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ON ESTIMATIONS OF SOLUTIONS OF ONE CONVOLUTION TYPE EQUATION

We consider a convolution type equation in a semi-strip for the functions belonging to the Hardy-Smirnov space. Estimations of solutions are obtained in terms of analytic continuation.

Key words and phrases: weighted Hardy space, convolution equation, outer function, cyclic function.

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INTRODUCTION

For $p \in [1;\infty)$ the Hardy space $H^p(\mathbb{C}_+)$ consists of analytic functions on the half-plane $\mathbb{C}_+ = \{z : \mathrm{Re} z > 0\}$ such that

$$||f||_* := \sup_{x>0} \left\{ \int_{-\infty}^{+\infty} |f(x+iy)|^p \, dy \right\}^{1/p} < +\infty.$$

A function $f \in H^p(\mathbb{C}_+)$ has almost everywhere (a.e.) on $i\mathbb{R}$ angular boundary values $f(iy)$ and $f(iy) \in L^p(-\infty;+\infty)$. Each $H^p(\mathbb{C}_+), 1 \le p < +\infty$, is a Banach space with respect to the above norm. Previous and other properties of this space are presented in [10].

The following classical result (see [11, 12]) about convolution equation on the ray has many important and elegant applications for the spectral theory of linear operators (see [3, 4, 7]).

Theorem A. *If $q \in L_2(-\infty;0)$ and $Q(z) = \int_{-\infty}^{0} q(t)e^{tz}\,dt$, then the following statements are equivalent:*

1) *the equation*

$$\int_{-\infty}^{0} \psi(t+\tau)q(t)\,dt = 0, \quad \tau \le 0, \tag{1}$$

has a nontrivial solution $\psi \in L_2(-\infty;0)$;

2) *the system $\{Q(z)e^{\tau z} : \tau \le 0\}$ is not complete in $H^2(\mathbb{C}_+)$;*

3) *Q is not outer function for $H^2(\mathbb{C}_+)$.*

A function $Q \in H^2(\mathbb{C}_+)$ is called an outer function for $H^2(\mathbb{C}_+)$ if $Q(z) \neq 0$ for all $z \in \mathbb{C}_+$,

$$\overline{\lim_{x\to+\infty}} \frac{\ln|Q(x)|}{x} = 0,$$

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and singular boundary function of Q is a constant.

The necessary part of Theorem A is based on the following result.

Theorem B. *Suppose* $q \in L_2(-\infty;0)$, $Q(z) = \int\limits_{-\infty}^{0} q(t)e^{tz}\,dt$. *Function* $\psi \in L_2(-\infty;0)$ *is a solution of equation* (1) *if and only if the function* $Q(iy)\Psi(iy)$, $y \in \mathbb{R}$, *is the angular boundary function on* $i\mathbb{R}$ *of some function* $P \in H^1(\mathbb{C}_+)$, where $\Psi(z) = \int\limits_{-\infty}^{0} \psi(t)e^{-tz}\,dt$.

For a generalization of the above results we introduce some spaces. Let $H^p_\sigma(\mathbb{C}_+)$, $\sigma \geq 0$, $1 \leq p < +\infty$, be the space of analytic functions on $\mathbb{C}_+$ satisfying

$$||f|| := \sup_{-\frac{\pi}{2}<\varphi<\frac{\pi}{2}} \left\{ \int\limits_0^{+\infty} |f(re^{i\varphi})|^p e^{-pr\sigma|\sin\varphi|}\,dr \right\}^{1/p} < +\infty.$$

A.M. Sedletskii proved (see [8]) the equality $H^p_0(\mathbb{C}_+) = H^p(\mathbb{C}_+)$. A function $f \in H^p_\sigma(\mathbb{C}_+)$ has a.e. on $i\mathbb{R}$ angular boundary values $f(iy)$ and $f(iy)e^{-\sigma|y|} \in L^p(-\infty;+\infty)$. Each $H^p_\sigma(\mathbb{C}_+)$, $1 \leq p < +\infty$, is a Banach space with respect to the above norm. Previous and other properties of these spaces are presented in [14]. The singular boundary function h of $G \in H^p_\sigma(\mathbb{C}_+)$ is defined up to an additive constant and to the values in points of continuity by equality

$$h(t_2) - h(t_1) = \lim_{x\to 0+} \int\limits_{t_1}^{t_2} \ln|G(x+iy)|dy - \int\limits_{t_1}^{t_2} \ln|G(iy)|dy.$$

Let $E^p[D_\sigma]$ and $E^p_*[D_\sigma], 1 \leq p < +\infty, \sigma > 0$, be the spaces of analytic functions f in the domains $D_\sigma = \{z : |\mathrm{Im} z| < \sigma, \mathrm{Re} z < 0\}$ and $D^*_\sigma = \mathbb{C}\backslash\overline{D}_\sigma$ respectively, satisfying

$$\sup \left\{ \int\limits_\gamma |f(z)|^p\,|dz| \right\}^{1/p} < +\infty,$$

where the supremum is over all segments γ lying in D_σ and D^*_σ respectively. The spaces $E^p[D_\sigma]$ and $E^p_*[D_\sigma]$ have been studied in [2]. Functions f belonging to these spaces have a.e. on ∂D_σ angular boundary values $f(z)$ and $f \in L^p[\partial D_\sigma]$. The paper [2] covers the following analogue of equation (1)

$$\int\limits_{\partial D_\sigma} f(w+\tau)g(w)\,dw = 0, \quad \tau \leq 0, \quad g \in E^2_*[D_\sigma]. \tag{2}$$

In [6, 9, 13] the following analogue of Theorem A is obtained.

Theorem C. *Let* $g \in E^2_*[D_\sigma]$. *Then the following conditions are equivalent:*

1) *equation* (2) *has a nontrivial solution* $f \in E^2[D_\sigma]$;

2) G *is not cyclic in* $H^2_\sigma(\mathbb{C}_+)$, *i.e. the system* $\{G(z)e^{\tau z} : \tau \leq 0\}$ *is not complete in* $H^2_\sigma(\mathbb{C}_+)$, *where*

$$G(z) = \frac{1}{i\sqrt{2\pi}} \int\limits_{\partial D_\sigma} g(w)e^{-zw}\,dw;$$

3) $G(z) = 0$ *for some* $z \in \mathbb{C}_+$ *or the singular boundary function of* G *is not a constant or*

$$\varliminf_{r\to+\infty} \left(\frac{1}{2\pi} \int\limits_{1<|t|<r} \left(\frac{1}{t^2} - \frac{1}{r^2}\right) \ln|G(it)|\,dt - \frac{\sigma}{\pi}\ln r \right) > -\infty.$$

The aim of this article is to search an analogue of Theorem B. We denote by F_j, $j \in \{1;2;3\}$, the functions

$$F_j(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{l_j} f(w) e^{-zw}\, dw, \quad j \in \{1;2;3\},$$

where l_1, l_3, and l_2 are the legs of ∂D_σ (the rays laying under and above of the real axis, and the segment $[-i\sigma; i\sigma]$ respectively) and their orientation corresponds to the positive orientation of D_σ.

1 THE MAIN RESULT

Theorem 1. *Let $f \in E_2[D_\sigma]$ be a solution of equation* (2). *Then there exists the analytic in $\mathbb{C}_+$ function P_1 such that*

$$\sup\left\{ \int\limits_0^{+\infty} \left|P_1(re^{i\varphi})\right| e^{-\sigma r|\sin\varphi|}\, dr : \varphi \in (-\pi/2; \pi/2)\backslash(-\delta;\delta) \right\} < +\infty \tag{3}$$

for all $\delta \in (0; \pi/2)$ and the angular boundary values of the function P_1 on $i\mathbb{R}$ coincides a.e. with $G(iy)F_1(iy)e^{\sigma y}$.

Proof. Let $f \in E_2[D_\sigma]$ be a solution of equation (2) and

$$S(z) = -\frac{1}{2\pi i} \int\limits_0^{+i\infty} \Phi_1(w) \frac{1}{w-z}\, dw + \frac{1}{2\pi i} \int\limits_{-i\infty}^{0} \Phi_3(w) \frac{1}{w-z}\, dw - \frac{1}{2\pi i} \int\limits_0^{+\infty} \Phi_2(w) \frac{1}{w-z}\, dw,$$

where $\Phi_j(it) = F_j(it)G(it)$, $j \in \{1,2,3\}$, $t \in \mathbb{R}$. Then the function

$$P_1(z) = e^{-i\sigma z} \begin{cases} S(z), & z \in \mathbb{C}(0;\pi/2), \\ S(z) - \Phi_2(z), & z \in \mathbb{C}(-\pi/2; 0) \end{cases}$$

is required. Since the equality

$$\int\limits_{\partial D_\sigma} f(w+\tau) g(w)\, dw = \int\limits_0^{+i\infty} \Phi_1(z) e^{\tau z}\, dz + \int\limits_{-i\infty}^{0} \Phi_3(z) e^{\tau z}\, dz + \int\limits_0^{+\infty} \Phi_2(z) e^{\tau z}\, dz,$$

holds (see [2]), we have

$$\int\limits_{-\infty}^{0} e^{-\tau z} \Bigg(\int\limits_0^{+i\infty} \Phi_1(z) e^{\tau z}\, dz + \int\limits_{-i\infty}^{0} \Phi_3(z) e^{\tau z}\, dz + \int\limits_0^{+\infty} \Phi_2(z) e^{\tau z}\, dz \Bigg) d\tau = 0, \quad \mathrm{Re} z < 0.$$

But by Fubini's theorem for $\mathrm{Re} z < 0$ we have

$$\int\limits_{-\infty}^{0} e^{-\tau z} \int\limits_0^{+\infty} \Phi_2(u) e^{\tau u}\, du\, d\tau = \int\limits_0^{+\infty} \Phi_2(u)\, du \int\limits_{-\infty}^{0} e^{\tau(u-z)}\, d\tau = -\int\limits_0^{+\infty} \frac{\Phi_2(u)}{u-z}\, du.$$

Analogously,

$$\int_{-\infty}^{0} e^{-\tau z} \int_{0}^{+\infty} i\Phi_1(iv)e^{\tau u}\, dv\, d\tau = -i \int_{0}^{+\infty} \frac{\Phi_1(iv)}{iv - z}\, dv, \quad \int_{-\infty}^{0} e^{-\tau z} \int_{-\infty}^{0} i\Phi_3(iv)e^{\tau u}\, dv\, d\tau = i \int_{-\infty}^{0} \frac{\Phi_3(iv)}{iv - z}\, dv.$$

Therefore

$$0 = -\int_{0}^{+i\infty} \Phi_1(w)\frac{1}{w-z}\, dw + \frac{1}{2\pi i}\int_{-i\infty}^{0} \Phi_3(w)\frac{1}{w-z}\, dw - \frac{1}{2\pi i}\int_{0}^{+\infty} \Phi_2(w)\frac{1}{w-z}\, dw, \quad \mathrm{Re} z < 0.$$

Then we have the for $\mathrm{Re} z > 0$

$$0 = -\frac{1}{2\pi i}\int_{0}^{+i\infty} \Phi_1(w)\frac{1}{w+\bar{z}}\, dw + \frac{1}{2\pi i}\int_{-i\infty}^{0} \Phi_3(w)\frac{1}{w+\bar{z}}\, dv - \frac{1}{2\pi i}\int_{0}^{+\infty} \Phi_2(w)\frac{1}{w+\bar{z}}\, dw,$$

$$0 = -\frac{1}{2\pi i}\int_{0}^{+i\infty} \Phi_1(w)\frac{1}{w+z}\, dw + \frac{1}{2\pi i}\int_{-i\infty}^{0} \Phi_3(w)\frac{1}{w+z}\, dv - \frac{1}{2\pi i}\int_{0}^{+\infty} \Phi_2(w)\frac{1}{w+z}\, dw.$$

As a consequence of the previous equalities we obtain for $z = x + iy$, $y \neq 0$,

$$0 = -\frac{1}{2\pi i}\int_{0}^{+i\infty} \Phi_1(w)\left(-\frac{1}{w+\bar{z}} + \frac{x}{iy}\left(\frac{1}{w+\bar{z}} - \frac{1}{w+z}\right)\right) dw + \frac{1}{2\pi i}\int_{-i\infty}^{0} \Phi_3(w)\Bigg(-\frac{1}{w+\bar{z}}$$
$$+\frac{x}{iy}\left(\frac{1}{w+z} - \frac{1}{w+\bar{z}}\right)\Bigg) dw - \frac{1}{2\pi i}\int_{0}^{+\infty} \Phi_2(w)\left(-\frac{1}{w+\bar{z}} + \frac{x}{iy}\left(\frac{1}{w+z} - \frac{1}{w+\bar{z}}\right)\right) dw.$$

Hence

$$S(z) = -\int_{0}^{+i\infty} \Phi_1(w)K_+(w;z)\, dw + \int_{-i\infty}^{0} \Phi_3(w)K_+(w;z)\, dw - \int_{0}^{+\infty} \Phi_2(w)K_+(w;z)\, dw,$$

where

$$K_+(w;z) := \frac{-2}{\pi i}\frac{wx}{(w+\bar{z})(w-z)(w+z)} = \frac{-1}{2\pi i}\left(\frac{1}{w-z} - \frac{1}{w+\bar{z}} + \frac{x}{iy}\left(\frac{1}{w+\bar{z}} - \frac{1}{w+z}\right)\right).$$

Obviously, $\frac{\pi}{2}\int_{0}^{+\infty} \left|K_+(it;re^{i\varphi})\right| dr = \int_{0}^{+\infty} \frac{|t|\, r\cos\varphi\, dr}{(t^2 - 2tr\sin\varphi + r^2)\sqrt{t^2 + 2tr\sin\varphi + r^2}}$. If $t > 0$, then

$$\int_{0}^{+\infty} \frac{|t|\, r\cos\varphi\, dr}{(t^2 - 2tr\sin\varphi + r^2)\sqrt{t^2 + 2tr\sin\varphi + r^2}} = \int_{0}^{+\infty} \frac{u\cos\varphi\, du}{(1 - 2u\sin\varphi + u^2)\sqrt{(1 + 2u\sin\varphi + u^2)}}.$$

If $\varphi \in (0;\pi/2)$, then by inequality $u < \sqrt{u^2+1}$ we have

$$\int_{0}^{+\infty} \frac{u\cos\varphi\, du}{(1 - 2u\sin\varphi + u^2)\sqrt{(1 + 2u\sin\varphi + u^2)}} \leq \int_{0}^{+\infty} \frac{\cos\varphi\, du}{1 - 2u\sin\varphi + u^2} = \frac{\pi}{2} + \mathrm{arctg}\,\mathrm{tg}\,\varphi \leq \pi.$$

If $\varphi \in (-\pi/2; 0)$, then

$$\int_0^{+\infty} \frac{u\cos\varphi\, du}{(1-2u\sin\varphi+u^2)\sqrt{1+2u\sin\varphi+u^2}} \le \int_0^{+\infty} \frac{u\cos\varphi\, du}{(1+u^2)\sqrt{2u(1+\sin\varphi)}}$$

$$= \int_0^{+\infty} \frac{u2\sin\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)\, du}{(1+u^2)\sqrt{4u\cos^2\left(\frac{\pi}{4}-\frac{\varphi}{2}\right)}} \le \int_0^{+\infty} \frac{\sqrt{u}\sin\left(\frac{\pi}{4}-\frac{\varphi}{2}\right) du}{1+u^2} \le \int_0^{+\infty} \frac{\sqrt{u}\, du}{1+u^2} = \pi\frac{\sqrt{2}}{2}.$$

If $t<0$, then analogously

$$\int_0^{+\infty} \frac{|t|\, r\cos\varphi\, dr}{(t^2-2tr\sin\varphi+r^2)\sqrt{t^2+2tr\sin\varphi+r^2}}$$

$$= \int_0^{+\infty} \frac{u\cos\varphi\, du}{(1+2u\sin\varphi+u^2)\sqrt{1-2u\sin\varphi+u^2}} \le \max\left\{\pi; \int_0^{+\infty} \frac{\sqrt{u}\, du}{1+u^2}\right\} \le \pi.$$

This means by Fubini's theorem that $\int\limits_0^{+\infty} dr \int\limits_0^{+i\infty} \left|\Phi_1(w)K_+(w;re^{i\varphi})\right| dw \le c < +\infty$ and $\int\limits_0^{+\infty} dr \int\limits_0^{+i\infty} \left|\Phi_3(w)K_+(w;re^{i\varphi})\right| dw \le c < +\infty$, $\varphi \in (-\pi/2;\pi/2)$. Also for $t>0$ we have

$$\frac{\pi}{2}\int_0^{+\infty} \left|K_+(t;re^{i\varphi})\right| dr = \int_0^{+\infty} \frac{tx}{|(t+\bar{z})(t-z)(t+z)|}\, dr$$

$$= \int_0^{+\infty} \frac{tr\cos\varphi\, dr}{(t^2+2tr\cos\varphi+r^2)\sqrt{(t^2-2tr\cos\varphi+r^2)}} = \int_0^{+\infty} \frac{u\cos\varphi\, dr}{(u^2+2u\cos\varphi+1)\sqrt{(u^2-2u\cos\varphi+1)}}.$$

Hence

$$\int_0^{+\infty} dr \int_0^{+\infty} \left|\Phi_2(w)K_+(w;re^{i\varphi})\right| dw \le c_1 < +\infty, \quad \varphi \in (-\pi/2+\delta; \pi/2-\delta),$$

therefore (3) is valid. Similar estimations are valid also for $\varphi \in (\pi/2; 3\pi/2)$. □

2 DISCUSSION

Analogously we can prove the following result.

Theorem 2. *Let $f \in E_2[D_\sigma]$ be a solution of equation* (2)*. Then there exists the analytic in $\mathbb{C}_+$ function P_3 such that*

$$\sup\left\{\int_0^{+\infty} \left|P_3(re^{i\varphi})\right| e^{-\sigma r|\sin\varphi|} dr : \varphi \in (-\pi/2;\pi/2)\backslash(-\delta;\delta)\right\} < +\infty$$

for all $\delta \in (0;\pi/2)$ and the angular boundary values on $i\mathbb{R}$ of the function P_3 coincides a.e. with $G(iy)F_3(iy)e^{-\sigma y}$.

In [5, 1] we obtain the inverse, in some sense, result to Theorem 1.

Theorem D. *Assume $f \in E^2[D_\sigma]$. If there exists a function $\widetilde{P}_1$, $\widetilde{P}_1(z)e^{-i\sigma z} \in H^1_\sigma(\mathbb{C}_+)$, such that the angular boundary values of $\widetilde{P}_1$ on $i\mathbb{R}$ coincides a.e. with $G(iy)F(iy)$, then f is a solution of equation* (2)*.*

We do not know, whether the estimation (3) for the case $\delta = 0$ is valid, i.e. $P_1 \in H^1_\sigma(\mathbb{C}_+)$.

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Винницький Б.В., Дільний В.М. *Про оцінки розв'язків одного рівняння типу згортки* // Карпатські математичні публікації. — 2013. — Т.5, №1. — C. 30–35.

Розглядається одне рівняння типу згортки у півсмузі для функцій, що належать класу Гарді-Смірнова. Одержано оцінки розв'язків в термінах аналітичного продовження.

Ключові слова і фрази: вагові простори Гарді, рівняння згортки, зовнішня функція, циклічна функція.

Винницкий Б.В., Дильный В.М. *Об оценках решений одного уравнения типа свертки* // Карпатские математические публикации. — 2013. — Т.5, №1. — C. 30–35.

Рассматривается одно уравнение типа свертки в полуполосе для функций, принадлежащих пространству Харди-Смирнова. Получено оценки решений в терминах аналитического продолжения.

Ключевые слова и фразы: весовые пространства Харди, уравнение свертки, внешняя функция, циклическая функция.