ISSN 2075-9827 **http://www.journals.pu.if.ua/index.php/cmp**

Carpathian Math. Publ. 2013, **5** (1), 129–133 Карпатські матем. публ. 2013, Т.5, №1, С.129–133

doi:10.15330/cmp.5.1.129-133

KHATS' R.V.

ASYMPTOTIC BEHAVIOR OF AVERAGING OF ENTIRE FUNCTIONS OF IMPROVED REGULAR GROWTH

Using the Fourier series method for entire functions, we investigate the asymptotic behavior of averaging of entire functions of improved regular growth.

Key words and phrases: entire function of completely regular growth, entire function of improved regular growth, Fourier series method.

Ivan Franko State Pedagogical University, 24 Franka str., 82100, Drohobych, Ukraine
E-mail: khats@ukr.net

It is well known that an entire function f of order $\rho \in (0, +\infty)$ with the indicator h is of *completely regular growth* in the Levin-Pflüger sense [5, p. 183] if the relation

$$\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^\rho), \quad z \to \infty, \quad \varphi = \arg z \in [0, 2\pi),$$

holds outside an exceptional set $C_0 \subset \mathbb{C}$ of disks of zero linear density. In the theory of entire functions of completely regular growth (see [5, pp. 182–217], [1], [2]) the following theorem is valid.

Theorem A. *([5, p. 194]) If an entire function f of order $\rho \in (0, +\infty)$ with the indicator h is of completely regular growth, then*

$$I_f^r(\varphi) := \int_1^r \frac{dt}{t} \int_1^t \frac{\log|f(ue^{i\varphi})|}{u}\, du = \frac{r^\rho}{\rho^2} h(\varphi) + o(r^\rho), \quad r \to +\infty,$$

holds uniformly in $\varphi \in [0, 2\pi]$.

Similar results for entire functions of ρ-regular growth were obtained by A. Grishin [3] and for meromorphic functions of completely regular growth of finite λ-type [10, p. 75] by A. Kondratyuk [10, p. 112] and Ya. Vasyl'kiv [12] (see also Yu. Lapenko [11]).

In [13, 6], the notion of an entire function of improved regular growth was introduced, and criteria for this regularity were established in terms of the distribution of zeros that are located on a finite number of rays. In [4] this notion was generalized to subharmonic functions. A criterion for the improved regular growth of entire functions of positive order with zeros on a finite system of rays in terms of their Fourier coefficients was established in [7].

An entire function f is called a function of *improved regular growth* (see [13, 6, 7]) if for some $\rho \in (0, +\infty)$ and $\rho_1 \in (0, \rho)$, and a 2π-periodic ρ-trigonometrically convex function

УДК 517.5
2010 *Mathematics Subject Classification:* 30D15.

$h(\varphi) \not\equiv -\infty$ there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that

$$\log|f(z)| = |z|^\rho h(\varphi) + O(|z|^{\rho_1}), \quad U \not\ni z = re^{i\varphi} \to \infty.$$

If f is an entire function of improved regular growth, then it has [13] the order ρ and indicator h.

In the present paper, using the Fourier series method [10] for the logarithm of the modulus of an entire function we obtain an analog of Theorem A for the class of entire functions of improved regular growth. Our principal result is the following theorem (see also [9]), which improves the results of papers [8, 14].

Theorem 1. *If an entire function f of order $\rho \in (0, +\infty)$ is of improved regular growth, then for some $\rho_2 \in (0, \rho)$*

$$I_f^r(\varphi) = \frac{r^\rho}{\rho^2} h(\varphi) + O(r^{\rho_2}), \quad r \to +\infty,$$

holds uniformly in $\varphi \in [0, 2\pi]$.

To prove Theorem 1, we need some preliminaries. Let f be an entire function with $f(0) = 1$, let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of its zeros, let Q_k be the coefficient of z^k in the exponential factor in the Hadamard-Borel representation [5, p. 38] of an entire function f of order $\rho \in (0, +\infty)$, and let

$$c_k(r, \log|f|) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} \log|f(re^{i\varphi})|\, d\varphi, \quad k \in \mathbb{Z},$$

$$c_k(r, I_f^r) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} I_f^r(\varphi)\, d\varphi, \quad k \in \mathbb{Z}, \quad r > 0.$$

Put ([10, p. 104], [12])

$$n_k(r, f) := \sum_{|\lambda_n| \le r} e^{-ik \arg \lambda_n}, \quad N_k(r, f) := \int_0^r \frac{n_k(t, f)}{t}\, dt, \quad k \in \mathbb{Z}.$$

Then (see [10, p. 107], [12])

$$c_k(r, I_f^r) = \int_0^r \frac{dt}{t} \int_0^t \frac{c_k(u, \log|f|)}{u}\, du, \quad k \in \mathbb{Z}, \tag{1}$$

and

$$N_k(r, f) = c_k(r, \log|f|) - k^2 c_k(r, I_f^r), \quad k \in \mathbb{Z}, \quad r > 0. \tag{2}$$

Lemma 1. *([7]) Let f be an entire function of improved regular growth of order $\rho \in (0, +\infty)$. Then for some $\rho_3 \in (0, \rho)$ and each $k \in \mathbb{Z}$*

$$c_k(r, \log|f|) = c_k r^\rho + O(r^{\rho_3}), \quad r \to +\infty,$$

$$N_k(r, f) = c_k(1 - k^2/\rho^2) r^\rho + O(r^{\rho_3}), \quad r \to +\infty, \tag{3}$$

where

$$c_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} h(\varphi)\, d\varphi.$$

By $M_1, M_2, \ldots$ we denote some positive constants.

Lemma 2. *If an entire function f of order $\rho \in (0, +\infty)$ is of improved regular growth, then*

$$|c_k(r, I_f^r)| \le \frac{M_1}{k^2} r^\rho + \frac{M_2}{k^2} r^{\rho_3}, \quad k \in \mathbb{Z} \setminus \{0\}, \tag{4}$$

for some $\rho_3 \in (0, \rho)$ and all $r > 0$.

Proof. Let ρ be noninteger and $p = [\rho]$. Then (see [10, pp. 10, 120, 124], [12])

$$c_k(r, \log|f|) = \overline{c_{-k}(r, \log|f|)}, \quad k \le -1, \quad c_0(r, \log|f|) = N_0(r, f), \tag{5}$$

$$c_k(r, \log|f|) = \frac{1}{2} Q_k r^k + \frac{k}{2} \int_0^r \left(\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k \right) \frac{N_k(t, f)}{t}\, dt + N_k(r, f), \quad 1 \le k \le p, \tag{6}$$

and

$$c_k(r, \log|f|) = -\frac{k}{2} \left(r^{-k} \int_0^r t^{k-1} N_k(t, f)\, dt + r^k \int_r^{+\infty} t^{-k-1} N_k(t, f)\, dt \right) + N_k(r, f), \tag{7}$$

for $k \ge p + 1$. Since $|N_k(r, f)| \le N_0(r, f)$ for each $k \in \mathbb{Z}$, then using (3) and (6), we obtain

$$|c_k(r, \log|f|)| \le \frac{1}{2}|Q_k| r^k + \frac{k}{2} \int_0^r \left(\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k \right) \frac{N_0(t, f)}{t}\, dt + N_0(r, f)$$

$$\le \frac{1}{2}|Q_k| r^k + \frac{k}{2} \int_0^r \left(\left(\frac{r}{t}\right)^k - \left(\frac{t}{r}\right)^k \right) (c_0 t^{\rho-1} + O(t^{\rho_3 - 1}))\, dt + c_0 r^{\rho-1} + O(r^{\rho_3 - 1})$$

$$= \frac{c_0 \rho^2}{\rho^2 - k^2} r^\rho + \frac{O(r^{\rho_3})}{\rho_3^2 - k^2}, \quad 1 \le k \le p < \rho_3, \tag{8}$$

as $r \to +\infty$. Similarly, using formulas (3) and (7), we get

$$|c_k(r, \log|f|)| \le \frac{2k^2 - \rho^2}{k^2 - \rho^2} c_0 r^\rho + \frac{2k^2 - \rho_3^2}{k^2 - \rho_3^2} O(r^{\rho_3}), \quad k \ge p + 1 > \rho, \tag{9}$$

as $r \to +\infty$. Therefore, from (2), (3) and (8) it follows

$$|c_k(r, I_f^r)| \le \frac{|c_k(r, \log|f|)| + N_0(r, f)}{k^2} \le \frac{2\rho^2 - k^2}{k^2(\rho^2 - k^2)} c_0 r^\rho + \frac{\rho_3^2 - k^2 + 1}{k^2(\rho_3^2 - k^2)} O(r^{\rho_3})$$

$$\le \frac{M_3}{k^2} r^\rho + \frac{M_4}{k^2} r^{\rho_3}, \quad r > 0, \quad 1 \le k \le p < \rho_3. \tag{10}$$

Similarly, from (2), (3) and (9), we get

$$|c_k(r, I_f^r)| \le \frac{3k^2 - 2\rho^2}{k^2(k^2 - \rho^2)} c_0 r^\rho + \frac{3k^2 - 2\rho_3^2}{k^2(k^2 - \rho_3^2)} O(r^{\rho_3})$$

$$\le \frac{M_5}{k^2} r^\rho + \frac{M_6}{k^2} r^{\rho_3}, \quad r > 0, \quad k \ge p + 1 > \rho. \tag{11}$$

Thus, relations (1), (5), (10) and (11) imply (4). The case $\rho \in \mathbb{N}$ is considered analogously. Lemma 2 is proved. □

Proof of Theorem 1. Using (1) and (2), we have

$$c_k(r, I_f^r) = \frac{c_k(r, \log|f|) - N_k(r, f)}{k^2}, \quad k \in \mathbb{Z} \setminus \{0\},$$

and

$$c_0(r, I_f^r) = \int\limits_0^r \frac{dt}{t} \int\limits_0^t \frac{c_0(u, \log|f|)}{u}\, du.$$

Moreover, according to Lemma 1, we get

$$c_0(r, I_f^r) = c_0 \frac{r^\rho}{\rho^2} + O(r^{\rho_3}),$$

$$c_k(r, I_f^r) = c_k \frac{r^\rho}{\rho^2} + \frac{O(r^{\rho_3})}{k^2}, \quad k \in \mathbb{Z} \setminus \{0\},$$

as $r \to +\infty$ for some $\rho_3 \in (0, \rho)$. Therefore, taking into account Lemma 2, we obtain

$$I_f^r(\varphi) := \sum_{k\in\mathbb{Z}} c_k(r, I_f^r) e^{ik\varphi} = \sum_{k\in\mathbb{Z}\setminus\{0\}} c_k(r, I_f^r) e^{ik\varphi} + c_0(r, I_f^r)$$

$$= \frac{r^\rho}{\rho^2} \sum_{k\in\mathbb{Z}} c_k e^{ik\varphi} + O(r^{\rho_3}) = \frac{r^\rho}{\rho^2} h(\varphi) + O(r^{\rho_3}),$$

as $r \to +\infty$ uniformly in $\varphi \in [0, 2\pi]$. □

REFERENCES

[1] Gol'dberg A.A. *B.Ya. Levin, the founder of the theory of entire functions of completely regular growth*. Mat. Fiz., Anal., Geom. 1994, **1** (2), 186–192. (in Russian)

[2] Gol'dberg A.A., Levin B.Ya., Ostrovskii I.V. *Entire and meromorphic functions*. VINITI 1991, **85**, 5–186. (in Russian)

[3] Grishin A.F. Subharmonic functions of finite order, Doctor-Degree Thesis (Physics and Mathematics). Kharkov, 1992. (in Russian)

[4] Hirnyk M.O. *Subharmonic functions of improved regular growth*. Dopov. Nats. Akad. Nauk Ukr. 2009, **4**, 13–18. (in Ukrainian) doi: 10.1007/s11253-012-0624-2

[5] Levin B.Ya. Distribution of zeros of entire functions, Gostekhizdat, Moscow, 1956. (in Russian)

[6] Khats' R.V. *On entire functions of improved regular growth of integer order with zeros on a finite system of rays*. Mat. Stud. 2006, **26** (1), 17–24.

[7] Khats' R.V. *Regularity of growth of Fourier coefficients of entire functions of improved regular growth*. Ukr. Mat. Zh. 2011, **63** (12), 1717–1723 (in Ukrainian); English translation: Ukr. Math. J. 2012, **63** (12), 1953–1960. doi: 10.1007/s11253-012-0624-2

[8] Khats' R.V. *Averaging of entire functions of improved regular growth with zeros on a finite system of rays*. Visn. Nats. Univ. L'viv. Politekh., Fiz.-Mat. Nauky 2011, **718** (718), 10–14.

[9] Khats' R.V. Asymptotic behavior of averaging of entire functions of improved regular growth. In: Proc. of Intern. Conf. dedicated to 120-th anniv. of Stefan Banach, Lviv, Ukraine, September 17–21 2012, 136–137.

[10] Kondratyuk A.A. Fourier series and meromorphic functions, Vyshcha shkola, Lviv, 1988. (in Russian)

[11] Lapenko Yu.P. On entire and meromorphic functions of completely regular growth and their defects, Candidate-Degree Thesis (Physics and Mathematics), Rostov-on-Don, 1981. (in Russian)

[12] Vasyl'kiv Ya.V. *Asymptotic behavior of logarithmic derivatives and logarithms of meromorphic functions of completely regular growth in the metric of* $L^p[0, 2\pi]$. *I*. Mat. Stud. 1999, **12** (1), 37–58. (in Ukrainian)

[13] Vynnyts'kyi B.V., Khats' R.V. *On growth regularity of entire function of noninteger order with zeros on a finite system of rays*. Mat. Stud. 2005 **24** (1), 31–38. (in Ukrainian)

[14] Vynnyts'kyi B.V., Khats' R.V. *On asymptotic properties of entire functions, similar to the entire functions of completely regular growth*. Visn. Nats. Univ. L'viv. Politekh., Fiz.-Mat. Nauky 2011, **718** (718), 5–9.

Received 14.11.2012

Хаць Р.В. *Асимптотична поведінка усереднення цілих функцій покращеного регулярного зростання* // Карпатські математичні публікації. — 2013. — Т.5, №1. — С. 129–133.

За допомогою методу рядів Фур'є для цілих функцій досліджено асимптотичну поведінку усереднення цілих функцій покращеного регулярного зростання.

Ключові слова і фрази: цілі функції цілком регулярного зростання, цілі функції покращеного регулярного зростання, метод рядів Фур'є.

Хаць Р.В. *Асимптотическое поведение усреднения целых функций улучшенного регулярного роста* // Карпатские математические публикации. — 2013. — Т.5, №1. — С. 129–133.

С помощью метода рядов Фурье для целых функций исследовано асимптотическое поведение усреднения целых функций улучшенного регулярного роста.

Ключевые слова и фразы: целые функции вполне регулярного роста, целые функции улучшенного регулярного роста, метод рядов Фурье.