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A REMARK ON STATIONARY FUZZY METRIC SPACES

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The main result states that the category of stationary fuzzy metric spaces (with respect to an archimedean t-norm) and nonexpanding maps is isomorphic to a full subcategory of the category of metric spaces and nonexpanding maps. The case of non-archimedean t-norms is also discussed.

Introduction

The notion of fuzzy metric space is tightly connected with the notion of probabilistic metric space. The latter is a generalization of the notion of metric space in which the distances take their values in the class of distribution functions. The fuzzy metric spaces found numerous applications, e.g., to the theory of image processing.

The theory of fuzzy metric spaces is developed in different directions. In particular, many authors considered the problem of existence of fixed points in the fuzzy setting (see, e.g. [3], [4]). Also, some functorial constructions in the categories of fuzzy metric spaces were investigated.

There are two main theories of the fuzzy metric spaces. One of them is based on the notion introduced by Kramosil and Michalek in [6]. Another class of fuzzy metric spaces, more restrictive, is defined by George and Veeramani [2]. In this note we deal with a subclass of the latter class, namely, with the so-called stationary fuzzy metric spaces. The obtained results demonstrate that the stationary fuzzy metric spaces with respect to given archimedean (and some non-archimedean) t-norm are tightly connected with the ordinary metric spaces.

In the last section we formulate some open problems related to the results of this note.

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1 Preliminaries

A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with the unit 1 such that $a*b\le c*d$ whenever $a*c$ and $b*d$ for all $a,b,c,d\in[0,1]$.

In the sequel, we will consider the following examples of t-norms:

1. $a*b=\min\{a,b\}$;
2. $a*b=ab$;
3. $a*b=\max\{a+b-1,0\}$ (Łukasiewicz t-norm);
4. $a*b=\dfrac{ab}{\max\{a,b,\alpha\}}$.

A t-norm $*$ is said to be *archimedean* provided for every $x,y\in(0,1)$ there exists $n\in\mathbb{N}$ such that $x*x*x\cdots*x<y$ (here $*$ appears $n-1$ times).

It is well-known (see, e.g. [7]) that any archimedean t-norm $*$ can be represented by means of a continuous additive generator, i.e. a continuous strictly decreasing function $t\colon[0,1]\to[0,\infty]$ with $t(1)=0$ such that

$$x*y=t^{(-1)}(t(x)+t(y)),\ \ x,y\in[0,1]$$

(hereafter, $t^{(-1)}(u)=t^{-1}(\min(u,t(0)))$ is the *pseudoinverse* of t).

Definition 1.1. *A triplet $(X,M,*)$ is a FM-space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $t,s>0$:*

(i) $M(x,y)>0$;

(ii) $M(x,y,t)=1$ *for all* $t>0$ *if and only if* $x=y$;

(iii) $M(x,y,t)=M(y,x,t)$;

(iv) $M(x,y,t)*M(y,z,t)\le M(x,z,t+s)$;

(v) $M(x,y,-)\colon[0,\infty)\to[0,1]$ *is continuous.*

Let $(X_i,M_i,*)$, $i=1,2$, be fuzzy metric spaces. A map $f\colon X_1\to X_2$ is called *non-expanding* if $M_1(x,y,t)\le M_2(f(x),f(y),t)$ for any $x,y\in X_1$ and $t>0$. The fuzzy metric spaces and non-expanding maps form a category, which we denote $\mathcal{FMS}(*)$.

If $(X,M,*)$ is a fuzzy metric space, $x\in X$, $r\in(0,1)$ and $t>0$, then the set

$$B(x,r,t)=\{y\in X\mid M(x,y,t)>1-r\}$$

is called the *ball of radius r centered at x for t*. It is known that the family of all balls is a base of a metrizable topology for every fuzzy metric space X (see [2]).

A fuzzy metric M is called *stationary* if the function $M(x,y,-)\colon [0,\infty) \to [0,1]$ is constant for every $x, y \in X$. We write $M(x,y)$ instead of $M(x,y,t)$, and $B(x,r)$ instead of $B(x,r,t)$ for any stationary fuzzy metric M. We denote by $\mathcal{SFMS}(*)$ the full subcategory of the category $\mathcal{FMS}(*)$ whose objects are stationary fuzzy metric spaces.

We say that a metric space (X,d) is of *strong diameter* c if $d(x,y) < c$ for all $x, y \in X$.

Theorem 1. *Let (X,M) be a stationary fuzzy metric space with respect to an archimedean t-norm $*$. Then the function $d = t \circ M$, where t is a continuous additive generator of M, is a metric on X of strong diameter $t(0)$. The topologies on X induced by M and d coincide.*

Moreover, this construction determines a functor from the category $\mathcal{SFMS}(*)$ into the category of metric spaces and nonexpanding maps.

Proof. Let $x, y, z \in X$.

If $d(x,y) = 0$, then, since t is strictly decreasing, $M(x,y) = 1$ and therefore $x = y$. Also $d(x,x) = t(M(x,y)) = t(1) = 0$.

Clearly, $d(x,y) = d(y,x)$.

Prove the triangle inequality. We have

$$\begin{aligned} M(x,y) * M(y,z) =& t^{(-1)}(t(M(x,y)) + t(M(y,z))) = \\ & t^{-1}(\min\{t(M(x,y)) + t(M(y,z)), 0\}) \le M(x,z), \end{aligned}$$

and therefore applying t to both sides of the above inequality we obtain

$$\begin{aligned} d(x,y) + d(x,z) = t(M(x,y)) + t(M(y,z)) \ge& \min\{t(M(x,y)) + t(M(y,z)), 0\} \ge \\ & t(M(x,z)) = d(x,z). \end{aligned}$$

Remark also that, since $M(x,y) > 0$, we obtain $d(x,y) < t(0)$.

Let $B^d(x,r) = \{y \in X \mid d(x,y) < r\}$. From the fact that $B^d(x,r) = B(x, 1 - t^{-1}(r))$ it follows that the topologies generated by M and d coincide.

If (X_i, M_i), $i = 1,2$, are stationary fuzzy metric spaces and a map $f\colon X_1 \to X_2$ is nonexpanding, then this map is easily seen to be nonexpanding with respect to the induced metrics. □

Recall that an ultrametric d on a set X is a metric satisfying

$$d(x,z) \le \max\{d(x,y), d(y,z)\},\ x, y, z \in X.$$

Proposition 1.1. *Let $* = \min$. The category $\mathcal{SFMS}(*)$ is isomorphic to the category of ultrametric spaces and nonexpanding maps.*

Proof. Define $d(x,y) = 1 - M(x,y)$. One can easily prove that d is an ultrametric on X. Moreover, the topologies on X induced by d and M coincide. □

Proposition 1.2. *Let $* = \cdot$. The category $\mathcal{SFMS}(*)$ is isomorphic to the category of metric spaces and nonexpanding maps.*

Proof. Since $t = -\ln$ is clearly the continuous additive generator for the t-norm $\cdot$, the assertion follows from Theorem 1. $\square$

Proposition 1.3. *Let $*$ be the Łukasiewicz t-norm. The category $\mathcal{SFMS}(*)$ is isomorphic to the category of metric spaces of strong diameter* 1 *and nonexpanding maps.*

Proof. It is known (and can be easily seen) that the function t defined by the formula $t(x) = 1 - x$ is the continuous additive generator for the Łukasiewicz t-norm. Since $t(0) = 1$, the assertion follows from Theorem 1. $\square$

2 K-ultrametric spaces

Let X be a set and $K \in [0, \infty]$. A metric d on X is called a K-*ultrametric* if $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ whenever $\min\{d(x, z), d(z, y)\} \leq K$.

Note that any 0-ultrametric is a metric and any ∞-ultrametric is an ultrametric.

Below we describe a construction which allows us to produce examples of K-ultrametric space. Let (X, ϱ) be an ultrametric space and let $\sim_K$ denote its decomposition into disjoint closed balls of radius K. Denote by $q\colon X \to X/\sim_K$ the quotient map. For any metric D on $X/\sim_K$, the function $d\colon X \times X \to \mathbb{R}$ defined by the formula $d(x, y) = \varrho(x, y) + D(q(x), q(y))$ is a K-ultrametric on X. Indeed, if $x, y, z \in X$ and $\varrho(x, z) \leq K$, $\varrho(z, y) \leq K$, then $q(x) = q(z) = q(y)$ and therefore

$$d(x, y) = \varrho(x, y) + D(q(x), q(y)) \leq \max\{\varrho(x, z), \varrho(x, z)\} + D(q(x), q(z)) + D(q(z), q(y)) \leq \max\{\varrho(x, z), \varrho(y, z)\} = \max\{d(x, z), d(y, z)\},$$

and, if, say, $\varrho(x, z) \leq K$, $\varrho(z, y) > K$, then $q(x) = q(z)$ and therefore

$$d(x, y) = \varrho(x, y) + D(q(x), q(y)) \leq \max\{\varrho(x, z), \varrho(y, z)\} + D(q(z), q(y)) = \varrho(y, z) + D(q(z), q(y)) = d(y, z) \leq \max\{d(x, z), d(y, z)\}.$$

Theorem 2. *Let $a * b = \frac{ab}{\max\{a,b,\alpha\}}$, where $\alpha \in (0, 1)$. Then the category $\mathcal{SFMS}(*)$ is isomorphic to the category of K-ultrametric spaces and nonexpanding maps.*

Proof. Let M be a stationary fuzzy metric. Define $d\colon X \times X \to \mathbb{R}$ by the formula $d(x, y) = -\ln M(x, y)$. Then, suppose that $x, y, z \in X$ and $\min\{d(x, z), d(z, y)\} \geq -\ln \alpha$. Then $M(x, y) \leq e^{\ln \alpha} = \alpha$, $M(y, z) \leq \alpha$ and therefore

$$M(x, z) \geq M(x, y) * M(y, z) = \frac{M(x, y)M(y, z)}{\alpha},$$

which in turn implies that $d(x, y) + d(y, z) \geq d(x, z) + \alpha \geq d(x, z)$.

We are going to prove that d is a K-ultrametric on X with $K = -\ln \alpha$. Suppose that $x, y, z \in X$ and $\min\{d(x, z), d(z, y)\} \leq K$. Then, say, $M(x, y) \geq \alpha$, whence

$$M(x, z) \geq M(x, y) * M(y, z) = \frac{M(x, y)M(y, z)}{\max\{M(x, y), M(y, z), \alpha\}} \geq \frac{M(x, y)M(y, z)}{M(x, y)} \geq \frac{M(x, y)M(y, z)}{\max\{M(x, y), M(y, z)\}} \geq \min\{M(x, y), M(y, z)\},$$

and therefore $d(x, y) \leq \max\{d(x, y), d(x, z)\}$. The rest of the proof is left to the reader. $\square$

By $\exp X$ we denote the hyperspace of a topological space X, i.e. the family of nonempty closed subsets in X. It is known [8] that every fuzzy metric M on X generates a fuzzy metric M_H on $\exp X$ (the fuzzy Hausdorff metric) as follows:

$$M_H(A,B,t) = \min\left\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\right\}$$

for every $A, B \in \exp X$ and $t > 0$. Here $M(a,B,t) = \sup\{M(a,b,t) \mid b \in B\}$, $a \in X$, $B \in \exp X$.

Corollary 2.1. *The hyperspace of any K-ultrametric space is again a K-ultrametric space.*

Proof. The result follows from the previous theorem and from the fact that for any stationary fuzzy metric space the fuzzy Hausdorff stationary metric is again a stationary fuzzy metric. □

3 Remarks

Let $(X_i, M_i, *)$, $i = 1, 2$, be fuzzy metric spaces. A map $f\colon X_1 \to X_2$ is called a *contraction* if $M_1(x,y,t) < M_2(f(x), f(y), t)$ for any $x, y \in X_1$ and $t > 0$. The results of the previous sections can be also formulated for the category of (stationary fuzzy) metric spaces and contractions.

For any $k \in (0,1)$ a map $f\colon X \to X$ of a fuzzy metric space $(X, M, *)$ is called a *k-contraction* if

$$\frac{1}{M(f(x), f(y), t)} - 1 \le k\left(\frac{1}{M(x,y,t)} - 1\right)$$

for every $x, y \in X$. A metric analogue of this notion is unknown.

It is known (see, e.g., [1]) that for any continuous t-norm $*$ there exists a unique (finite or countable) index set A, unique pairwise disjoint intervals $(a_\alpha, e_\alpha) \subset [0,1]$, and unique continuous archimedean t-norms $*_\alpha$ such that

$$x * y = \begin{cases} a_\alpha + (e_\alpha - a_\alpha)\left(\frac{x-a_\alpha}{e_\alpha - a_\alpha} *_\alpha \frac{y - a_\alpha}{e_\alpha - a_\alpha}\right), & \text{if } (x,y) \in (a_\alpha, e_\alpha),\\ \min\{x,y\}, & \text{otherwise.}\end{cases}$$

The above results lead us to the following question. Is there a category of metric spaces which is isomorphic to the category $\mathcal{SFMS}(*)$?

In the theory of metric spaces, one of the most important roles is played by the Urysohn universal metric spaces [10]. Recall that a metric space (U, d) is called Urysohn universal if U is separable and complete and has the following property: given any finite metric space X, any point $x \in X$, and any isometric embedding $f\colon X \setminus \{x\} \to U$, there exists an isometric embedding $F\colon X \to U$ that extends f. The classical Urysohn theorem asserts that an Urysohn universal metric space exists and is unique up to isometry.

A counterpart of the notion of universal Urysohn metric space in the class of ultrametric spaces is discussed in [9]. Also, there are Urysohn universal spaces for the class of metric spaces of diameter ≤ 1 (see, e.g., [5]).

This allows us to formulate the following problem. Is there a counterpart of the universal Urysohn space for (some subclasses of) the class of separable fuzzy metric spaces?

References

1. Clifford A.H., Preston G.B. The Algebraic Theory of Semigroups. Amer. Math. Soc., Providence, RI, 1961.
2. George A., Veeramani P. *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems, **90** (1997), 365–368.
3. Grabiec M. *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, **27** (1988), 385–389.
4. Gregori V., Sapena A. *On fixed-point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems, **125**, 2 (2002), 245–252.
5. Katětov M. *On universal metric spaces*, in: Frolik Z.(ed.), General Topology and its Relations to Modern Analysis and Algebra VI. Proceedings of the Sixth Prague Topological Symposium 1986, Heldermann Verlag, Berlin, 1988, 323–330.
6. Kramosil I., Michalek J. *Fuzzy metric and statistical metric spaces*, Kybernetica, **11** (1975), 326–334.
7. Ling C.M. *Representation of associative functions*. Publ. Math. Debrecen, **12** (1965), 189–212.
8. Rodríguez-López J., Romaguera S. *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, **147**, 2 (2004), 273–283.
9. Su Gao, Chuang Shao. *Polish ultrametric Urysohn spaces and their isometry groups*, Topology and its Applications, **158** (2011), 492–508.
10. Urysohn P. *Sur un espace métrique universel*, Bull. Sci. Math., **51** (1927), 43–64, 74–90.

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Доведено, що категорія стаціонарних розмитих метричних просторів (відносно архімедової t-норми) і нерозтягуючих відображень ізоморфна повній підкатегорії категорії метричних просторів і нерозтягуючих відображень. Розглянуто також випадок неархімедової t-норми.

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Доказано, что категория стационарных нечетких метрических просторанств (по отношению к архимедовой t-норме) и нерастягивающих отображений изоморфна полной подкатегории категории метрических пространств и нерастягивающих отображений. Рассмотрено также случай неархимедовой t-нормы.