightarrow 0$, $S_2 : H_{0,0} \longrightarrow H_0$, we obtain that $R_2 \in Q_0(I)$.

Lemma 5.2. *Suppose that the Assumptions P_1, P_2 hold. Then the system V_2 forms the Riesz basis in the space H_0.*

The proof is carried out analogously in Lemma 3.1.

Remark 5.3. *Suppose that the Assumptions P_1, P_2 hold. Then the following relations hold*

$$v_{2q-1}(x, L_{3,k}) = v_{2q-1,2}(x) + \sum_{j=2}^{n} c_{j,q}^1 y_j(x, \rho_q) + (\rho_q)^{-1} c_q^2 (1 - 2x)) v_{2q-1}(x), \tag{61}$$

where

$$|c_{j,q}^1| \leq K_{12}, \ |c_q^2| \leq K_{12} < \infty, \quad q = 0, 1, \ldots .$$

Therefore, the systems $V(L_{3,k})$ and V_2 are squarely close in the space H_0.

Lemma 5.3. *Suppose that the Assumptions P_1, P_2 hold. Then, for any fixed $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $p = 1, 2, \dots, n$, the system $V(L_{3,k})$ forms the Riesz basis in the space H_0.*

The statement follows from Lemma 5.1, Lemma 5.2, Remark 5.4 and the Bari Theorem (see [9]).

Theorem 7. *Suppose that the Assumption P_1 holds. Then, for any $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $r, p = 1, 2, \dots, n$, the following assertions are true*

1) the eigenvalues of the operators $L_{0,k}$ and $L_{3,k}$ coincide;

2) the system $V(L_{3,k})$ is complete and minimal in the space H_0;

3) if in addition the Assumption P_2 holds, then the system $V(L_{3,k})$ forms the Riesz basis of the space H_0.

Proof. The proof of part 1 of the theorem follows from Theorem 3, the second statement follows from Lemma 5.1 and the third statement follows from Lemma 5.3. □

Let
$$V(L_3) \equiv \left\{ v_{q,k}(x, L_3) \equiv v_q(x, L_{3,k}) v_k \in W_1 : q = \overline{0, \infty},\ k = \overline{1, \infty} \right\}$$
forms the system of the root functions of the operator L_3. Let
$$W(L_{3,k}) \equiv \left\{ w_{q,k}(x, L_3) \in H_1 : q = \overline{0, \infty} \right\}$$
be the biorthogonal system of functions to the system $V(L_{3,k})$ in the space H_0. Let
$$W(L_3) \equiv \left\{ w_{q,k}(x, L_3) \equiv w_q(x, L_{3,k}) w_k \in H_1 : q = \overline{0, \infty},\ k = \overline{1, \infty} \right\},$$
be the biorthogonal system of functions to the system $V(L_3)$ in the space W_1 and $R(L_3)$ be the operator which acts as $V(L_0) \to V(L_3)$.

Theorem 8. *Suppose that the Assumption P_1 holds. Then, for all numbers $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $p = 1, 2, \dots, n$, the following assertions are true*

1) the eigenvalues of the operators L_0 and L_3 coincide;

2) the system $V(L_3)$ is complete and minimal in the space H_1;

3) if in addition the Assumption P_2 holds, then the system $V(L_3)$ forms the Riesz basis of the space H_1.

Proof. The proof of part 1 and 2 of the theorem follows from Theorem 7. Taking the relations (60), (61) into account we obtain the equality
$$R(L_3) \equiv \prod_{j=1}^{n} R(L_{2,j}) \equiv \prod_{j=1}^{n} \left(E + S(L_{2,j}) \right) = E + \sum_{j=1}^{n} S(L_{2,j}). \tag{62}$$

Let Assumptions P_1 and P_2 hold. Then from the equality (62) and the assertion 3 of Theorem 7 we obtain
$$R(L_3) \in [H_1],\ R(L_3)^{-1} = E - S(L_3) \in [H_1].$$
Therefore, the system $V(L_3)$ forms the Riesz basis of the space H_1. □

Remark 5.4. *The operator L_3 has the system of root functions in the means of the equalities*
$$L_3 v_{2q-1,k}(x, L_3) = \lambda_{q,k} v_{2q-1,k}(x, L_3) + \xi^0_{q,k} v_{2q,k}(x, L_3),$$
$$\xi^0_{q,k} = 2\sqrt{2}\tau_2 n \left(\rho_q\right)^{2n-1} \sum_{p=1}^{n} c_{p,q}, \quad q = 1, 2, \dots,$$
$$L_3 v_{2q,k}(x, L_3) = \lambda_{q,k} v_{2q-1,k}(x, L_3), \quad q = 0, 1, \dots .$$

We consider the system of functions

$$V_3 \equiv \left\{ v_{q,k,3}(x) \in H_1 : v_{q,k,3}(x) \equiv v_{q,1}(x) v_k,\ q = \overline{0, \infty},\ k = \overline{1, \infty} \right\}.$$

Remark 5.5. *The systems $V(L_3)$ and V_3 are squarely close in the space H_1.*

6 The spectral boundary value problem for a differential-operator equation with involution

Consider the spectral problem

$$(-1)^n D_x^{2n} u + A^{2n} u + \sum_{s=1}^{n} a_s \left(D_x^{2s-1} u(x) + D_x^{2s-1} u(1-x) \right) = \lambda u, \tag{63}$$

$$\ell_j u \equiv D_x^{2j-1} u(0) - D_x^{2j-1} u(1) + l_j^1 u = 0, \tag{64}$$

$$\ell_{n+j} u \equiv D_x^{2j-2} u(0) - D_x^{2j-2} u(1) = 0, \tag{65}$$

with

$$\ell_j^1 u \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r u(0) + b_{j,r,1} D_x^r u(1)), \quad j = 1, 2, \ldots, n. \tag{66}$$

Let L be the operator of the problem (63)–(66) and

$$Lu \equiv (-1)^n D_x^{2n} u + A^{2n} u + \sum_{s=1}^{n} a_s \left(D_x^{2s-1} u(x) + D_x^{2s-1} u(1-x) \right);$$

$$u \in D(L), \quad D(L) \equiv \{u \in H_2 : l_j u = 0,\ j = 1, 2, \ldots, 2n\}.$$

We can prove that

$$Lv_{2q-1,k}(x, L_3) = \lambda_{q,k} v_{2q-1,k}(x, L_3) + \xi_q^1 v_{2q,k}(x, L_3), \quad q = 1, 2, \ldots,$$

$$\xi_{q,k}^1 = \xi_{q,k}^0 + 2\sqrt{2}\tau_2 \sum_{j=1}^{n} (-1)^{j-1} a_j \left(\rho_q\right)^{2j-1} (-4j+2), \quad q = 1, 2, \ldots,$$

$$Lv_{2q,k}(x, L_3) = \lambda_{q,k} v_{2q-1,k}(x, L_3), \quad q = 0, 1, \ldots.$$

Consequently, $V(L) \equiv V(L_3)$ and the following theorem is true.

Theorem 9. *Suppose that the Assumption P_1 holds. Then for the any numbers $b_{p,r,0}$, $a_j \in \mathbb{R}$, $r = 0, 1, \ldots, m_p$, j, $p = 1, 2, \ldots, n$ we have 1) the eigenvalues of the operators L_0 and L coincide;*

2) the system $V(L)$ of the root functions of the operator L is complete and minimal in the space H_1 .

3) if in addition the Assumption P_2 holds, then the system $V(L)$ forms the Riesz basis in the space H_1.

Let

$$f = \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} f_{q,k} v_{q,k}(x, L), \quad f_{q,k} = (f, w_{q,k}(x, L); H_1).$$

Remark 6.1. *From the definition of the Riesz basis of Hilbert space and the third statement of Theorem 9 for any $f \in H_1$ we obtain the relation*

$$K_{13} \|f; H_1\|^2 \le \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} |f_{q,k}|^2 \le K_{14} \|f; H_1\|^2. \tag{67}$$

7 THE BOUNDARY VALUE PROBLEM WITH HOMOGENEOUS CONDITIONS WITH INVOLUTION

We consider the following boundary problem

$$(-1)^n D_x^{2n} w + A^{2n} w + \sum_{s=1}^{n} a_s \left(D_x^{2s-1} w(x) + D_x^{2s-1} w(1-x) \right) = f, \tag{68}$$

$$\ell_j w \equiv D_x^{2j-1} w(0) - D_x^{2j-1} w(1) + l_j^1 w = 0, \tag{69}$$

$$\ell_{n+j} w \equiv D_x^{2j-2} w(0) - D_x^{2j-2} w(1) = 0, \tag{70}$$

where

$$\ell_j^1 w \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)), \quad j = 1, 2, \ldots, n. \tag{71}$$

Let $W(L)$ be a biorthogonal system of functions from $V(L)$ in the space H_1.

Theorem 10. *Suppose that the Assumption P_1 holds. Then for arbitrary numbers $b_{p,r,0}, a_j \in \mathbb{R}$, $r = 0, 1, \ldots, m_p$, $j, p \in \{1, 2, \ldots, n\}$, and function $f \in H_1$ there exists a unique solution of the problem* (68)–(71).

Proof. The solution of the problem (68)–(71) can be determined by the relation

$$w = \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} w_{q,k} v_{q,k}(x, L). \tag{72}$$

Substituting the relations (67), (72) into the equation (68) we obtain

$$w_{2q-1,k} = \lambda_{q,k}^{-1} f_{2q-1,k}, \; w_{2q,k} = \lambda_{q,k}^{-1} f_{2q,k} - \lambda_{q,k}^{-2} \xi_{q,k}^{1} f_{2q-1,k}, \quad q, k = 1, 2, \ldots .$$

Therefore,

$$|w_{2q-1,k}|^2 \leq K_{15} |f_{2q-1,k}|^2, \tag{73}$$

$$|w_{2q,k}|^2 \leq K_{16} (|f_{2q-1,k}|^2 + |f_{2q,k}|^2), \quad q, k = 1, 2, \ldots . \tag{74}$$

Taking the assumption $f \in H_1$ and the inequalities (67) into account we obtain that $\|w; H_1\| \leq K_{17} \|f; H_1\|$, $w \in H_1$.

Consider the function $h_1 \equiv A^{2n} w$

$$\begin{aligned} h_1 = \sum_{k=1}^{\infty} (f_{0,k} v_{0,k}(x, L) + \sum_{q=1}^{\infty} (z_k^{2n} \; \lambda_{q,k}^{-1} f_{2q-1,k} v_{2q-1,k}(x, L) \\ + (z_k^{2n} \lambda_{q,k}^{-1} f_{2q,k} - \lambda_{q,k}^{-2} \; z_k^{2n} \; \xi_{q,k}^{1} f_{2q-1,k}) v_{2q,k}(x, L))). \end{aligned} \tag{75}$$

Taking the assumption $f \in H_1$ and the inequalities (67) into account we obtain that $\|h_1; H_1\| \leq K_{18} \|f; H_1\|$, $h_1 \in H_1$.

Consider the function $h_2 \equiv (-1)^n D_2^{2n} w$

$$\begin{aligned} h_2 = \sum_{k=1}^{\infty} \sum_{q=1}^{\infty} (\rho_q^{2n} \; \lambda_{q,k}^{-1} f_{2q-1,k} v_{2q-1,k}(x, L) \\ + (\rho_q^{2n} \lambda_{q,k}^{-1} f_{2q,k} - 2\rho_q^{2n} \; \lambda_{q,k}^{-2} \xi_{q,k}^{1} f_{2q-1,k}) v_{2q,k}(x, L)). \end{aligned} \tag{76}$$

From the equalities (76) we get $\|h_2; H_1\| \leq K_{19} \|f; H_1\|$, $h_2 \in H_1$.

Consider the function $h_3 = \sum\limits_{s=1}^{n} a_s(D_x^{2s-1}w(x) - D_x^{2s-1}w(1-x))$

$$h_3 = 2\sum_{k=1}^{\infty}\sum_{q=1}^{\infty}\sum_{j=1}^{n} a_j(-1)^{j-1}\rho_q^{2j-1}(\lambda_{q,k}^{-1}f_{2q,k} - \lambda_{q,k}^{-2}\xi_{q,k}^{1}f_{2q-1,k})v_{2q-1,k}(x,L). \tag{77}$$

Taking the assumption $f \in H_1$ and the equalities (77) into account we obtain that

$$\|h_3; H_1\| \leq K_{20}\|f; H_1\|.$$

From the definition of the space H_2, inequalities (75)–(77) and Cauchy's inequality we get

$$\|w; H_2 \ \leq K_{21}\|f; H_1\| < \infty,\ K_{21} = 3(\max(K_{17}^2,\ K_{18}^2,\ K_{19}^2))^{\frac{1}{2}}.$$

Thus $w \in H_2$. □

REFERENCES

[1] Ashyralyev A., Sarsenbi A.M. *Well–posedness of an elliptic equations with an involution*. Electr. J. Diff. Eq. 2015, **284**, 1–8.

[2] Baranetskij Ya.O., Yarka U.B., Fedushko S.A. *Abstract perturbations of the Dirichlet differential operator. Spectral properties*. Scientific Bull. Uzhg. Univ. Ser. Mat. Inf. 2012, **3** (1), 12–16. (in Ukrainian)

[3] Baranetskij Ya., Basha A. *Nonlocal multipoint problem for differential-operator equations of order* $2n$. J. Math. Sci. 2016, **217** (2), 176–186. doi:10.1007/s10958-016-2965-0

[4] Baranetskij Ya.O., Kalenyuk P.I., Kolyasa L.I. *Boundary-value problem for abstract differential equations with operator involution*. Bukov. Math. J. 2016, **4** (3-4), 22–29.

[5] Baranetskij Ya., Kolyasa L. *Boundary-value problem for abstract second-order differential equation with involution*. Visn. Lviv Polytech. National Univ. Ser. Phys. Math. Sci. 2017, **871**, 20–27.

[6] Baranetskij Ya. O., Kalenyuk P.I., Kolyasa L.I., Kopach M.I. *The nonlocal problem for the differential-operator equation of the even order with the involution*. Carpathian Math. Publ. 2017, **9** (2), 109–119. doi: 10.15330/cmp.9.2.109-119

[7] Burlutskaya M.Sh., Khromov A.P. *Initial–boundary value problems for first-order hyperbolic equations with involution*. Dokl. Math. 2011, **84** (3), 783–786. doi:10.1134/S1064562411070088 (translation of Dokl. Akad. Nauk 2011, **441** (2), 156–159. (in Russian))

[8] Cabada A., Tojo F.A.F. *Existence results for a linear equation with reflection, non-constant coefficient and periodic boundary conditions*. J. Math. Anal. Appl. 2014, **412** (1), 529–546. doi: 10.1016/j.jmaa.2013.10.067

[9] Gokhberg I. Ts., Krein M.G. Introduction to the Theory of Linear Non Self-Adjoint Operators. Nauka, Moscow, 1965. (in Russian)

[10] Gupta C.P. *Two-point boundary value problems involving reflection of the argument*. Int. J. Math. Math. Sci. 1987, **10** (2), 361–371. doi:10.1155/S0161171287000425

[11] Kirane M., Al-Salti N. *Inverse problems for a nonlocal wave equation with an involution perturbation*. J. Nonlinear Sci. Appl. 2016, **9**, 1243–1251.

[12] Kritskov L.V., Sarsenbi A.M. *Spectral properties of a nonlocal problem for the differential equation with involution*. Differ. Equ. 2015, **51** (8), 984–990. doi:10.1134/S0012266115080029

[13] Kurdyumov V.P. *On Riescz bases of eigenfunction of 2-nd order differential operator with involution and integral boundary conditions*. Izv. Saratov Univ. (N.S.), Ser. Math. Mech. Inform. 2015, **15** (4), 392–405. doi:10.18500/1816-9791-2015-15-4-392-405 (in Russian).

[14] Moiseev E.I., Ambartsumyan V.E. *On the Basis Property of the Eigenfunctions of the Frankl Problem with Nonlocal Evenness and Oddness Conditions of the Second Kind*. Dokl. Math. 2010, **432** (4), 451–455. doi: 10.1134/S1064562410030257 (in Russian)

[15] Naimark M.A. Linear differential operators. Frederick Ungar Publ. Co., New York, 1967.

[16] O'Regan D. *Existence results for differential equations with reflection of the argument*. J. Aust. Math. Soc. 1994, **57** (2), 237–260. doi:10.1017/S1446788700037538

[17] Sadybekov M.A., Sarsenbi A.M. *Mixed problem for a differential equation with involution under boundary conditions of general form*. AIP Conf. Proc. 2012, **1470**. doi:10.1063/1.4747681

[18] Sadybekov M.A., Turmetov B.K. *On analogues of periodic boundary value problems for the Laplace operator in a ball*. Eurasian Math J. 2012, **3** (1), 143–146.

Received 04.02.2018

Revised 03.05.2018

Баранецький Я.О., Демків І.І., Івасюк І.Я., Копач М.І. *Нелокальна задача для диференціально-операторного рівняння порядку* $2n$ *з необмеженими операторними коефіцієнтами з інволюцією* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 14–30.

Вивчається нелокальна крайова задача для диференціально-операторного рівняння парного порядку, який містить оператор іноволюції. Досліджується задача з періодичними крайовими умовами для диференціального рівняння, коефіцієнти якого є несамоспрженими операторами. Встановлено, що оператор задачі має два інваріантні підпростори, породжені оператором інволюції та дві підсистеми системи власних функцій, які є базисами Рісса в кожному з підпросторів. Для диференціально-операторного рівняння парного порядку вивчається задача з несамоспряженими крайовими умовами, які є збуреннями періодичних умов. Вивчено випадки, коли збурені умови є регулярними, але не сильно регулярними за Біркгофом та нерегулярними за Біркгофом. Визначено власні значення і елементи системи кореневих функцій V оператора задачі, яка є повною та містить нескінченне число приєднаних функцій. Отримано достатні умови, при яких система V є базисом Рісса. Визначено умови існування та єдиності розв'язку задачі з однорідними крайовими умовами, який побудовано у вигляді ряду за системою V.

Ключові слова і фрази: оператор інволюції, диференціально-операторне рівняння, власні функції, базис Рісса.