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ON WICK CALCULUS ON SPACES OF NONREGULAR GENERALIZED FUNCTIONS OF LÉVY WHITE NOISE ANALYSIS

Development of a theory of test and generalized functions depending on infinitely many variables is an important and actual problem, which is stipulated by requirements of physics and mathematics. One of successful approaches to building of such a theory consists in introduction of spaces of the above-mentioned functions in such a way that the dual pairing between test and generalized functions is generated by integration with respect to some probability measure. First it was the Gaussian measure, then it were realized numerous generalizations. In particular, important results can be obtained if one uses the Lévy white noise measure, the corresponding theory is called the *Lévy white noise analysis*.

In the Gaussian case one can construct spaces of test and generalized functions and introduce some important operators (e.g., stochastic integrals and derivatives) on these spaces by means of a so-called *chaotic representation property* (CRP): roughly speaking, any square integrable random variable can be decomposed in a series of repeated Itô's stochastic integrals from nonrandom functions. In the Lévy analysis there is no the CRP, but there are different generalizations of this property.

In this paper we deal with one of the most useful and challenging generalizations of the CRP in the Lévy analysis, which is proposed by E. W. Lytvynov, and with corresponding spaces of nonregular generalized functions. The goal of the paper is to introduce a natural product (a Wick product) on these spaces, and to study some related topics. Main results are theorems about properties of the Wick product and of Wick versions of holomorphic functions. In particular, we prove that an operator of stochastic differentiation satisfies the Leibniz rule with respect to the Wick multiplication. In addition we show that the Wick products and the Wick versions of holomorphic functions, defined on the spaces of regular and nonregular generalized functions, constructed by means of Lytvynov's generalization of the CRP, coincide on intersections of these spaces.

Our research is a contribution in a further development of the Lévy white noise analysis.

Key words and phrases: Lévy process, Wick product, stochastic differentiation.

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INTRODUCTION

Development of a theory of test and generalized functions depending on infinitely many variables (i.e., with arguments belonging to infinite-dimensional spaces) is an important and actual problem, which is stipulated by requirements of physics and mathematics (in particular, of the quantum field theory, of the mathematical physics, of the theory of random processes). A successful (but, of course, not the only) approach to building of such a theory consists in introduction of spaces of the above-mentioned functions in such a way that the dual pairing

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between test and generalized functions is generated by integration with respect to some probability measure. First it was the Gaussian measure, the corresponding theory is called the *Gaussian white noise analysis* (e.g., [7, 19, 33, 35, 37]), then it were realized numerous generalizations. In particular, important results can be obtained if one uses the Lévy white noise measure (e.g., [10, 11, 38]), the corresponding theory is called the *Lévy white noise analysis*.

In the Gaussian white noise analysis one can construct spaces of test and generalized functions and introduce some important operators (e.g., stochastic integrals and derivatives) on these spaces by means of a so-called *chaotic representation property* (CRP). This property consists, roughly speaking, in the following: any square integrable random variable can be decomposed in a series of repeated Itô's stochastic integrals from nonrandom functions (see, e.g., [39] for a detailed presentation). In the Lévy white noise analysis there is no the CRP (more exactly, the only Lévy processes with the CRP are Wiener and Poisson processes) [44]; but there are different generalizations of this property: Itô's generalization [21], Nualart-Schoutens' generalization [40, 41], Lytvynov's generalization [38], Oksendal's generalization [10, 11], etc. The interconnections between these generalizations are described in, e.g., [4, 10, 11, 29, 38, 43, 45]. Now, depending on problems under consideration, one can select a most suitable generalization of the CRP and construct corresponding spaces of test and generalized functions.

In this paper we deal with one of the most useful and challenging generalizations of the CRP in the Lévy analysis, which is proposed by E. W. Lytvynov [38] (see also [9]). The idea of this generalization is to decompose square integrable with respect to the Lévy white noise measure random variables in series of special orthogonal functions (see Subsection 1.2), by analogy with decompositions of square integrable random variables by Hermite polynomials in the Gaussian analysis (remind that the last decompositions are equivalent to decompositions by repeated stochastic integrals). In a sense, the most natural spaces that can be constructed using Lytvynov's generalization of the CRP, are spaces of *regular* test and generalized functions [25]. In a moment these spaces are well studied. In particular, the extended stochastic integral and the Hida stochastic derivative on them are introduced and studied in [14, 25], operators of stochastic differentiation — in [12, 13, 16], some elements of a Wick calculus — in [15]. But, as in the Gaussian analysis, in connection with some problems of the mathematical physics and of the stochastic analysis (in particular, of the theory of stochastic equations with Wick-type nonlinearities), it is necessary to introduce into consideration so-called spaces of *nonregular* test and generalized functions in terms of Lytvynov's generalization of the CRP [25], and to study operators and operations on these spaces. Note that, as distinct from the Gaussian analysis, now the spaces of regular generalized functions are not embedded into the spaces of nonregular generalized functions, and, accordingly, the spaces of nonregular test functions are not embedded into the spaces of regular test functions. Moreover, one can widen the extended stochastic integral from the space of square integrable random variables to the spaces of nonregular generalized functions, and, accordingly, to restrict the Hida stochastic derivative and the operators of stochastic differentiation to the spaces of nonregular test functions; but the extended stochastic integral cannot be naturally restricted to the spaces of nonregular test functions, and, accordingly, it is impossible to widen in a natural way the Hida stochastic derivative and the operators of stochastic differentiation to the spaces of nonregular generalized functions. Therefore it is necessary to introduce and to study natural analogs of the above-mentioned operators on the corresponding spaces. The stochastic integrals, derivatives, operators of stochastic differentiation, and their analogs on the spaces of nonregular test and generalized functions are studied in detail in [25, 30, 31]. The goal of the present paper is to make the next natural step — to introduce a natural product (a Wick product) on the spaces of nonregular generalized functions, by analogy with the Gaussian analysis [34] and with the Lévy analysis on the spaces of regular generalized functions [15], and to study some related topics (Wick versions of holomorphic functions, an interconnection between the Wick calculus and the operators of stochastic differentiation). Main results of the paper are theorems about properties of the Wick product and of the Wick versions of holomorphic functions. In particular, we prove that, as in the regular case, the operator of stochastic differentiation is a differentiation (satisfies the Leibniz rule) with respect to the Wick multiplication. In addition we show that the Wick products and the Wick versions of holomorphic functions, defined on the spaces of regular and nonregular generalized functions, constructed by means of Lytvynov's generalization of the CRP, coincide on intersections of these spaces.

Note that some results of the paper can be transferred to weighted symmetric Fock spaces, by analogy with [32]. This gives an opportunity to extend an area of possible applications of these results. In particular, one can transfer them to any spaces isomorphic to the above-mentioned Fock spaces.

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a probability triplet connected with L, convenient for our considerations; then we describe Lytvynov's generalization of the CRP; and construct a nonregular rigging of the space of square integrable random variables (the positive and negative spaces of this rigging are the spaces of nonregular test and generalized functions respectively). The second section is devoted to the Wick calculus: in the first subsection we introduce and study the Wick product and the Wick versions of holomorphic functions on the spaces of nonregular generalized functions; in the second subsection we consider a question about an interconnection between Wick calculuses in the regular and nonregular cases; in the third subsection we study an interconnection between the Wick calculus and the operator of stochastic differentiation.

1 PRELIMINARIES

In this paper we denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the *real*, i.e., *bilinear* scalar product in a space H; and by $\langle\cdot,\cdot\rangle_H$ or $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ the dual pairing generated by the scalar product in a space H.

1.1 A Lévy process and its probability space

Denote $\mathbb{R}_+ := [0,+\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_u)_{u\in\mathbb{R}_+}$ (a random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift. As is well known (e.g., [11]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_u}] = \exp\left[u\int_{\mathbb{R}}(e^{i\theta x} - 1 - i\theta x)\nu(dx)\right], \tag{1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, here and below $\mathcal{B}$ denotes the Borel σ-algebra; $\mathbb{E}$ denotes the expectation. We assume that *ν is a Radon measure whose support contains an infinite number of points, $\nu(\{0\}) = 0$, there exists $\varepsilon > 0$ such that $\int_{\mathbb{R}} x^2 e^{\varepsilon|x|}\nu(dx) < \infty$, and $\int_{\mathbb{R}} x^2\nu(dx) = 1$.*

Let us define a measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [8]; see also Subsection 1.3). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle\omega,\varphi\rangle$; note that actually $\langle\cdot,\cdot\rangle$ is the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on $\mathbb{R}_+$ [8]. The notation $\langle\cdot,\cdot\rangle$ will be preserved for dual pairings in tensor powers of the complexification of a rigging $\mathcal{D}' \supset L^2(\mathbb{R}_+) \supset \mathcal{D}$.

Definition 1. *A probability measure μ on $(\mathcal{D}',\mathcal{C}(\mathcal{D}'))$, where $\mathcal{C}$ denotes the cylindrical σ-algebra, with the Fourier transform*

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+\times\mathbb{R}}(e^{i\varphi(u)x} - 1 - i\varphi(u)x)du\nu(dx)\Big], \quad \varphi \in \mathcal{D}, \tag{2}$$

is called the measure of a Lévy white noise.

The existence of μ follows from the Bochner-Minlos theorem (e.g., [20]), see [38]. Below we assume that the *σ-algebra $\mathcal{C}(\mathcal{D}')$ is completed with respect to μ.*

Denote by $(L^2) := L^2(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$ the space of (classes of) complex-valued square integrable with respect to μ functions on $\mathcal{D}'$ (in what follows, this notation will be used very often). Let $f \in L^2(\mathbb{R}_+)$ and a sequence $(\varphi_k \in \mathcal{D})_{k\in\mathbb{N}}$ converge to f in $L^2(\mathbb{R}_+)$ as $k \to \infty$ (as is well known (e.g., [8]), $\mathcal{D}$ is a dense set in $L^2(\mathbb{R}_+)$). One can show [10, 11, 29, 38] that $\langle\circ, f\rangle := (L^2) - \lim_{k\to\infty}\langle\circ,\varphi_k\rangle$ is well-defined as an element of (L^2).

Denote by 1_A the indicator of a set A. Put $1_{[0,0)} \equiv 0$ and consider $\langle\circ, 1_{[0,u)}\rangle \in (L^2)$, $u \in \mathbb{R}_+$. It follows from (1) and (2) that $\big(\langle\circ, 1_{[0,u)}\rangle\big)_{u\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space (triplet) $(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu)$ (see [10,11]). So, one can write $L_u = \langle\circ, 1_{[0,u)}\rangle \in (L^2)$.

Remark 1. *The derivative in the sense of generalized functions (e.g., [17]) of a Lévy process (a Lévy white noise) is $L'_\cdot(\omega) = \langle\omega,\delta_\cdot\rangle = \omega(\cdot)$, where δ is the Dirac delta-function. Therefore L' is a generalized random process (in the sense of [17]) with trajectories from $\mathcal{D}'$, and μ is the measure of L' in the classical sense of this notion [18].*

Remark 2. *A Lévy process L without Gaussian part and drift is a* Poisson process *if its Lévy measure ν is a point mass at 1, i.e., if for each $\Delta \in \mathcal{B}(\mathbb{R})$ $\nu(\Delta) = \delta_1(\Delta)$. This measure does not satisfy the conditions accepted above (the support of δ_1 does not contain an infinite number of points); nevertheless, all results of the present paper have natural (and often strong) analogs in the Poissonian analysis. The reader can find more information about peculiarities of the Poissonian case in [29], Subsection 1.2.*

1.2 Lytvynov's generalization of the CRP

Denote by $\widehat{\otimes}$ the symmetric tensor multiplication, by a subscript $\mathbb{C}$ — complexifications of spaces. Set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Denote by $\mathcal{P}$ the set of complex-valued polynomials on $\mathcal{D}'$ that consists of zero and elements of the form

$$f(\omega) = \sum_{n=0}^{N_f}\langle\omega^{\otimes n}, f^{(n)}\rangle, \quad \omega \in \mathcal{D}',\ f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n},\ N_f \in \mathbb{Z}_+,\ f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle\omega^{\otimes 0}, f^{(0)}\rangle := f^{(0)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}0} := \mathbb{C}$. The measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (2) and properties of the measure ν, see also [38]), then $\mathcal{P}$ is a dense set in (L^2) [42]. Denote by $\mathcal{P}_n$, $n \in \mathbb{Z}_+$, the set of polynomials of power smaller or equal to n, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2) = \mathop{\oplus}_{n=0}^{\infty} \mathbf{P}_n. \tag{3}$$

Let $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n \in \mathbb{Z}_+$. Denote by $:\langle\circ^{\otimes n}, f^{(n)}\rangle:$ the orthogonal projection of a monomial $\langle\circ^{\otimes n}, f^{(n)}\rangle$ onto $\mathbf{P}_n$. Let us define *real*, i.e., *bilinear* scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!}\int_{\mathcal{D}'} :\langle\omega^{\otimes n}, f^{(n)}\rangle::\langle\omega^{\otimes n}, g^{(n)}\rangle:\mu(d\omega). \tag{4}$$

The proof of the well-posedness of this definition coincides up to obvious modifications with the proof of the corresponding statement in [38].

By $|\cdot|_{ext}$ we denote the norms corresponding to scalar products (4), i.e.,

$$|f^{(n)}|_{ext} := \sqrt{(f^{(n)}, \overline{f^{(n)}})_{ext}}.$$

Denote by $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, the completions of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ with respect to the norms $|\cdot|_{ext}$. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\langle\circ^{\otimes n}, F^{(n)}\rangle: \stackrel{\text{def}}{=} (L^2){-}\lim_{k\to\infty} :\langle\circ^{\otimes n}, f_k^{(n)}\rangle:$, where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n} \ni f_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$ (the well-posedness of this definition can be proved by the method of "mixed sequences"). One can show that $:\langle\circ^{\otimes 0}, F^{(0)}\rangle: = \langle\circ^{\otimes 0}, F^{(0)}\rangle = F^{(0)}$ and $:\langle\circ, F^{(1)}\rangle: = \langle\circ, F^{(1)}\rangle$ (cf. [38]).

Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\langle\circ^{\otimes n}, f^{(n)}\rangle: | f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}\}$ is dense in $\mathbf{P}_n$, the next statement from (3) follows.

Theorem 1. (Lytvynov's generalization of the CRP, cf. [38]) *A random variable $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ such that*

$$F = \sum_{n=0}^{\infty} :\langle\circ^{\otimes n}, F^{(n)}\rangle: \tag{5}$$

(the series converges in (L^2)) and $\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2\mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n!|F^{(n)}|_{ext}^2 < \infty$.

Remark 3. *In order to consider many problems of the Lévy white noise analysis, in terms of Lytvynov's generalization of the CRP, it is necessary to know an explicit formula for the scalar products $(\cdot,\cdot)_{ext}$. Such a formula is calculated in [38]; in another record form (more convenient for some calculations) it is given in, e.g., [13, 15, 16].*

Denote $\mathcal{H} := L^2(\mathbb{R}_+)$, then $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}_+)_{\mathbb{C}}$ (in what follows, this notation will be used very often). It follows from the explicit formula for $(\cdot,\cdot)_{ext}$ that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, and for $n \in \mathbb{N}\setminus\{1\}$ one can identify $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (roughly speaking, such that $F^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k, j \in \{1,\dots,n\}$: $k \neq j$, but $u_k = u_j$). In this sense the *space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$* (this explains why we use the subscript "*ext*" in our designations).

1.3 A nonregular rigging of (L^2)

Denote by T the set of indexes $\tau = (\tau_1, \tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Let $\mathcal{H}_\tau$ be the real Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., $\mathcal{H}_\tau$ is the completion of $\mathcal{D}$ with respect to the norm generated by the scalar product

$$(\varphi, \psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+} \Big(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1} \varphi^{[k]}(u)\psi^{[k]}(u)\Big)\tau_2(u)du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ respectively. It is well known (e.g., [8]) that $\mathcal{D} = \operatorname{pr}\lim_{\tau \in T} \mathcal{H}_\tau$ (moreover, for any $n \in \mathbb{N}$ $\mathcal{D}^{\widehat{\otimes} n} = \operatorname{pr}\lim_{\tau \in T} \mathcal{H}_\tau^{\widehat{\otimes} n}$, see, e.g., [6] for details), and for each $\tau \in T$ $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$. Therefore one can consider the chain

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D},$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Note that by the Schwartz theorem [8] $\mathcal{D}' = \operatorname{ind}\lim_{\tau \in T} \mathcal{H}_{-\tau}$ (it is convenient for us to consider $\mathcal{D}'$ as a topological space with the inductive limit topology). By analogy with [28] one can easily show that the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$ with some $\widetilde{\tau} \in T$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$. Excepting from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$, we will assume, in what follows, that *for each $\tau \in T$ $\mu(\mathcal{H}_{-\tau}) = 1$.*

Denote the norms in $\mathcal{H}_{\tau,\mathbb{C}}$ and its tensor powers by $|\cdot|_\tau$, i.e., for $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, $|f^{(n)}|_\tau = \sqrt{(f^{(n)}, \overline{f^{(n)}})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}}$ (note that $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$ and $|f^{(0)}|_\tau = |f^{(0)}|$).

The next statement easily follows from results of [25].

Lemma 1. *There exists $\tau' \in T$ such that for each $n \in \mathbb{Z}_+$ the space $\mathcal{H}_{\tau',\mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$. Moreover, for all $f^{(n)} \in \mathcal{H}_{\tau',\mathbb{C}}^{\widehat{\otimes} n}$ $|f^{(n)}|_{ext}^2 \leq n!c^n|f^{(n)}|_{\tau'}^2$, where $c > 0$ is some constant.*

It follows from this lemma that if for some $\tau \in T$ the space $\mathcal{H}_\tau$ is continuously embedded into the space $\mathcal{H}_{\tau'}$ then for each $n \in \mathbb{Z}_+$ the space $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$

$$|f^{(n)}|_{ext}^2 \leq n!c(\tau)^n|f^{(n)}|_\tau^2. \tag{6}$$

In what follows, it will be convenient to assume that the *indexes τ such that $\mathcal{H}_\tau$ is not continuously embedded into $\mathcal{H}_{\tau'}$, are removed from T.*

Accept on default $q \in \mathbb{Z}_+$ and $\tau \in T$. Denote $\mathcal{P}_W := \{f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, N_f \in \mathbb{Z}_+\} \subset (L^2)$. Define real scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \; g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle: \in \mathcal{P}_W$$

$$(f, g)_{\tau,q} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}. \tag{7}$$

Let $\|\cdot\|_{\tau,q}$ be the corresponding norms, i.e., $\|f\|_{\tau,q} = \sqrt{(f,\overline{f})_{\tau,q}}$. In order to verify the well-posedness of this definition, i.e., that formula (7) defines *scalar*, and not just quasiscalar products, we note that if $f \in \mathcal{P}_W$ and $\|f\|_{\tau,q} = 0$ then by (7) for each kernel $f^{(n)}$ we have $|f^{(n)}|_\tau = 0$ and therefore by (6) $|f^{(n)}|_{ext} = 0$. So, in this case $f = 0$ in (L^2).

Let $(\mathcal{H}_\tau)_q$ be completions of $\mathcal{P}_W$ with respect to the norms $\|\cdot\|_{\tau,q}$, $(\mathcal{H}_\tau) := \operatorname{pr}\lim_{q\to\infty}(\mathcal{H}_\tau)_q$, $(\mathcal{D}) := \operatorname{pr}\lim_{\tau\in T, q\to\infty}(\mathcal{H}_\tau)_q$. As is easy to see, $f \in (\mathcal{H}_\tau)_q$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle:, \; f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \tag{8}$$

(the series converges in $(\mathcal{H}_\tau)_q$), with

$$\|f\|_{\tau,q}^2 := \|f\|_{(\mathcal{H}_\tau)_q}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_\tau^2 < \infty \tag{9}$$

(since for each $n \in \mathbb{Z}_+$ $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \subseteq \mathcal{H}_{ext}^{(n)}$, for $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ is a well defined Wick monomial, see Subsection 1.2). Further, $f \in (\mathcal{H}_\tau)$ ($f \in (\mathcal{D})$) if and only if f can be presented in form (8) and norm (9) is finite for each $q \in \mathbb{Z}_+$ (for each $\tau \in T$ and each $q \in \mathbb{Z}_+$).

Lemma 2. *For each $\tau \in T$ there exists $q_0(\tau) \in \mathbb{Z}_+$ such that the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into (L^2) for each $q \in \mathbb{N}_{q_0(\tau)} := \{q_0(\tau), q_0(\tau)+1, \dots\}$.*

The proof coincides up to obvious modifications with the proof of the corresponding statement in the real case [25]. In view of this lemma one can consider a chain

$$(\mathcal{D}') \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}), \; \tau \in T, \; q \in \mathbb{N}_{q_0(\tau)}, \tag{10}$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \operatorname{ind}\lim_{q'\to\infty}(\mathcal{H}_{-\tau})_{-q'}$ and $(\mathcal{D}') = \operatorname{ind}\lim_{\widehat{\tau}\in T, q'\to\infty}(\mathcal{H}_{-\widehat{\tau}})_{-q'}$ are the spaces dual of $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ with respect to (L^2).

Definition 2. *Chain* (10) *is called a nonregular rigging of the space (L^2). The positive spaces of this chain $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ are called Kondratiev spaces of nonregular test functions. The negative spaces of this chain $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$ and $(\mathcal{D}')$ are called Kondratiev spaces of nonregular generalized functions.*

Finally, we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider chains

$$\mathcal{D}_{\mathbb{C}}'^{(m)} \supset \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \supset \mathcal{H}_{ext}^{(m)} \supset \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \supset \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}, \tag{11}$$

$m \in \mathbb{Z}_+$ (for $m = 0$ $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} 0} = \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} 0} = \mathcal{H}_{ext}^{(0)} = \mathcal{H}_{-\tau,\mathbb{C}}^{(0)} = \mathcal{D}_{\mathbb{C}}'^{(0)} = \mathbb{C}$), where $\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ and $\mathcal{D}_{\mathbb{C}}'^{(m)} = \operatorname{ind}\lim_{\widehat{\tau}\in T} \mathcal{H}_{-\widehat{\tau},\mathbb{C}}^{(m)}$ are the spaces dual of $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}$ and $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}$ with respect to $\mathcal{H}_{ext}^{(m)}$. In what follows, we denote by $\langle\cdot,\cdot\rangle_{ext}$ the real dual pairings between elements of negative and positive spaces from chains (11), these pairings are generated by the scalar products in $\mathcal{H}_{ext}^{(m)}$. The next statement follows from the definition of the spaces $(\mathcal{H}_{-\tau})_{-q}$ and the general duality theory (cf. [25,28]).

Proposition 1. *There exists a system of generalized functions*

$$\{:\langle\circ^{\otimes m}, F_{ext}^{(m)}\rangle: \in (\mathcal{H}_{-\tau})_{-q} \mid F_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)},\ m \in \mathbb{Z}_+\}$$

such that

1) for $F_{ext}^{(m)} \in \mathcal{H}_{ext}^{(m)} \subset \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ $:\langle\circ^{\otimes m}, F_{ext}^{(m)}\rangle:$ is a Wick monomial that is defined in Subsection 1.2;

2) any generalized function $F \in (\mathcal{H}_{-\tau})_{-q}$ can be presented as a series

$$F = \sum_{m=0}^{\infty} :\langle\circ^{\otimes m}, F_{ext}^{(m)}\rangle:,\ F_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}, \tag{12}$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$, i.e.,

$$\|F\|_{-\tau,-q}^2 := \|F\|_{(\mathcal{H}_{-\tau})_{-q}}^2 = \sum_{m=0}^{\infty} 2^{-qm} |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}}^2 < \infty; \tag{13}$$

and, vice versa, any series (12) *with finite norm* (13) *is a generalized function from $(\mathcal{H}_{-\tau})_{-q}$ (i.e., such a series converges in $(\mathcal{H}_{-\tau})_{-q}$);*

3) the dual pairing between $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_{\tau})_q$ that is generated by the scalar product in (L^2), has the form

$$\langle\langle F, f\rangle\rangle_{(L^2)} = \sum_{m=0}^{\infty} m! \langle F_{ext}^{(m)}, f^{(m)}\rangle_{ext}, \tag{14}$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ and $f^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}$ are the kernels from decompositions (12) *and* (8) *for F and f respectively.*

It is clear that $F \in (\mathcal{H}_{-\tau})$ ($F \in (\mathcal{D}')$) if and only if F can be presented in form (12) and norm (13) is finite for some $q \in \mathbb{N}_{q_0(\tau)}$ (for some $\tau \in T$ and some $q \in \mathbb{N}_{q_0(\tau)}$).

2 ELEMENTS OF A WICK CALCULUS

In this paper we construct a Wick calculus on the spaces $(\mathcal{H}_{-\tau})$; but, as is easy to verify, all our results hold true up to obvious modifications on the space $(\mathcal{D}')$.

2.1 A Wick product and Wick versions of holomorphic functions

One can introduce a Wick product and Wick versions of holomorphic functions on $(\mathcal{H}_{-\tau})$ by different ways. We use the most natural and convenient from technical point of view classical way, based on a so-called S-transform.

Definition 3. *Let $F \in (\mathcal{H}_{-\tau})$. We define an S-transform $(SF)(\lambda)$, $\lambda \in \mathcal{D}_{\mathbb{C}}$, as a formal series*

$$(SF)(\lambda) := \sum_{m=0}^{\infty} \langle F_{ext}^{(m)}, \lambda^{\otimes m}\rangle_{ext} \equiv F_{ext}^{(0)} + \sum_{m=1}^{\infty} \langle F_{ext}^{(m)}, \lambda^{\otimes m}\rangle_{ext}, \tag{15}$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ are the kernels from (12) *for F. In particular, $(SF)(0) = F_{ext}^{(0)}$, $S1 \equiv 1$.*

Remark 4. *As is easily seen, each term in series* (15) *is well-defined, but the series can diverge. However, the last is not an obstruction in order to construct the Wick calculus (cf. [15]); moreover, it is easy to obtain a simple sufficient condition under which series* (15) *converges. Namely, by the generalized and classical Cauchy-Bunyakovsky inequalities*

$$\begin{aligned}|(SF)(\lambda)| &\leq \sum_{m=0}^{\infty} |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}} |\lambda|_{\tau}^{m} = \sum_{m=0}^{\infty} \left(2^{-qm/2} |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}}\right)\left(2^{qm/2}|\lambda|_{\tau}^{m}\right) \\ &\leq \sqrt{\sum_{m=0}^{\infty} 2^{-qm} |F_{ext}^{(m)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}}} \sqrt{\sum_{m=0}^{\infty} 2^{qm}|\lambda|_{\tau}^{2m}} = \|F\|_{-\tau,-q} \sqrt{\sum_{m=0}^{\infty} 2^{qm}|\lambda|_{\tau}^{2m}}\end{aligned}$$

(see (13)*). Therefore series* (15) *converges if* $F \in (\mathcal{H}_{-\tau})_{-q}$ *and* $\lambda \in \mathcal{D}_{\mathbb{C}}$ *is such that* $|\lambda|_\tau < 2^{-q/2}$*. Note that the last inequality is true if and only if a function* $f_\lambda(\circ) := \sum_{m=0}^{\infty} \frac{1}{m!} {:}\langle \circ^{\otimes m}, \lambda^{\otimes m}\rangle{:} \in (\mathcal{H}_\tau)_q$*, in this case* $\|f_\lambda\|_{\tau,q} = \sqrt{\sum_{m=0}^{\infty} 2^{qm}|\lambda|_\tau^{2m}} < \infty$ *(see* (9)*). Now one can define the* S*-transform of* F *by the formula* $(SF)(\lambda) = \langle\!\langle F, f_\lambda \rangle\!\rangle_{(L^2)}$ *(cf. [34]), see* (14)*. Note that in the Gaussian (and Poissonian) analysis* $f_\lambda(\circ) = \exp^{\Diamond}\{\langle \circ, \lambda\rangle\}$*, where* $\exp^{\Diamond}$ *is a Wick version of the exponential function (e.g., [34]), and therefore* f_λ *is called a* Wick exponential*; in the Lévy analysis this representation for* f_λ *does not hold.*

Definition 4. *For* $F, G \in (\mathcal{H}_{-\tau})$ *and a holomorphic at* $(SF)(0)$ *function* $h : \mathbb{C} \to \mathbb{C}$ *we define a Wick product* $F \Diamond G$ *and a Wick version* $h^{\Diamond}(F)$ *by setting formally*

$$F \Diamond G := S^{-1}(SF \cdot SG), \quad h^{\Diamond}(F) := S^{-1}h(SF). \tag{16}$$

It is obvious that the Wick multiplication $\Diamond$ is commutative, associative and distributive over a field $\mathbb{C}$.

Remark 5. *A function* h *can be decomposed in a Taylor series*

$$h(u) = \sum_{m=0}^{\infty} h_m \big(u - (SF)(0)\big)^m. \tag{17}$$

Using this decomposition, it is easy to calculate that

$$h^{\Diamond}(F) = \sum_{m=0}^{\infty} h_m \big(F - (SF)(0)\big)^{\Diamond m}, \tag{18}$$

where $F^{\Diamond m} := \underbrace{F \Diamond \cdots \Diamond F}_{m \text{ times}}$*,* $F^{\Diamond 0} := 1$*.*

Let us write out "coordinate formulas" for the Wick product and for the Wick versions of holomorphic functions (i.e., representations of $F \Diamond G$ and $h^{\Diamond}(F)$ via kernels from decompositions (12) for F and G and coefficients from decomposition (17) for h). We need a small preparation: it is necessary to introduce an analog of the symmetric tensor multiplication on the spaces $\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$, $m \in \mathbb{Z}_+$.

Consider a family of chains

$$\mathcal{D}_{\mathbb{C}}'^{\widehat{\otimes} m} \supset \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes} m} \supset \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} m} \supset \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \supset \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}, \ m \in \mathbb{Z}_+ \tag{19}$$

(as is well known (e.g., [6,8]), $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}m}$ and $\mathcal{D}_{\mathbb{C}}'^{\widehat{\otimes}m} = \operatorname{ind\,lim}_{\widehat{\tau}\in T} \mathcal{H}_{-\widehat{\tau},\mathbb{C}}^{\widehat{\otimes}m}$ are the spaces dual of $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}m}$ and $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}m}$ respectively; in the case $m = 0$ all spaces from chain (19) are equal to $\mathbb{C}$). Since the spaces of test functions in chains (19) and (11) coincide, there exists a family of natural isomorphisms $U_m : \mathcal{D}_{\mathbb{C}}'^{(m)} \to \mathcal{D}_{\mathbb{C}}'^{\widehat{\otimes}m}$, $m \in \mathbb{Z}_+$, such that for all $F_{ext}^{(m)} \in \mathcal{D}_{\mathbb{C}}'^{(m)}$ and $f^{(m)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}m}$

$$\langle F_{ext}^{(m)}, f^{(m)} \rangle_{ext} = \langle U_m F_{ext}^{(m)}, f^{(m)} \rangle. \tag{20}$$

It is easy to see that the restrictions of U_m to $\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ are isometric isomorphisms between the spaces $\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ and $\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}m}$.

Remark 6. *As we saw above, $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, therefore in the case $m = 1$ chains* (19) *and* (11) *coincide. Thus U_1 is the identity operator on $\mathcal{D}_{\mathbb{C}}'^{(1)} = \mathcal{D}_{\mathbb{C}}'^{\widehat{\otimes}1} = \mathcal{D}_{\mathbb{C}}'$. In the case $m = 0$ U_0 is, obviously, the identity operator on $\mathbb{C}$.*

For $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ and $G_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$, $n, m \in \mathbb{Z}_+$, set

$$F_{ext}^{(n)} \diamond G_{ext}^{(m)} := U_{n+m}^{-1}\big[(U_n F_{ext}^{(n)}) \widehat{\otimes} (U_m G_{ext}^{(m)})\big] \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)}. \tag{21}$$

It follows from properties of operators U_m and of the symmetric tensor multiplication that the multiplication $\diamond$ is commutative, associative and distributive over a field $\mathbb{C}$. Further, since $U_m : \mathcal{H}_{-\tau,\mathbb{C}}^{(m)} \to \mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}m}$, $m \in \mathbb{Z}_+$, are isometric isomorphisms,

$$\begin{aligned} |F_{ext}^{(n)} \diamond G_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)}} &= |(U_n F_{ext}^{(n)}) \widehat{\otimes} (U_m G_{ext}^{(m)})|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n+m}} \\ &\le |U_n F_{ext}^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}n}} |U_m G_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{\widehat{\otimes}m}} = |F_{ext}^{(n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}} |G_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}}. \end{aligned} \tag{22}$$

Finally, by (20) and (21) for $\lambda \in \mathcal{D}_{\mathbb{C}}$

$$\begin{aligned} \langle F_{ext}^{(n)}, \lambda^{\otimes n} \rangle_{ext} \langle G_{ext}^{(m)}, \lambda^{\otimes m} \rangle_{ext} &= \langle U_n F_{ext}^{(n)}, \lambda^{\otimes n} \rangle \langle U_m G_{ext}^{(m)}, \lambda^{\otimes m} \rangle \\ &= \langle (U_n F_{ext}^{(n)}) \otimes (U_m G_{ext}^{(m)}), \lambda^{\otimes n+m} \rangle = \langle (U_n F_{ext}^{(n)}) \widehat{\otimes} (U_m G_{ext}^{(m)}), \lambda^{\otimes n+m} \rangle \\ &= \langle U_{n+m}^{-1}\big[(U_n F_{ext}^{(n)}) \widehat{\otimes} (U_m G_{ext}^{(m)})\big], \lambda^{\otimes n+m} \rangle_{ext} = \langle F_{ext}^{(n)} \diamond G_{ext}^{(m)}, \lambda^{\otimes n+m} \rangle_{ext}. \end{aligned}$$

Using (16), (15) and this equality, by analogy with the Meixner analysis [28] one can prove the following statement.

Proposition 2. *For $F_1, \dots, F_n \in (\mathcal{H}_{-\tau})$*

$$F_1 \Diamond \cdots \Diamond F_n = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \sum_{\substack{k_1,\dots,k_n \in \mathbb{Z}_+: \\ k_1+\cdots+k_n=m}} F_1^{(k_1)} \diamond \cdots \diamond F_n^{(k_n)} \rangle :; \tag{23}$$

in particular, for $F, G \in (\mathcal{H}_{-\tau})$

$$F \Diamond G = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \sum_{k=0}^{m} F_{ext}^{(k)} \diamond G_{ext}^{(m-k)} \rangle :. \tag{24}$$

Here $F_j^{(k_j)} \in \mathcal{H}^{(k_j)}_{-\tau,\mathbb{C}}$, $j \in \{1,\dots,n\}$, *are the kernels from decompositions* (12) *for* F_j; $F_{ext}^{(k)} \in \mathcal{H}^{(k)}_{-\tau,\mathbb{C}}$, $G_{ext}^{(m-k)} \in \mathcal{H}^{(m-k)}_{-\tau,\mathbb{C}}$, *are the kernels from the same decompositions for* F *and* G *respectively. Further, for* $F \in (\mathcal{H}_{-\tau})$ *and a holomorphic at* $(SF)(0) = F_{ext}^{(0)}$ *function* $h : \mathbb{C} \to \mathbb{C}$

$$h^{\Diamond}(F) = h_0 + \sum_{m=1}^{\infty} :\langle \circ^{\otimes m}, \sum_{n=1}^{m} h_n \sum_{\substack{k_1,\dots,k_n \in \mathbb{N}:\\ k_1+\cdots+k_n=m}} F_{ext}^{(k_1)} \diamond \cdots \diamond F_{ext}^{(k_n)} \rangle :, \tag{25}$$

where $F_{ext}^{(k)} \in \mathcal{H}^{(k)}_{-\tau,\mathbb{C}}$, $k \in \mathbb{Z}_+$, *are the kernels from decomposition* (12) *for* F; $h_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$, *are the coefficients from decomposition* (17) *for* h.

Remark 7. *Formulas* (24) *and* (25) *can be used as alternative definitions of the Wick product and of the Wick version of a holomorphic function respectively.*

It is clear that in order to give an informal sense to notions "the Wick product" and "the Wick version of a holomorphic function", it is necessary to study a question about convergence of series (23) and (25) in the spaces $(\mathcal{H}_{-\tau})$.

Theorem 2. *1) Let* $F_1,\dots,F_n \in (\mathcal{H}_{-\tau})$. *Then* $F_1 \Diamond \cdots \Diamond F_n \in (\mathcal{H}_{-\tau})$. *Moreover, the Wick multiplication is continuous in the sense that*

$$\|F_1 \Diamond \cdots \Diamond F_n\|_{-\tau,-q} \le \sqrt{\max_{m\in\mathbb{Z}_+} [2^{-m}(m+1)^{n-1}]}\, \|F_1\|_{-\tau,-(q-1)} \cdots \|F_n\|_{-\tau,-(q-1)}, \tag{26}$$

where $q \in \mathbb{N}$ *is such that* $F_1,\dots,F_n \in (\mathcal{H}_{-\tau})_{-(q-1)}$. *2) Let* $F \in (\mathcal{H}_{-\tau})$ *and a function* $h : \mathbb{C} \to \mathbb{C}$ *be holomorphic at* $(SF)(0)$. *Then* $h^{\Diamond}(F) \in (\mathcal{H}_{-\tau})$.

Proof. 1) It is sufficient to prove (26), the fact that $F_1 \Diamond \cdots \Diamond F_n \in (\mathcal{H}_{-\tau})$ follows from this estimate. Let $F_j^{(k)} \in \mathcal{H}^{(k)}_{-\tau,\mathbb{C}}$ be the kernels from decompositions (12) for F_j, $j \in \{1,\dots,n\}$; and $q \in \mathbb{N}$ be such that $F_1,\dots,F_n \in (\mathcal{H}_{-\tau})_{-(q-1)}$ (such q exists because by Schwartz's theorem $(\mathcal{H}_{-\tau}) = \bigcup_{q\in\mathbb{N}_{q_0(\tau)}} (\mathcal{H}_{-\tau})_{-q}$ ($\mathbb{N}_{q_0(\tau)}$ is defined in Lemma 2), see, e.g., [8] for details). Using (23), (13), a known estimate for a norm $\left\|\sum_{l=1}^{p} a_l\right\|^2 \le p\sum_{l=1}^{p}\|a_l\|^2$ and (22), we obtain

$$\begin{aligned}
\|F_1 \Diamond \cdots \Diamond F_n\|^2_{-\tau,-q} &= \sum_{m=0}^{\infty} 2^{-qm} \Big| \sum_{\substack{k_1,\dots,k_n\in\mathbb{Z}_+:\\ k_1+\cdots+k_n=m}} F_1^{(k_1)} \diamond \cdots \diamond F_n^{(k_n)} \Big|^2_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}} \\
&= \sum_{m=0}^{\infty} 2^{-qm} \Big| \sum_{k_1=0}^{m} \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{n-1}=0}^{m-k_1-\cdots-k_{n-2}} F_1^{(k_1)} \diamond \cdots \diamond F_{n-1}^{(k_{n-1})} \diamond F_n^{(m-k_1-\cdots-k_{n-1})} \Big|^2_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}} \\
&\le \sum_{m=0}^{\infty} 2^{-qm}(m+1) \sum_{k_1=0}^{m} \Big| \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{n-1}=0}^{m-k_1-\cdots-k_{n-2}} F_1^{(k_1)} \diamond \cdots \diamond F_n^{(m-k_1-\cdots-k_{n-1})} \Big|^2_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}} \\
&\le \cdots \le \sum_{m=0}^{\infty} 2^{-qm}(m+1)^{n-1} \sum_{k_1=0}^{m} \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{n-1}=0}^{m-k_1-\cdots-k_{n-2}} \big| F_1^{(k_1)} \diamond \cdots \diamond F_n^{(m-k_1-\cdots-k_{n-1})} \big|^2_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}} \\
&\le \sum_{m=0}^{\infty} \big[2^{-m}(m+1)^{n-1}\big] 2^{-(q-1)m}
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{k_1=0}^{m} \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{n-1}=0}^{m-k_1-\cdots-k_{n-2}} |F_1^{(k_1)}|^2_{\mathcal{H}^{(k_1)}_{-\tau,\mathbb{C}}} \cdots |F_n^{(m-k_1-\cdots-k_{n-1})}|^2_{\mathcal{H}^{(m-k_1-\cdots-k_{n-1})}_{-\tau,\mathbb{C}}} \\
&\le C(n) \sum_{k_1=0}^{\infty} 2^{-(q-1)k_1} |F_1^{(k_1)}|^2_{\mathcal{H}^{(k_1)}_{-\tau,\mathbb{C}}} \sum_{m=k_1}^{\infty} \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{n-1}=0}^{m-k_1-\cdots-k_{n-2}} 2^{-(q-1)k_2} |F_2^{(k_2)}|^2_{\mathcal{H}^{(k_2)}_{-\tau,\mathbb{C}}} \\
&\cdots 2^{-(q-1)(m-k_1-\cdots-k_{n-1})} |F_n^{(m-k_1-\cdots-k_{n-1})}|^2_{\mathcal{H}^{(m-k_1-\cdots-k_{n-1})}_{-\tau,\mathbb{C}}} \\
&= C(n) \|F_1\|^2_{-\tau,-(q-1)} \sum_{m=0}^{\infty} \sum_{k_2=0}^{m} \cdots \sum_{k_{n-1}=0}^{m-k_2-\cdots-k_{n-2}} 2^{-(q-1)k_2} |F_2^{(k_2)}|^2_{\mathcal{H}^{(k_2)}_{-\tau,\mathbb{C}}} \\
&\cdots 2^{-(q-1)(m-k_2-\cdots-k_{n-1})} |F_n^{(m-k_2-\cdots-k_{n-1})}|^2_{\mathcal{H}^{(m-k_2-\cdots-k_{n-1})}_{-\tau,\mathbb{C}}} \\
&= \cdots = C(n) \|F_1\|^2_{-\tau,-(q-1)} \cdots \|F_n\|^2_{-\tau,-(q-1)},
\end{aligned} \tag{27}
$$

where $C(n) := \max_{m\in\mathbb{Z}_+} \left[2^{-m}(m+1)^{n-1}\right]$.

2) Let us establish that for some $q \in \mathbb{Z}_+$ $\|h^\Diamond(F)\|_{-\tau,-q} < \infty$, it is enough to assert that $h^\Diamond(F) \in (\mathcal{H}_{-\tau})$. Let $F^{(k)}_{ext} \in \mathcal{H}^{(k)}_{-\tau,\mathbb{C}}$, $k \in \mathbb{Z}_+$, be the kernels from decomposition (12) for F. Since by Schwartz's theorem for some $\widetilde{q} \in \mathbb{Z}_+$ $F \in (\mathcal{H}_{-\tau})_{-\widetilde{q}}$, by (13) for each k we have $|F^{(k)}_{ext}|_{\mathcal{H}^{(k)}_{-\tau,\mathbb{C}}} \le \|F\|_{-\tau,-\widetilde{q}} 2^{\widetilde{q}k/2}$. Further, it follows from the holomorphy of h that there exists $q' \in \mathbb{Z}_+$ such that for each $n \in \mathbb{Z}_+$ $|h_n| \le 2^{q'n}$, where $h_n \in \mathbb{C}$ are the coefficients from decomposition (17) for h. Using these estimates, (13), (25), (22) and the estimate $\sum\limits_{\substack{k_1,\dots,k_n\in\mathbb{N}:\\ k_1+\cdots+k_n=m}} 1 = C^{n-1}_{m-1} \le 2^{m-1}$, we obtain

$$
\begin{aligned}
\|h^\Diamond(F)\|^2_{-\tau,-q} &= |h_0|^2 + \sum_{m=1}^{\infty} 2^{-qm} \Big| \sum_{n=1}^{m} h_n \sum_{\substack{k_1,\dots,k_n\in\mathbb{N}:\\ k_1+\cdots+k_n=m}} F^{(k_1)}_{ext} \diamond \cdots \diamond F^{(k_n)}_{ext} \Big|^2_{\mathcal{H}^{(m)}_{-\tau,\mathbb{C}}} \\
&\le |h_0|^2 + \sum_{m=1}^{\infty} 2^{-qm} \Big(\sum_{n=1}^{m} |h_n| \sum_{\substack{k_1,\dots,k_n\in\mathbb{N}:\\ k_1+\cdots+k_n=m}} |F^{(k_1)}_{ext}|_{\mathcal{H}^{(k_1)}_{-\tau,\mathbb{C}}} \cdots |F^{(k_n)}_{ext}|_{\mathcal{H}^{(k_n)}_{-\tau,\mathbb{C}}} \Big)^2 \\
&\le |h_0|^2 + \sum_{m=1}^{\infty} 2^{-qm} \Big(\sum_{n=1}^{m} 2^{q'n} \sum_{\substack{k_1,\dots,k_n\in\mathbb{N}:\\ k_1+\cdots+k_n=m}} \|F\|^n_{-\tau,-\widetilde{q}} 2^{\widetilde{q}m/2} \Big)^2 \\
&\le |h_0|^2 + \frac{1}{4} \sum_{m=1}^{\infty} 2^{(\widetilde{q}+2-q)m} \Big(\sum_{n=1}^{m} (2^{q'} \|F\|_{-\tau,-\widetilde{q}})^n \Big)^2 < \infty,
\end{aligned} \tag{28}
$$

if $q \in \mathbb{Z}_+$ is sufficiently large. □

Remark 8. *Let $h^\Diamond_N(F)$, $N \in \mathbb{N}$, be the Wick version of the N-th partial sum of decomposition* (17) *for h. It follows from calculation* (28) *that $h^\Diamond_N(F) \to h^\Diamond(F)$ as $N \to \infty$ in $(\mathcal{H}_{-\tau})$.*

Remark 9. *One of generalizations of the Gaussian white noise analysis is a so-called* biorthogonal analysis *(see [1,2,5,23,24,36]) that developed actively in 90th of the last century. Its main idea is to use as orthogonal bases in spaces of test functions so-called generalized Appell polynomials (or their generalizations), in this case orthogonal bases in spaces of generalized functions are biorthogonal to the above-mentioned polynomials generalized functions. Over time the interest to the biorthogonal analysis went down because of the lack of interesting applications.*

But methods developed within its framework, and some its results can be successfully used in another generalizations of the Gaussian analysis, in particular, in the Lévy analysis. For example, the proof of Theorem 2 is adopted from the biorthogonal analysis, cf. [24].

2.2 Interconnection between the Wick calculuses in the regular and nonregular cases

In the paper [15], in particular, a Wick product and Wick versions of holomorphic functions are introduced and studied on so-called *parametrized Kondratiev-type spaces of regular generalized functions* of the Lévy white noise analysis [15,25]. As distinct from the Gaussian or Poissonian analysis, these spaces are not embedded into the spaces of nonregular generalized functions, but have with the last wide intersections (for example, (L^2) is a part of all these intersections). So, it is natural to consider a question about interconnection between the Wick calculuses on the spaces of regular and nonregular generalized functions. The answer is very simple: actually, on the above-mentioned intersections the Wick products and the Wick versions of holomorphic functions, introduced in [15] and in this paper, coincide. Now we'll explain this in detail.

Definition 5. *Accept on default $\beta \in [0,1]$. Parametrized Kondratiev-type spaces of regular generalized functions $(L^2)^{-\beta}_{-q}$ and $(L^2)^{-\beta}$ can be defined as follows: $(L^2)^{-\beta}_{-q}$ consists of formal series* (5) *such that $\|F\|^2_{(L^2)^{-\beta}_{-q}} = \sum_{n=0}^{\infty}(n!)^{1-\beta}2^{-qn}|F^{(n)}|^2_{ext} < \infty$; $(L^2)^{-\beta} := \operatorname*{ind\,lim}_{q\to\infty}(L^2)^{-\beta}_{-q}$.*

The well-posedness of this definition is proved in [15, 25]. Note that the space of square integrable random variables $(L^2) = (L^2)^0_0$ is densely and continuously embedded into each $(L^2)^{-\beta}_{-q}$ and therefore into $(L^2)^{-\beta}$.

Remark 10. *Let $(L^2)^{\beta}_q$, $(L^2)^{\beta} = \operatorname*{pr\,lim}_{q\to\infty}(L^2)^{\beta}_q$ be parametrized Kondratiev-type spaces of regular test functions [15,25], i.e., the positive spaces of a chain $(L^2)^{-\beta} \supset (L^2)^{-\beta}_{-q} \supset (L^2) \supset (L^2)^{\beta}_q \supset (L^2)^{\beta}$. It is not difficult to understand that $(L^2)^{\beta}_q$ consist of elements of form* (5) *such that $\|F\|^2_{(L^2)^{\beta}_q} = \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}|F^{(n)}|^2_{ext} < \infty$. By analogy one can introduce spaces $(\mathcal{H}_\tau)^{\beta}_q$ that consist of formal series* (8) *such that $\|f\|^2_{(\mathcal{H}_\tau)^{\beta}_q} = \sum_{n=0}^{\infty}(n!)^{1+\beta}2^{qn}|f^{(n)}|^2_{\tau} < \infty$. It is possible to study properties of these spaces and of its projective limits, to introduce and to study operators and operations on them; such considerations are interesting by itself and can be useful for applications. But, in contrast to the Gaussian and Poissonian analysis, in the Lévy analysis $(\mathcal{H}_\tau)^{\beta}_q \not\subset (L^2)$ if $\beta < 1$, generally speaking, so, we cannot consider $(\mathcal{H}_\tau)^{\beta}_q$ with $\beta < 1$ as spaces of test functions.*

Definition 6 ([15])**.** *For $F \in (L^2)^{-\beta}$ we define an $\widetilde{S}$-transform $(\widetilde{S}F)(\lambda)$, $\lambda \in \mathcal{D}_{\mathbb{C}}$, as a formal series*

$$(\widetilde{S}F)(\lambda) := \sum_{m=0}^{\infty}(F^{(m)}, \lambda^{\otimes m})_{ext} \equiv F^{(0)} + \sum_{m=1}^{\infty}(F^{(m)}, \lambda^{\otimes m})_{ext}, \tag{29}$$

where $F^{(m)} \in \mathcal{H}^{(m)}_{ext}$ are the kernels from decomposition (5) *for F (cf.* (15)*). In particular, $(\widetilde{S}F)(0) = F^{(0)}$, $\widetilde{S}1 \equiv 1$.*

Definition 7 ([15])**.** *For $F, G \in (L^2)^{-\beta}$ and a holomorphic at $(\widetilde{S}F)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ we define a Wick product $F\widetilde{\Diamond}G$ and a Wick version $h^{\widetilde{\Diamond}}(F)$ by setting formally (cf.* (16)*)*

$$F\widetilde{\Diamond}G := \widetilde{S}^{-1}(\widetilde{S}F \cdot \widetilde{S}G), \quad h^{\widetilde{\Diamond}}(F) := \widetilde{S}^{-1}h(\widetilde{S}F). \tag{30}$$

As in the nonregular case, the Wick multiplication $\widetilde{\Diamond}$ is commutative, associative and distributive over a field $\mathbb{C}$, and the following statement is fulfilled (cf. Theorem 2).

Theorem 3 ([15]). *1) Let $F_1, \dots, F_n \in (L^2)^{-\beta}$. Then $F_1 \widetilde{\Diamond} \cdots \widetilde{\Diamond} F_n \in (L^2)^{-\beta}$. Moreover, the Wick multiplication is continuous in the sense that for any $q, q' \in \mathbb{Z}_+$ such that $F_1, \dots, F_n \in (L^2)^{-\beta}_{-q'}$ and $q > q' + (1-\beta)\log_2 n + 1$*

$$\|F_1 \widetilde{\Diamond} \cdots \widetilde{\Diamond} F_n\|_{(L^2)^{-\beta}_{-q}} \le \sqrt{\max_{m \in \mathbb{Z}_+} [2^{-m}(m+1)^{n-1}]} \|F_1\|_{(L^2)^{-\beta}_{-q'}} \cdots \|F_n\|_{(L^2)^{-\beta}_{-q'}}$$

(cf. (26)). 2) Let $F \in (L^2)^{-\beta}$ and a function $h : \mathbb{C} \to \mathbb{C}$ be holomorphic at $(\widetilde{S}F)(0)$. Then $h^{\widetilde{\Diamond}}(F) \in (L^2)^{-1}$.

Remark 11. *Theorem 3 can be proved with the use of "coordinate formulas" for the Wick product and for the Wick versions of holomorphic functions on the spaces $(L^2)^{-\beta}$ [15]. Formally these formulas coincide with the corresponding formulas in the nonregular case, see Proposition 2. Actually, this coincidence is not accidental: the restriction of the multiplication $\diamond$ to the spaces $\mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{Z}_+$, is an analog of the symmetric tensor multiplication on these spaces, the proof of this fact coincides up to obvious modifications with the proof of the corresponding statement in the real case [31].*

Comparing (15) with (29), (16) with (30), and taking into account Theorems 2 and 3, we obtain the following statement.

Theorem 4. *1) Let $F_1, \dots, F_n \in (\mathcal{H}_{-\tau}) \cap (L^2)^{-\beta}$. Then*

$$F_1 \Diamond \cdots \Diamond F_n = F_1 \widetilde{\Diamond} \cdots \widetilde{\Diamond} F_n \in (\mathcal{H}_{-\tau}) \cap (L^2)^{-\beta}.$$

2) Let $F \in (\mathcal{H}_{-\tau}) \cap (L^2)^{-\beta}$ and a function $h : \mathbb{C} \to \mathbb{C}$ be holomorphic at $(SF)(0) = (\widetilde{S}F)(0)$. Then $h^{\Diamond}(F) = h^{\widetilde{\Diamond}}(F) \in (\mathcal{H}_{-\tau}) \cap (L^2)^{-1}$.

2.3 Interconnection between the Wick calculus and operators of stochastic differentiation

As is well known, a very important role in the Gaussian white noise analysis and its generalizations belongs to the extended stochastic integral and to its adjoint operator — the Hida stochastic derivative. Together with these operators, it is natural and useful to introduce and to study so-called *operators of stochastic differentiation*, which are closely related with the stochastic integral and derivative. Roughly speaking, one can understand the stochastic differentiation as a "differentiation" with respect to a "stochastic argument", i.e., the operator of stochastic differentiation acts on an orthogonal decomposition of a (generalized) random variable in common with an action of the differentiation operator on Taylor's decomposition of a function. The operators of stochastic differentiation can be used, in particular, in order to study some properties of the extended stochastic integral and of solutions of stochastic equations with Wick-type nonlinearities.

As is known [3], in the Gaussian analysis the operator of stochastic differentiation of order 1 is a differentiation (i.e., satisfies the Leibniz rule) with respect to the Wick multiplication. This important for applications property holds true in a Gamma-analysis (i.e., a white noise analysis connected with a so-called Gamma-measure) [22], in a Meixner analysis [26,27], and in the Lévy analysis on the spaces of regular generalized functions [15]. But, in contrast to the

Gaussian case, in the Lévy analysis (in the same way as in the Gamma- and Meixner analysis) the operators of stochastic differentiation (in the same way as the Hida stochastic derivative) cannot be naturally continued from (L^2) to the spaces of nonregular generalized functions, see [30] for details. Nevertheless, one can introduce on these spaces natural analogs of the above-mentioned operators. These analogs are introduced and studied (in a real case) in [30]. They have properties similar to properties of "classical" operators of stochastic differentiation [13], and can be accepted as operators of stochastic differentiation on the spaces of nonregular generalized functions. Now we'll recall the definition of such operator of order 1, and will show that this operator satisfies the Leibniz rule with respect to the Wick multiplication $\lozenge$.

Let $F_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$, $m \in \mathbb{N}\backslash\{1\}$, $g \in \mathcal{H}_{\tau,\mathbb{C}}$. We define a generalized partial pairing $\langle F_{ext}^{(m)}, g\rangle_{ext} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m-1)}$ by setting for any $f^{(m-1)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}m-1}$

$$\langle\langle F_{ext}^{(m)}, g\rangle_{ext}, f^{(m-1)}\rangle_{ext} = \langle F_{ext}^{(m)}, g\widehat{\otimes} f^{(m-1)}\rangle_{ext}. \tag{31}$$

Since by the generalized Cauchy-Bunyakovsky inequality

$$|\langle F_{ext}^{(m)}, g\widehat{\otimes} f^{(m-1)}\rangle_{ext}| \leq |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}} |g\widehat{\otimes} f^{(m-1)}|_\tau \leq |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}} |g|_\tau |f^{(m-1)}|_\tau,$$

this definition is well posed and

$$|\langle F_{ext}^{(m)}, g\rangle_{ext}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m-1)}} \leq |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}} |g|_\tau. \tag{32}$$

Definition 8. *Let $g \in \mathcal{H}_{\tau,\mathbb{C}}$. We define (the analog of) the operator of stochastic differentiation*

$$(D\circ)(g) : (\mathcal{H}_{-\tau}) \to (\mathcal{H}_{-\tau}) \tag{33}$$

as a linear continuous operator that is given by the formula

$$(DF)(g) := \sum_{m=1}^{\infty} m \colon\langle \circ^{\otimes m-1}, \langle F_{ext}^{(m)}, g\rangle_{ext}\rangle\colon, \tag{34}$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ are the kernels from decomposition (12) *for $F \in (\mathcal{H}_{-\tau})$.*

The proof of the well-posedness of this definition is based on estimate (32) and coincides up to obvious modifications with the proof of the corresponding statement in a real case [30].

Let us define a characterization set of the space $(\mathcal{H}_{-\tau})$ in terms of the S-transform, setting $B_\tau := S(\mathcal{H}_{-\tau}) \equiv \{SF : F \in (\mathcal{H}_{-\tau})\}$ (cf. [15]). It is clear that B_τ is a linear space, which consists of formal series $\sum_{m=0}^{\infty}\langle F_{ext}^{(m)}, \cdot^{\otimes m}\rangle_{ext}$ (see (15)) with the kernels $F_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ satisfying a condition: there exists $q \in \mathbb{N}_{q_0(\tau)} \subseteq \mathbb{Z}_+$ such that $\sum_{m=0}^{\infty} 2^{-qm}|F_{ext}^{(m)}|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}} < \infty$. It follows from Definition 4 and Theorem 2 that B_τ is an algebra with respect to the pointwise multiplication. Moreover, if we introduce on B_τ a topology induced by the topology of $(\mathcal{H}_{-\tau})$, then the S-transform becomes a topological isomorphism between a topological algebra $(\mathcal{H}_{-\tau})$ with the Wick multiplication and a topological algebra B_τ with the pointwise multiplication.

Denote by

$$d_g : B_\tau \to B_\tau, \quad g \in \mathcal{H}_{\tau,\mathbb{C}}, \tag{35}$$

a directional derivative, i.e., for $(SF)(\cdot) = \sum_{m=0}^{\infty}\langle F_{ext}^{(m)}, \cdot^{\otimes m}\rangle_{ext} = \sum_{m=0}^{\infty}\langle U_m F_{ext}^{(m)}, \cdot^{\otimes m}\rangle \in B_\tau$ (see (15), (20); $F \in (\mathcal{H}_{-\tau})$, $F_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$ are the kernels from decomposition (12) for F)

$$\begin{aligned} d_g(SF)(\cdot) &= \sum_{m=1}^{\infty} m\langle U_m F_{ext}^{(m)}, g\widehat{\otimes}(\cdot^{\otimes m-1})\rangle = \sum_{m=1}^{\infty} m\langle F_{ext}^{(m)}, g\widehat{\otimes}(\cdot^{\otimes m-1})\rangle_{ext} \\ &= \sum_{m=1}^{\infty} m\langle\langle F_{ext}^{(m)}, g\rangle_{ext}, \cdot^{\otimes m-1}\rangle_{ext} = \big(S(DF)(g)\big)(\cdot) \in B_\tau \end{aligned} \tag{36}$$

(see (20), (31), (34) and (15)). As we see, directional derivative (35) is the image on B_τ of operator of stochastic differentiation (33) under the S-transform (in particular, (35) is a linear *continuous* operator). Vice versa, operator of stochastic differentiation (33) is a pre-image of directional derivative (35) under the S-transform, i.e., for all $F \in (\mathcal{H}_{-\tau})$ and $g \in \mathcal{H}_{\tau,\mathbb{C}}$

$$(DF)(g) = S^{-1}d_g SF \in (\mathcal{H}_{-\tau}). \tag{37}$$

Now we are ready to prove the main result of this subsection.

Theorem 5. *Operator of stochastic differentiation* (33) *is a differentiation (i.e., satisfies the Leibniz rule) with respect to the Wick multiplication, i.e., for all* $F, G \in (\mathcal{H}_{-\tau})$ *and* $g \in \mathcal{H}_{\tau,\mathbb{C}}$

$$\big(D(F\Diamond G)\big)(g) = (DF)(g)\Diamond G + F\Diamond(DG)(g) \in (\mathcal{H}_{-\tau}). \tag{38}$$

Proof. First we note that the expressions in the left hand side and in the right hand side of (38) belong to $(\mathcal{H}_{-\tau})$, this follows from the definition of operator (33) and Theorem 2. As for *equality* (38), using (37), (16), the fact that the directional derivative satisfies the Leibniz rule, and (36), we obtain

$$\begin{aligned} \big(D(F\Diamond G)\big)(g) &= S^{-1}d_g\big(S(F\Diamond G)\big) = S^{-1}d_g(SF\cdot SG) = S^{-1}\big[(d_g SF)\cdot SG + SF\cdot(d_g SG)\big] \\ &= S^{-1}\big[\big(S(DF)(g)\big)\cdot SG + SF\cdot\big(S(DG)(g)\big)\big] = (DF)(g)\Diamond G + F\Diamond(DG)(g), \end{aligned}$$

which is what had to be proved. □

Corollary. *Let* $F \in (\mathcal{H}_{-\tau})$*,* $g \in \mathcal{H}_{\tau,\mathbb{C}}$*, and* $h : \mathbb{C} \to \mathbb{C}$ *be a holomorphic at* $(SF)(0)$ *function. Then*

$$\big(Dh^{\Diamond}(F)\big)(g) = h'^{\Diamond}(F)\Diamond(DF)(g) \in (\mathcal{H}_{-\tau}), \tag{39}$$

where $h'^{\Diamond}$ *is the Wick version of the usual derivative of a function* h.

Proof. Using (38), one can prove by the mathematical induction method that for each $m \in \mathbb{Z}_+$

$$\Big(D\big(F-(SF)(0)\big)^{\Diamond m}\Big)(g) = m\big(F-(SF)(0)\big)^{\Diamond m-1}\Diamond(DF)(g). \tag{40}$$

Further, let $h_N^{\Diamond}(F)$, $N \in \mathbb{N}$, be the Wick version of the N-th partial sum of decomposition (17) for h, i.e., $h_N^{\Diamond}(F) = \sum_{m=0}^{N} h_m\big(F-(SF)(0)\big)^{\Diamond m}$, see (18). It follows from the linearity of the operator D, (40), Theorem 2 and Remark 8 that

$$\begin{aligned} \big(Dh_N^{\Diamond}(F)\big)(g) &= \sum_{m=0}^{N} h_m\Big(D\big(F-(SF)(0)\big)^{\Diamond m}\Big)(g) \\ &= \sum_{m=1}^{N} h_m m\big(F-(SF)(0)\big)^{\Diamond m-1}\Diamond(DF)(g) \underset{N\to\infty}{\to} h'^{\Diamond}(F)\Diamond(DF)(g) \end{aligned}$$

in $(\mathcal{H}_{-\tau})$. On the other hand, it follows from Remark 8 and the continuity of the operator $(D\circ)(g)$ on $(\mathcal{H}_{-\tau})$ that $\big(Dh_N^{\Diamond}(F)\big)(g) \to \big(Dh^{\Diamond}(F)\big)(g)$ as $N \to \infty$ in $(\mathcal{H}_{-\tau})$. Therefore equality (39) is valid. □

In a forthcoming paper we'll consider an interconnection between the Wick calculus and the stochastic integration on the spaces of nonregular generalized functions, and give examples of integral stochastic equations with Wick-type nonlinearities.

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Качановський М.О. *Про Віківське числення на просторах нерегулярних узагальнених функцій аналізу білого шуму Леві* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 114–132.

Розвиток теорії основних і узагальнених функцій, що залежать від нескінченної кількості змінних, є важливою та актуальною задачею, яка обумовлена потребами фізики і математики. Один з успішних підходів до побудови такої теорії полягає у введенні просторів вищезгаданих функцій таким чином, що дуальне спарювання між основними і узагальненими функціями породжується інтегруванням за деякою ймовірнісною мірою. Спочатку це була гауссівська міра, згодом були зроблені численні узагальнення. Зокрема, важливі результати можна отримати, використовуючи міру білого шуму Леві, відповідна теорія називається *аналізом білого шуму Леві*.

У гауссівському випадку можна будувати простори основних і узагальнених функцій та уводити деякі важливі оператори (наприклад, стохастичні інтеграли і похідні) на цих просторах за допомогою так званої *властивості хаотичного розкладу* (ВХР): грубо кажучи, кожну квадратично інтегровну випадкову величину можна розкласти у ряд повторних стохастичних інтегралів Іто від невипадкових функцій. У аналізі Леві нема ВХР, але є різні узагальнення цієї властивості.

У цій статті ми маємо справу з одним з найбільш корисних і перспективних узагальнень ВХР у аналізі Леві, запропонованим Є. В. Литвиновим, та з відповідними просторами нерегулярних узагальнених функцій. Метою статті є увести природний добуток (віківський добуток) на цих просторах, та вивчити деякі пов'язані питання. Основними результатами є теореми про властивості віківського добутку і віківських версій голоморфних функцій. Зокрема, ми доводимо, що оператор стохастичного диференціювання задовольняє правило Лейбніца відносно віківського множення. Крім того, ми показуємо, що віківські добутки і віківські версії голоморфних функцій, визначені на просторах регулярних і нерегулярних узагальнених функцій, побудованих за допомогою литвинівського узагальнення ВХР, співпадають на перетинах цих просторів.

Наші дослідження є внеском у подальший розвиток аналізу білого шуму Леві.

Ключові слова і фрази: процес Леві, віківський добуток, стохастичне диференціювання.