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(p,q)TH ORDER ORIENTED GROWTH MEASUREMENT OF COMPOSITE p-ADIC ENTIRE FUNCTIONS

Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and let $\mathcal{A}(\mathbb{K})$ be the $\mathbb{K}$-algebra of entire functions on $\mathbb{K}$. For any p-adic entire function $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$, where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. For any two entire functions $f \in \mathcal{A}(\mathbb{K})$ and $g \in \mathcal{A}(\mathbb{K})$ the ratio $\frac{|f|(r)}{|g|(r)}$ as $r \to \infty$ is called the comparative growth of f with respect to g in terms of their multiplicative norms. Likewise to complex analysis, in this paper we define the concept of (p,q)th order (respectively (p,q)th lower order) of growth as $\rho^{(p,q)}(f) = \limsup\limits_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}$ (respectively $\lambda^{(p,q)}(f) = \liminf\limits_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}$), where p and q are any two positive integers. We study some growth properties of composite p-adic entire functions on the basis of their (p,q)th order and (p,q)th lower order.

Key words and phrases: p-adic entire function, growth, (p,q)th order, (p,q)th lower order, composition.

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INTRODUCTION AND DEFINITIONS

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to a p-adic absolute value $|\cdot|$ (example $\mathbb{C}_p$). For any $\alpha \in \mathbb{K}$ and $R \in (0,+\infty)$, the closed disk $\{x \in \mathbb{K} : |x-\alpha| \le R\}$ and the open disk $\{x \in \mathbb{K} : |x-\alpha| < R\}$ are denoted by $d(\alpha,R)$ and $d(\alpha,R^-)$ respectively. Also $C(\alpha,r)$ denotes the circle $\{x \in \mathbb{K} : |x-\alpha| = r\}$. Moreover $\mathcal{A}(\mathbb{K})$ represent the $\mathbb{K}$-algebra of analytic functions on $\mathbb{K}$, i.e. the set of power series with an infinite radius of convergence. For the most comprehensive study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we refer the reader to the books [9, 10, 15, 18]. During the last several years the ideas of p-adic analysis have been studied from different aspects and many important results were gained (see [1–6], [8, 11–14, 19]).

Let $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$, then we denote by $|f|(r)$ the number $\sup\{|f(x)| : |x| = r\}$ where $|\cdot|(r)$ is a multiplicative norm on $\mathcal{A}(\mathbb{K})$. For any two entire functions $f \in \mathcal{A}(\mathbb{K})$ and $g \in \mathcal{A}(\mathbb{K})$ the ratio $\frac{|f|(r)}{|g|(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their multiplicative norms.

For any $x \in [0,\infty)$ and $k \in \mathbb{N}$, we define recursively $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$, where $\mathbb{N}$ stands for the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Throughout the paper, log denotes the Neperian logarithm.

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Taking this into account the order (resp. lower order) of an entire function $f \in \mathcal{A}(\mathbb{K})$ is given by (see [4])

$$\begin{matrix} \rho(f) \\ \lambda(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]} |f|(r)}{\log r}.$$

The above definition of order (resp. lower order) does not seem to be feasible if an entire function $f \in \mathcal{A}(\mathbb{K})$ is of order zero. To overcome this situation and in order to study the growth of an entire function $f \in \mathcal{A}(\mathbb{K})$ precisely, one may introduce the concept of logarithmic order (resp. logarithmic lower order) by increasing $\log^+$ once in the denominator following the classical definition of logarithmic order (see, for example, [7]). Therefore the logarithmic order $\rho_{\log}(f)$ and logarithmic lower order $\lambda_{\log}(f)$ of an entire function $f \in \mathcal{A}(\mathbb{K})$ are define as

$$\begin{matrix} \rho_{\log}(f) \\ \lambda_{\log}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[2]} |f|(r)}{\log^{[2]} r}.$$

Further the concept of (p,q)th order (p and q are any two positive integers with $p \geq q$) is not new and was first introduced by Juneja et al. [16,17]. In the line of Juneja et al. [16,17], now we shall introduce the definitions of (p,q)th order and (p,q)th lower order respectively of an entire function $f \in \mathcal{A}(\mathbb{K})$ where $p,q \in \mathbb{N}$. In order to keep accordance with the definition of logarithmic order we will give a minor modification to the original definition of (p,q)-order introduced by Juneja et al. [16,17].

Definition 1. *Let $f \in \mathcal{A}(\mathbb{K})$ and $p,q \in \mathbb{N}$. Then the (p,q)th order and (p,q)th lower order of f are respectively defined as:*

$$\begin{matrix} \rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} |f|(r)}{\log^{[q]} r}.$$

These definitions extend the generalized order $\rho^{[l]}(f)$ and generalized lower order $\lambda^{[l]}(f)$ of $f \in \mathcal{A}(\mathbb{K})$ for each integer $l \geq 2$ since these correspond to the particular case $\rho^{[l]}(f) = \rho^{(l,1)}(f)$ and $\lambda^{[l]}(f) = \lambda^{(l,1)}(f)$. Clearly $\rho^{(2,1)}(f) = \rho(f)$ and $\lambda^{(2,1)}(f) = \lambda(f)$. The above definition avoid the restriction $p > q$ and give the idea of generalized logarithmic order.

However in this connection we just introduce the following definition which is analogous to the definition of Juneja et al. [16,17].

Definition 2. *An entire function $f \in \mathcal{A}(\mathbb{K})$ is said to have index-pair (p,q), where p and $q \in \mathbb{N}$, if $b < \rho^{(p,q)}(f) < \infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise. Moreover if $0 < \rho^{(p,q)}(f) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1,2,\ldots. \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1,2,\ldots. \end{cases}$$

The main aim of this paper is to establish some results related to the growth properties of composite p-adic entire functions on the basis of (p,q)th order and (p,q)th lower order, where $p,q \in \mathbb{N}$.

1 LEMMA

In this section we present the following lemma which can be found in [4] or [5] and will be needed in the sequel.

Lemma 1. *Let $f, g \in \mathcal{A}(\mathbb{K})$. Then for all sufficiently large values of r the following equality holds*

$$|f \circ g|(r) = |f|(|g|(r)).$$

2 MAIN RESULTS

Theorem 1. *Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $\rho^{(m,n)}(g) < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$, where $p, q, m, n \in \mathbb{N}$. Then*

$$(i) \quad \lim_{r\to+\infty} \frac{\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p-1]} |f|(\exp^{[q-1]} r)} = 0 \quad \text{if } q \geq m$$

and

$$(ii) \quad \lim_{r\to+\infty} \frac{\log^{[p+m-q-1]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p-1]} |f|(\exp^{[q-1]} r)} = 0 \quad \text{if } q < m.$$

Proof. We get from Lemma 1, for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) = \log^{[p]} |f| \left(|g| \left(\exp^{[n-1]} r\right)\right)$$

i.e.,

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \log^{[q]} |g| \left(\exp^{[n-1]} r\right). \tag{1}$$

Now the following two cases may arise.

Case I. Let $q \geqslant m$. Then we have from (1) for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \log^{[m-1]} |g| \left(\exp^{[n-1]} r\right) \tag{2}$$

i.e.,

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}. \tag{3}$$

Case II. Let $q < m$. Then for all sufficiently large positive numbers of r we get from (1) that

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leqslant \left(\rho^{(p,q)}(f) + \varepsilon\right) \exp^{[m-q]} \log^{[m]} |g| \left(\exp^{[n-1]} r\right). \tag{4}$$

Further for all sufficiently large positive numbers of r, it follows that

$$\log^{[m]} |g| \left(\exp^{[n-1]} r\right) \leqslant \log \left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)$$

i.e.,

$$\exp^{[m-q]} \log^{[m]} |g| \left(\exp^{[n-1]} r\right) \leqslant \exp^{[m-q-1]} \left(r^{\rho^{(m,n)}(g)+\varepsilon}\right). \tag{5}$$

Now from (4) and (5) we have for all sufficiently large positive numbers of r that

$$\log^{[p]} |f \circ g| \left(\exp^{[n-1]} r\right) \leq \left(\rho^{(p,q)}(f) + \varepsilon\right) \exp^{[m-q-1]} \left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)$$

i.e.,

$$\log^{[p+1]}|f\circ g|\left(\exp^{[n-1]}r\right)\leqslant \exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)+O(1)$$

i.e.,

$$\log^{[p+1]}|f\circ g|\left(\exp^{[n-1]}r\right)\leqslant \exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)\left(1+\frac{O(1)}{\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)}\right)$$

i.e.,

$$\log^{[p+m-q-1]}|f\circ g|\left(\exp^{[n-1]}r\right)\leqslant r^{\rho_g(m,n)+\varepsilon}\left(1+\frac{O(1)}{\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)}\right). \tag{6}$$

Also from the definition of $\lambda^{(p,q)}(f)$, we get for all sufficiently large positive numbers of r that

$$\log^{[p-1]}|f|(\exp^{[q-1]}r)\geqslant r^{(\lambda^{(p,q)}(f)-\varepsilon)}. \tag{7}$$

Now combining (3) of Case I and (7) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}\leq\frac{\left(\rho_f\left(p,q\right)+\varepsilon\right)r^{\left(\rho^{(m,n)}(g)+\varepsilon\right)}}{r^{(\lambda^{(p,q)}(f)-\varepsilon)}}. \tag{8}$$

Since $\rho^{(m,n)}(g)<\lambda^{(p,q)}(f)$ we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho^{(m,n)}(g)+\varepsilon<\lambda^{(p,q)}(f)-\varepsilon. \tag{9}$$

Therefore in view of (9) it follows from (8) that

$$\lim_{r\to+\infty}\frac{\log^{[p]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}=0.$$

Hence the first part of the theorem follows.

Further combining (6) of Case II and (7) we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}\leq\frac{r^{\rho^{(m,n)}(g)+\varepsilon}\left(1+\frac{O(1)}{\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)}\right)}{r^{(\lambda^{(p,q)}(f)-\varepsilon)}}. \tag{10}$$

Therefore in view of (9) we get from above that

$$\lim_{r\to+\infty}\frac{\log^{[p+m-q-1]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}=0\,.$$

Thus the theorem follows. □

Theorem 2. *Let* $f,g\in\mathcal{A}(\mathbb{K})$ *be such that* $\lambda^{(m,n)}(g)<\lambda^{(p,q)}(f)\leq\rho^{(p,q)}(f)<\infty$, *where* $p,q,m,n\in\mathbb{N}$. *Then*

$$(i)\qquad \varliminf_{r\to+\infty}\frac{\log^{[p]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}=0\quad \textit{if } q\geq m$$

and

$$(ii)\qquad \varliminf_{r\to+\infty}\frac{\log^{[p+m-q-1]}|f\circ g|\left(\exp^{[n-1]}r\right)}{\log^{[p-1]}|f|(\exp^{[q-1]}r)}=0\quad \textit{if } q<m.$$

The proof of Theorem 2 is omitted as it can be carried out in the line of Theorem 1.

Theorem 3. *Let* $f, g \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$ *and* $\rho^{(m,n)}(g) < \infty$, *where* $p, q, m, n \in \mathbb{N}$. *Then*

$$(i) \quad \overline{\lim_{r\to+\infty}} \frac{\log^{[p+1]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)} \quad \text{if } q \geq m$$

and

$$(ii) \quad \overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)} \quad \text{if } q < m.$$

Proof. In view of the definition $\lambda^{(p,q)}(f)$, we have for all sufficiently large positive numbers of r that

$$\log^{[p]} |f|(\exp^{[q-1]} r) \geq \left(\lambda^{(p,q)}(f) - \varepsilon\right) \log r. \tag{11}$$

Case I. If $q \geqslant m$, then from (3) and (11) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+1]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\left(\rho^{(m,n)}(g) + \varepsilon\right) \log r + \log \left(\rho^{(p,q)}(f) + \varepsilon\right)}{\left(\lambda^{(p,q)}(f) - \varepsilon\right) \log r}.$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+1]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}.$$

This proves the first part of the theorem.

Case II. If $q < m$ then from (6) and (11) we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\left(\rho^{(m,n)}(g) + \varepsilon\right) \log r + \log \left(1 + \frac{O(1)}{\exp^{[m-q-2]}\left(r^{\rho^{(m,n)}(g)+\varepsilon}\right)}\right)}{\left(\lambda^{(p,q)}(f) - \varepsilon\right) \log r}.$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-1]} r\right)}{\log^{[p]} |f|(\exp^{[q-1]} r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}.$$

Thus the second part of the theorem is established. □

Theorem 4. *Let* $f, g \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$ *and* $\lambda^{(m,n)}(g) > 0$, *where* $p, q, m, n \in \mathbb{N}$. *Then for any positive integer* l, *we have*

$$(i) \quad \lim_{r\to\infty} \frac{\log^{[p]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[p+1]} |f| \left(\exp^{[l]} r\right)} = \infty \quad \text{if } q < m \text{ and } q \geqslant l;$$

$$(ii) \quad \lim_{r\to\infty} \frac{\log^{[p]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[p-q-l+1]} |f| \left(\exp^{[l]} r\right)} = \infty \quad \text{if } q < m \text{ and } q < l;$$

$$(iii) \quad \lim_{r\to\infty} \frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]}|f|\left(\exp^{[l]} r\right)} = \infty \quad \text{if } q > m \text{ and } q < l;$$

and

$$(iv) \quad \lim_{r\to\infty} \frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[p+1]}|f|\left(\exp^{[l]} r\right)} = \infty \quad \text{if } q > m \text{ and } q \geqslant l.$$

Proof. Let us choose $0 < \varepsilon < \min\left\{\lambda^{(p,q)}(f), \lambda^{(m,n)}(g)\right\}$. Now for all sufficiently large positive numbers of r we get from Lemma 1,

$$\log^{[p]}|f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f)-\varepsilon)\log^{[q]}|g|\left(\exp^{[n-1]} r\right). \tag{12}$$

Further from the definition of (m,n)th lower order of g we have for all sufficiently large positive numbers of r that

$$\log^{[m]}|g|\left(\exp^{[n-1]} r\right) \geqslant \log r^{(\lambda^{(m,n)}(g)-\varepsilon)}. \tag{13}$$

Now the following two cases may arise.

Case I. Let $q < m$. Then from (12) and (13) we obtain for all sufficiently large positive numbers of r that

$$\log^{[p]}|f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q]}\log^{[m]}|g|\left(\exp^{[n-1]} r\right) \tag{14}$$

i.e.,

$$\log^{[p]}|f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q]}\log r^{(\lambda^{(m,n)}(g)-\varepsilon)}$$

$$\log^{[p]}|f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}. \tag{15}$$

Case II. Let $q > m$. Then from (12) and (13) it follows for all sufficiently large positive numbers of r that

$$\log^{[p]}|f\circ g|(\exp^{[n-1]} r) \geqslant (\lambda^{(p,q)}(f)-\varepsilon)\log^{[q-m]}\log r^{(\lambda^{(m,n)}(g)-\varepsilon)}$$

i.e.,

$$\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r) \geqslant r^{(\lambda^{(m,n)}(g)-\varepsilon)}. \tag{16}$$

Again from the definition of $\rho^{(p,q)}(f)$ we get for all sufficiently large positive numbers of r that

$$\log^{[p]}|f|\left(\exp^{[l]} r\right) \leq \left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}\exp^{[l]} r. \tag{17}$$

Now the following two cases may arise.

Case III. Let $q \geqslant l$. Then we have from (17) for all sufficiently large positive numbers of r that

$$\log^{[p]}|f|\left(\exp^{[l]} r\right) \leq \left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-l]} r$$

i.e.,

$$\log^{[p+1]}|f|\left(\exp^{[l]} r\right) \leq \log^{[q-l+1]} r + \log\left(\rho^{(p,q)}(f)+\varepsilon\right). \tag{18}$$

Case IV. Let $q < l$. Then we have from (17) for all sufficiently large positive numbers of r that

$$\log^{[p]}|f|\left(\exp^{[l]} r\right) \leq \left(\rho^{(p,q)}(f)+\varepsilon\right)\exp^{[l-q]} r$$

i.e.,

$$\log^{[p+1]}|f|\left(\exp^{[l]}r\right)\leq\exp^{[l-q-1]}r+\log\left(\rho^{(p,q)}(f)+\varepsilon\right)$$

i.e.,

$$\log^{[p-q+l+1]}|f|\left(\exp^{[l]}r\right)\leq\log r+O(1). \tag{19}$$

Now combining (15) of Case I and (18) of Case III it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]}r)}{\log^{[p+1]}|f|\left(\exp^{[l]}r\right)}\geq\frac{(\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q-1]}r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\log^{[q-l+1]}r+\log\left(\rho^{(p,q)}(f)+\varepsilon\right)}.$$

Since $q<m$, we get from the above that

$$\lim_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]}r)}{\log^{[p+1]}|f|\left(\exp^{[l]}r\right)}=\infty.$$

This proves the first part of the theorem.

Again in view of (15) of Case I and (19) of Case IV we have for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]}r)}{\log^{[p-q+l+1]}|f|\left(\exp^{[l]}r\right)}\geq\frac{(\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q-1]}r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\log r+O(1)}. \tag{20}$$

When $q<m$ and $q<l$ then we get from (20) that

$$\lim_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]}r)}{\log^{[p-q+l+1]}|f|\left(\exp^{[l]}r\right)}=\infty.$$

This establishes the second part of the theorem.

Now in view of (16) of Case II and (18) of Case III we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]}r)}{\log^{[p+1]}|f|\left(\exp^{[l]}r\right)}\geq\frac{r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\log^{[q-l+1]}r+\log\left(\rho^{(p,q)}(f)+\varepsilon\right)}$$

i.e.,

$$\lim_{r\to+\infty}\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]}r)}{\log^{[p+1]}|f|\left(\exp^{[l]}r\right)}=\infty,$$

from which the third part of the theorem follows.

Again from (16) of Case II and (19) of Case IV we have for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]}r)}{\log^{[p-q+l+1]}|f|\left(\exp^{[l]}r\right)}\geq\frac{r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\log r+O(1)}$$

i.e.,

$$\lim_{r\to+\infty}\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]}r)}{\log^{[p-q+l+1]}|f|\left(\exp^{[l]}r\right)}=\infty.$$

This proves the fourth part of the theorem. Thus the theorem follows. □

Theorem 5. *Let* $f, g, h, k \in \mathcal{A}(\mathbb{K})$ *be such that* $0 < \rho^{(a,b)}(h) < \infty$, $\lambda^{(p,q)}(f) > 0$, $\lambda^{(m,n)}(g) > 0$ *and* $\rho^{(c,d)}(k) < \lambda^{(m,n)}(g)$, *where* $a, b, c, d, p, q, m, n \in \mathbb{N}$. *Then*

$$(i) \quad \lim_{r\to+\infty} \frac{\log^{[p]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a]} |h \circ k|(r)} = \infty \quad \text{if } b \geqslant c \text{ and } q < m,$$

$$(ii) \quad \lim_{r\to+\infty} \frac{\log^{[p]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]} |h \circ k|(r)} = \infty \quad \text{if } b < c \text{ and } q < m,$$

$$(iii) \quad \lim_{r\to+\infty} \frac{\log^{[p+m-q-1]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a]} |h \circ k|(r)} = \infty \quad \text{if } b \geqslant c \text{ and } q \geqslant m,$$

$$\text{and } (iv) \quad \lim_{r\to+\infty} \frac{\log^{[p+m-q-1]} |f \circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]} |h \circ k|(r)} = \infty \quad \text{if } b < c \text{ and } q \geqslant m.$$

Proof. In view of Lemma 1 we obtain for all sufficiently large positive numbers of r that

$$\log^{[a]} |h \circ k|(r) \leqslant \left(\rho^{(a,b)}(h) + \varepsilon\right) \log^{[b]} |k|(r). \tag{21}$$

Now from the definition of (c,d)th order of k we get for arbitrary positive ε and for all sufficiently large positive numbers of r that

$$\log^{[c]} |k|(r) \leqslant \left(\rho^{(c,d)}(k) + \varepsilon\right) \log^{[d]} r$$

i.e.,

$$\log^{[c]} |k|(r) \leqslant \left(\rho^{(c,d)}(k) + \varepsilon\right) \log r \tag{22}$$

i.e.,

$$\log^{[c-1]} |k|(r) \leqslant r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}. \tag{23}$$

Now the following cases may arise.

Case I. Let $b \geqslant c$. Then we have from (21) for all sufficiently large positive numbers of r that

$$\log^{[a]} |h \circ k|(r) \leqslant \left(\rho^{(a,b)}(h) + \varepsilon\right) \log^{[c-1]} |k|(r). \tag{24}$$

So from (23) and (24), it follows for all sufficiently large positive numbers of r that

$$\log^{[a]} |h \circ k|(r) \leqslant \left(\rho^{(a,b)}(h) + \varepsilon\right) r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}. \tag{25}$$

Case II. Let $b < c$. Then we get from (21) for all sufficiently large positive numbers of r that

$$\log^{[a]} |h \circ k|(r) \leqslant \left(\rho^{(a,b)}(h) + \varepsilon\right) \exp^{[c-b]} \log^{[c]} |k|(r). \tag{26}$$

Now from (22) and (26) we obtain for all sufficiently large positive numbers of r that

$$\log^{[a]} |h \circ k|(r) \leqslant \left(\rho^{(a,b)}(h) + \varepsilon\right) \exp^{[c-b]} \log r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}$$

i.e.,

$$\log^{[a+c-b-1]} |h \circ k|(r) \leqslant r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)} + O(1). \tag{27}$$

Since $\rho^{(c,d)}(k) < \lambda^{(m,n)}(g)$ we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho^{(c,d)}(k) + \varepsilon < \lambda^{(m,n)}(g) - \varepsilon. \tag{28}$$

Now combining (25) of Case I, (15) and in view of (28) it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[a]}|h\circ k|(r)} \geq \frac{(\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\left(\rho^{(a,b)}(h)+\varepsilon\right) r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}}$$

i.e.,

$$\lim_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[a]}|h\circ k|(r)} = \infty,$$

from which the first part of the theorem follows.

Again combining (27) of Case II, (15) and in view of (28) we obtain for all sufficiently large positive numbers of r that

$$\frac{\log^{[p]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]}|h\circ k|(r)} \geq \frac{(\lambda^{(p,q)}(f)-\varepsilon)\exp^{[m-q-1]} r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)} + O(1)}$$

i.e.,

$$\lim_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]}|h\circ k|(r)} = \infty.$$

This establishes the second part of the theorem.

Further in view of (25) of Case I and (16) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[a]}|h\circ k|(r)} \geq \frac{r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{\left(\rho^{(a,b)}(h)+\varepsilon\right) r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)}}. \tag{29}$$

So from (28) and (29) we obtain that

$$\lim_{r\to+\infty} \frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[a]}|h\circ k|(r)} = \infty,$$

from which the third part of the theorem follows.

Again combining (27) of Case II and (16) it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]}|h\circ k|(r)} \geq \frac{r^{(\lambda^{(m,n)}(g)-\varepsilon)}}{r^{\left(\rho^{(c,d)}(k)+\varepsilon\right)} + O(1)}. \tag{30}$$

Now in view of (28) we obtain from (30) that

$$\lim_{r\to+\infty} \frac{\log^{[p+m-q-1]}|f\circ g|(\exp^{[n-1]} r)}{\log^{[a+c-b-1]}|h\circ k|(r)} = \infty.$$

This proves the fourth part of the theorem. Thus the theorem follows. □

Theorem 6. *Let $f,\ g \in \mathcal{A}(\mathbb{K})$ be such that $\rho^{(a,b)}(f\circ g) < \infty$ and $\lambda^{(m,n)}(g) > 0$, where $a,b,m,n \in \mathbb{N}$. Then*

$$\lim_{r\to+\infty} \frac{\left[\log^{[a]}|f\circ g|(\exp^{[b-1]} r)\right]^2}{\log^{[m-1]}|g|(\exp^{[n]} r)\cdot \log^{[m]}|g|(\exp^{[n-1]} r)} = 0.$$

Proof. For any $\varepsilon > 0$ we have $\log^{[a]}|f\circ g|(\exp^{[b-1]} r) \le \left(\rho^{(a,b)}(f\circ g)+\varepsilon\right)\log^{[b]}\exp^{[b-1]} r$, i.e.,

$$\log^{[a]}|f\circ g|(\exp^{[b-1]} r) \le \left(\rho^{(a,b)}(f\circ g)+\varepsilon\right)\log r. \tag{31}$$

Again we obtain that $\log^{[m]}|g|(\exp^{[n-1]} r) \ge \left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}\exp^{[n-1]} r$, i.e.,

$$\log^{[m]}|g|(\exp^{[n-1]} r) \ge \left(\lambda^{(m,n)}(g)-\varepsilon\right)\log r. \tag{32}$$

Similarly we have $\log^{[m]}|g|(\exp^{[n]} r) \ge \left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}\exp^{[n]} r$, i.e.,

$$\log^{[m-1]}|g|(\exp^{[n]} r) \ge \exp\left[\left(\lambda^{(m,n)}(g)-\varepsilon\right) r\right]. \tag{33}$$

From (31) and (32) we have for all sufficiently large positive numbers of r that

$$\frac{\log^{[a]}|f\circ g|(\exp^{[b-1]} r)}{\log^{[m]}|g|(\exp^{[n-1]} r)} \le \frac{\left(\rho^{(a,b)}(f\circ g)+\varepsilon\right)\log r}{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log r}.$$

As $\varepsilon\,(>0)$ is arbitrary we obtain from the above that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[a]}|f\circ g|(\exp^{[b-1]} r)}{\log^{[m]}|g|(\exp^{[n-1]} r)} \le \frac{\rho^{(a,b)}(f\circ g)}{\lambda^{(m,n)}(g)}. \tag{34}$$

Again from (31) and (33) we get for all sufficiently large positive numbers of r that

$$\frac{\log^{[a]}|f\circ g|(\exp^{[b-1]} r)}{\log^{[m-1]}|g|(\exp^{[n]} r)} \le \frac{\left(\rho^{(a,b)}(f\circ g)+\varepsilon\right)\log r}{\exp\left[\left(\lambda^{(m,n)}(g)-\varepsilon\right) r\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows from the above that

$$\lim_{r\to+\infty} \frac{\log^{[a]}|f\circ g|(\exp^{[b-1]} r)}{\log^{[m-1]}|g|(\exp^{[n]} r)} = 0. \tag{35}$$

Thus the theorem follows from (34) and (35). □

Theorem 7. *Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda^{(p,q)}(f) \le \rho^{(p,q)}(f) < \infty$ and $0 < \lambda^{(m,n)}(g) \le \rho^{(m,n)}(g) < \infty$, where $p,q,m,n \in \mathbb{N}$. Then*

$$(i)\quad \frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \underline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \min\left\{\rho^{(m,n)}(g), \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\};$$

$$\max\left\{\lambda^{(m,n)}(g), \frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \overline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \le \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $q = m = n$,

$$(ii)\quad \frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \min\left\{\rho^{(m,n)}(g), \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\};$$

$$\max\left\{\lambda^{(m,n)}(g), \frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \le \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $q = m >$ *or* $< n$,

$$(iii)\quad \frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)} \leqslant \min\left\{1, \frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)}\right\};$$

$$\max\left\{1, \frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)} \le \frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)},$$

when $q > m$,

$$(iv)\quad \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \min\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}$$
$$\le \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \le \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $m > q = n$,

$$(v)\quad \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \min\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}$$
$$\le \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \le \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $m > q > n$, *and*

$$(vi)\quad \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)} \le \varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \min\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}$$
$$\le \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\} \le \varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}$$
$$\le \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)},$$

when $m > q < n$.

Proof. From the definitions of (p,q)th order and (p,q)th lower order of f, we have for all sufficiently large positive numbers of r that

$$\log^{[p]}|f| \le \left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r\,, \tag{36}$$
$$\log^{[p]}|f| \ge \left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r \tag{37}$$

and also for a sequence of positive numbers of r tending to infinity we get that

$$\log^{[p]}|f| \ge \left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r\,, \tag{38}$$
$$\log^{[p]}|f| \le \left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r\,. \tag{39}$$

Now in view of Lemma 1, we have for all sufficiently large positive numbers of r that

$$\log^{[p]}|f\circ g|(r)\leqslant\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}|g|(r) \tag{40}$$

and also we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}|f\circ g|(r)\leqslant\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]}|g|(r). \tag{41}$$

Similarly, in view of Lemma 1, it follows for all sufficiently large positive numbers of r that

$$\log^{[p]}|f\circ g|(r)\geq\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}|g|(r) \tag{42}$$

and also we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}|f\circ g|(r)\geq\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}|g|(r). \tag{43}$$

Now the following two cases may arise.

Case I. Let $q=m=n$. Then we have from (40) for all sufficiently large positive numbers of r that

$$\log^{[p]}|f\circ g|(r)\leqslant\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r, \tag{44}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}|f\circ g|(r)\leqslant\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r. \tag{45}$$

Also we obtain from (41) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}|f\circ g|(r)\leqslant\left(\lambda^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r. \tag{46}$$

Further it follows from (42) for all sufficiently large positive numbers of r that

$$\log^{[p]}|f\circ g|(r)\geq\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r, \tag{47}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}|f\circ g|(r)\geq\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r. \tag{48}$$

Moreover, we obtain from (43) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}|f\circ g|(r)\geq\left(\rho^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r. \tag{49}$$

Therefore from (37) and (44), we have for all sufficiently large positive numbers of r that

$$\begin{aligned}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)}&\leqslant\frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}\\&=\frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}\end{aligned}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}. \tag{50}$$

Similarly from (38) and (44), for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r},$$

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \rho^{(m,n)}(g). \tag{51}$$

Also from (37) and (45), we obtain for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r},$$

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{52}$$

Further from (37) and (46), for a sequence of positive numbers of r tending to infinity we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r},$$

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \rho^{(m,n)}(g). \tag{53}$$

Thus from (51), (52) and (53) it follows that

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \min\left\{\rho^{(m,n)}(g), \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}. \tag{54}$$

Further from (36) and (47), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r},$$

i.e.,

$$\underline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{55}$$

Similarly, from (39) and (47) we obtain that

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \lambda^{(m,n)}(g). \tag{56}$$

Also from (36) and (48), for a sequence of positive numbers of r tending to infinity we obtain that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \tag{57}$$

and from (36) and (49), for a sequence of positive numbers of r tending to infinity we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\rho^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \lambda^{(m,n)}(g). \tag{58}$$

Thus from (56), (57) and (58) it follows that

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \max\left\{\lambda^{(m,n)}(g), \frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{59}$$

Therefore the first part of the theorem follows from (50), (54), (55) and (59).

Case II. Let $q = m$ and $m > n$ or $n < m$. Now from (37) and (44), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}. \tag{60}$$

Similarly, from (38) and (44) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\rho^{(m,n)}(g). \tag{61}$$

Also from (37) and (45), for a sequence of positive numbers of r tending to infinity we obtain that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{62}$$

and from (37) and (46), for a sequence of positive numbers of r tending to infinity we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\rho^{(m,n)}(g). \tag{63}$$

Thus from (61), (62) and (63) it follows that

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\min\left\{\rho^{(m,n)}(g),\frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}. \tag{64}$$

Further from (36) and (47), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\underline{\lim}_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{65}$$

Similarly, from (39) and (47) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\overline{\lim}_{r\to+\infty}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\lambda^{(m,n)}(g). \tag{66}$$

Also from (36) and (48), for a sequence of positive numbers of r tending to infinity we obtain that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f| \left(\exp^{[q-n]} r\right)} \geq \frac{\lambda^{(p,q)}(f)\cdot \rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{67}$$

Similarly from (36) and (49), we get that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f| \left(\exp^{[q-n]} r\right)} \geq \lambda^{(m,n)}(g). \tag{68}$$

Thus from (66), (67) and (68) it follows that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f| \left(\exp^{[q-n]} r\right)} \geq \max\left\{\lambda^{(m,n)}(g), \frac{\lambda^{(p,q)}(f)\cdot \rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{69}$$

Thus the second part of the theorem follows from (60), (64), (65) and (69).

Case III. Let $q > m$. Then from (40) for all sufficiently large positive numbers of r we have

$$\log^{[p]} |f\circ g|(r) \leqslant \left(\rho^{(p,q)}(f)+\varepsilon\right) \log^{[q-m]} \left[\left(\rho^{(m,n)}(g)+\varepsilon\right) \log^{[n]} r\right]$$

i.e.,

$$\log^{[p]} M(r, f\circ g) \leqslant \left(\rho^{(p,q)}(f)+\varepsilon\right) \log^{[q-m+n]} r + O(1) \tag{70}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} |f\circ g|(r) \leqslant \left(\rho^{(p,q)}(f)+\varepsilon\right) \log^{[q-m+n]} r + O(1). \tag{71}$$

Also for the same reasoning, from (41) for a sequence of positive numbers of r tending to infinity we obtain that

$$\log^{[p]} |f\circ g|(r) \leqslant \left(\lambda^{(p,q)}(f)+\varepsilon\right) \log^{[q-m+n]} r + O(1). \tag{72}$$

Further from (42), for all sufficiently large positive numbers of r it follows that

$$\log^{[p]} |f\circ g|(r) \geq \left(\lambda^{(p,q)}(f)-\varepsilon\right) \log^{[q-m+n]} r + O(1), \tag{73}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} |f\circ g|(r) \geq \left(\lambda^{(p,q)}(f)-\varepsilon\right) \log^{[q-m+n]} r + O(1). \tag{74}$$

Moreover from (43) for a sequence of positive numbers of r tending to infinity we obtain that

$$\log^{[p]} |f\circ g|(r) \geq \left(\rho^{(p,q)}(f)-\varepsilon\right) \log^{[q-m+n]} r + O(1). \tag{75}$$

Now from (37) and (70), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f| \left(\exp^{[m-n]} r\right)} \leqslant \frac{\left(\rho^{(p,q)}(f)+\varepsilon\right) \log^{[q-m+n]} r + O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right) \log^{[q-m+n]} r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p]} |f\circ g|(r)}{\log^{[p]} |f| \left(\exp^{[m-n]} r\right)} \leqslant \frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)}. \tag{76}$$

Similarly, from (38) and (70) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant 1. \tag{77}$$

Also from (37) and (71) for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\frac{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}}\frac{\log^{[p]}M(r,f\circ g)}{\log^{[p]}M\left(\exp^{[m-n]}r,f\right)}\leqslant\frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)}, \tag{78}$$

and from (37) and (72) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\frac{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}}\frac{\log^{[p]}M(r,f\circ g)}{\log^{[p]}M\left(\exp^{[m-n]}r,f\right)}\leqslant 1. \tag{79}$$

Thus from (77), (78) and (79) it follows that

$$\underline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\leqslant\min\left\{1,\frac{\rho^{(p,q)}(f)}{\lambda^{(p,q)}(f)}\right\}. \tag{80}$$

Further from (36) and (73), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)}. \tag{81}$$

Similarly, from (39) and (73) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq 1. \tag{82}$$

Also from (36) and (74), for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)},\tag{83}$$

and from (36) and (75) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\frac{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q-m+n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q-m+n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq 1.\tag{84}$$

Thus from (82), (83) and (84) it follows that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[m-n]}r\right)}\geq\max\left\{1,\frac{\lambda^{(p,q)}(f)}{\rho^{(p,q)}(f)}\right\}.\tag{85}$$

Hence the third part of the theorem follows from (76), (80), (65) and (85).

Case IV. Let $m>q=n$. Then from (40) for all sufficiently large positive numbers of r we have

$$\log^{[p+m-q]}|f\circ g|(r)\leqslant\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1),\tag{86}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p+m-q]}|f\circ g|(r)\leqslant\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1).\tag{87}$$

Also from (41) for a sequence of positive numbers of r tending to infinity we obtain that

$$\log^{[p+m-q]}|f\circ g|(r)\leqslant\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1).\tag{88}$$

Further, from (42) for all sufficiently large positive numbers of r it follows that

$$\log^{[p+m-q]}|f\circ g|(r)\geq\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1),\tag{89}$$

and for a sequence of positive numbers of r tending to infinity that

$$\log^{[p+m-q]}|f\circ g|(r)\geq\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1).\tag{90}$$

Moreover, from (43) for a sequence of positive numbers of r tending to infinity we obtain that

$$\log^{[p+m-q]}|f\circ g|(r)\geq\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1).\tag{91}$$

Therefore from (37) and (86), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{92}$$

Similarly, from (38) and (86) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{93}$$

Also from (37) and (87) for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{94}$$

and from (37) and (88) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{95}$$

Thus from (93), (94) and (95) it follows that

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \leqslant \min\left\{\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}. \tag{96}$$

Further from (36) and (89), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r} = \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,. \tag{97}$$

Similarly, from (39) and (89) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r} = \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda_g(m,n)}{\lambda^{(p,q)}(f)}. \tag{98}$$

Also from(36) and (90) for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r} = \frac{\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \tag{99}$$

and from (36) and (91) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r} = \frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{100}$$

Thus from (98), (99) and (100) it follows that

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|(r)} \geq \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)},\frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{101}$$

Therefore the fourth part of the theorem follows from (92), (96), (98) and (101).

Case V. Let $m > q > n$. Currently from (37) and (86), we have for all sufficiently large positive numbers of r that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}. \tag{102}$$

Similarly, from (38) and (86) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)} \leqslant \frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,. \tag{103}$$

Also from (37) and (87), for a sequence of positive numbers of r tending to infinity we obtain that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\left(\lambda^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{104}$$

and from (37) and (88) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{105}$$

Thus from (103), (104) and (105) it follows that

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\leqslant\min\left\{\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)},\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}\,. \tag{106}$$

Further from (36) and (89), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,. \tag{107}$$

Similarly, from (39) and (89) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{108}$$

Also from (36) and (90), for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[p]}|f|\left(\exp^{[q-n]}r\right)}\geq\frac{\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[n]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[n]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f \circ g| (r)}{\log^{[p]} |f| \left(\exp^{[q-n]} r\right)} \geq \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,, \tag{109}$$

and from (36) and (91) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]} |f \circ g| (r)}{\log^{[p]} |f| \left(\exp^{[q-n]} r\right)} \geq \frac{\left(\lambda^{(m,n)}(g) - \varepsilon\right) \log^{[n]} r + O(1)}{\left(\rho^{(p,q)}(f) + \varepsilon\right) \log^{[n]} r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f \circ g| (r)}{\log^{[p]} |f| \left(\exp^{[q-n]} r\right)} \geq \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,. \tag{110}$$

Thus from (98), (99), and (100) it follows that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f \circ g| (r)}{\log^{[p]} |f| \left(\exp^{[q-n]} r\right)} \geq \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{111}$$

Thus the fifth part of the theorem follows from (102), (106), (107) and (111).

Case VI. Let $m > q < n$. At this instant case from (37) and (86) for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-q]} r\right)}{\log^{[p]} |f| (r)} \leqslant \frac{\left(\rho^{(m,n)}(g) + \varepsilon\right) \log^{[q]} r + O(1)}{\left(\lambda^{(p,q)}(f) - \varepsilon\right) \log^{[q]} r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-q]} r\right)}{\log^{[p]} |f| (r)} \leqslant \frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{112}$$

Similarly, from (38) and (86) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-q]} r\right)}{\log^{[p]} |f| (r)} \leqslant \frac{\left(\rho^{(m,n)}(g) + \varepsilon\right) \log^{[q]} r + O(1)}{\left(\rho^{(p,q)}(f) - \varepsilon\right) \log^{[q]} r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-q]} r\right)}{\log^{[p]} |f| (r)} \leqslant \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}\,. \tag{113}$$

Also from (37) and (87) for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-q]} r\right)}{\log^{[p]} |f| (r)} \leqslant \frac{\left(\lambda^{(m,n)}(g) + \varepsilon\right) \log^{[q]} r + O(1)}{\left(\lambda^{(p,q)}(f) - \varepsilon\right) \log^{[q]} r}$$

i.e.,

$$\underline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]} |f \circ g| \left(\exp^{[n-q]} r\right)}{\log^{[p]} |f| (r)} \leqslant \frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \tag{114}$$

and from (37) and (88) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\leqslant\frac{\left(\rho^{(m,n)}(g)+\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)-\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\leqslant\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\,. \tag{115}$$

Thus from (113), (114) and (115) it follows that

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\leqslant\min\left\{\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)},\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)},\frac{\rho^{(m,n)}(g)}{\lambda^{(p,q)}(f)}\right\}\,. \tag{116}$$

Further from (36) and (89), for all sufficiently large positive numbers of r we have that

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varliminf_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{117}$$

Similarly, from (39) and (89) for a sequence of positive numbers of r tending to infinity it follows that

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\lambda^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}. \tag{118}$$

Also from (36) and (90), for a sequence of positive numbers of r tending to infinity we obtain

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\left(\rho^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\varlimsup_{r\to+\infty}\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \tag{119}$$

and from (36) and (91) for a sequence of positive numbers of r tending to infinity also we have

$$\frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]}r\right)}{\log^{[p]}|f|(r)}\geq\frac{\left(\lambda^{(m,n)}(g)-\varepsilon\right)\log^{[q]}r+O(1)}{\left(\rho^{(p,q)}(f)+\varepsilon\right)\log^{[q]}r}$$

i.e.,

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]} r\right)}{\log^{[p]}|f|(r)} \geq \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}. \tag{120}$$

Thus from (98), (99) and (100) it follows that

$$\overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|\left(\exp^{[n-q]} r\right)}{\log^{[p]}|f|(r)} \geq \max\left\{\frac{\lambda^{(m,n)}(g)}{\lambda^{(p,q)}(f)}, \frac{\rho^{(m,n)}(g)}{\rho^{(p,q)}(f)}, \frac{\lambda^{(m,n)}(g)}{\rho^{(p,q)}(f)}\right\}. \tag{121}$$

Hence the sixth part of the theorem follows from (112), (116), (118) and (121). □

Theorem 8. *Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $0 < \lambda^{(p,q)}(f) \leq \rho^{(p,q)}(f) < \infty$ and $0 < \lambda^{(m,n)}(g) \leq \rho^{(m,n)}(g) < \infty$, where $p, q, m, n \in \mathbb{N}$. Then*

$$(i) \quad \frac{\lambda^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(m,n)}(g)} \leq \underline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[m]}|g|(r)} \leq \min\left\{\rho^{(p,q)}(f), \frac{\lambda^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(m,n)}(g)}\right\};$$

$$\max\left\{\lambda^{(p,q)}(f), \frac{\rho^{(p,q)}(f)\cdot\lambda^{(m,n)}(g)}{\rho^{(m,n)}(g)}\right\} \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|(r)}{\log^{[m]}|g|(r)} \leq \frac{\rho^{(p,q)}(f)\cdot\rho^{(m,n)}(g)}{\lambda^{(m,n)}(g)},$$

when $q = m$,

$$(ii) \quad \frac{\lambda^{(p,q)}(f)}{\rho^{(m,n)}(g)} \leq \underline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|\left(\exp^{[q-m]} r\right)}{\log^{[m]}|g|(r)} \leqslant \min\left\{\frac{\rho^{(p,q)}(f)}{\rho^{(m,n)}(g)}, \frac{\rho^{(p,q)}(f)}{\lambda^{(m,n)}(g)}, \frac{\lambda^{(p,q)}(f)}{\lambda^{(m,n)}(g)}\right\};$$

$$\max\left\{\frac{\rho^{(p,q)}(f)}{\rho^{(m,n)}(g)}, \frac{\lambda^{(p,q)}(f)}{\rho^{(m,n)}(g)}, \frac{\lambda^{(p,q)}(f)}{\lambda^{(m,n)}(g)}\right\} \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p]}|f\circ g|\left(\exp^{[q-m]} r\right)}{\log^{[m]}|g|(r)} \leq \frac{\rho^{(p,q)}(f)}{\lambda^{(m,n)}(g)},$$

when $q > m$, and

$$(iii) \quad \frac{\lambda^{(m,n)}(g)}{\rho^{(m,n)}(g)} \leq \underline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[m]}|g|(r)} \leq 1 \leq \overline{\lim_{r\to+\infty}} \frac{\log^{[p+m-q]}|f\circ g|(r)}{\log^{[m]}|g|(r)} \leq \frac{\rho^{(m,n)}(g)}{\lambda^{(m,n)}(g)},$$

when $m > q$.

We omit the proof of Theorem 8 as it can easily be deduced in the line of Theorem 7.

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Бісвас Т. *Оцінка орієнтованого росту складених p-адичних цілих функцій, що залежить від (p,q)-го порядку* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 248–272.

Нехай $\mathbb{K}$ — повне ультраметричне алгебраїчно замкнуте поле, $\mathcal{A}(\mathbb{K})$ — $\mathbb{K}$-алгебра цілих функцій на $\mathbb{K}$. Для довільної p-адичної цілої функції $f \in \mathcal{A}(\mathbb{K})$ і $r > 0$ позначимо $|f|(r)$ число $\sup\{|f(x)| : |x| = r\}$, де $|\cdot|(r)$ є мультиплікативною нормою на $\mathcal{A}(\mathbb{K})$. Для довільних двох цілих функцій $f \in \mathcal{A}(\mathbb{K})$ та $g \in \mathcal{A}(\mathbb{K})$ співвідношення $\frac{|f|(r)}{|g|(r)}$ при $r \to \infty$ називають порівняльним ростом f відносно g в сенсі їхніх мультиплікативних норм. Аналогічно до того, як це роблять в комплексному аналізі, в цій статті ми визначаємо поняття (p,q)-го порядку (відповідно (p,q)-го нижнього порядку) росту наступним чином $\rho^{(p,q)}(f) = \limsup\limits_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}$ (відпоідно $\lambda^{(p,q)}(f) = \liminf\limits_{r\to+\infty} \frac{\log^{[p]}|f|(r)}{\log^{[q]} r}$), де p і q два довільні натуральні числа. Ми досліджуємо деякі властивості росту складених p-адичних цілих функцій на основі їхнього (p,q)-го порядку і (p,q)-го нижнього порядку.

Ключові слова і фрази: p-адична ціла функція, ріст, (p,q)-й порядок, (p,q)-й нижній порядок, композиція.