ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2020, **12** (1), 46–54
doi:10.15330/cmp.12.1.46-54

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2020, Т.12, №1, С.46–54

KHATS' R.V.

SUFFICIENT CONDITIONS FOR THE IMPROVED REGULAR GROWTH OF ENTIRE FUNCTIONS IN TERMS OF THEIR AVERAGING

Let f be an entire function of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$ and $h(\varphi)$ be its indicator. In 2011, the author of the article has been proved that if f is of improved regular growth (an entire function f is called a function of improved regular growth if for some $\rho \in (0,+\infty)$, $\rho_1 \in (0,\rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that $\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$), then for some $\rho_3 \in (0,\rho)$ the relation

$$\int_1^r \frac{\log|f(te^{i\varphi})|}{t}\,dt = \frac{r^\rho}{\rho}h(\varphi) + o(r^{\rho_3}), \quad r \to +\infty,$$

holds uniformly in $\varphi \in [0,2\pi]$. In the present paper, using the Fourier coefficients method, we establish the converse statement, that is, if for some $\rho_3 \in (0,\rho)$ the last asymptotic relation holds uniformly in $\varphi \in [0,2\pi]$, then f is a function of improved regular growth. It complements similar results on functions of completely regular growth due to B. Levin, A. Grishin, A. Kondratyuk, Ya. Vasyl'kiv and Yu. Lapenko.

Key words and phrases: entire function of completely regular growth, entire function of improved regular growth, indicator, Fourier coefficients, averaging, finite system of rays.

Drohobych Ivan Franko State Pedagogical University, 24 Franko Str., 82100, Drohobych, Ukraine
E-mail: khats@ukr.net

1 INTRODUCTION

It is well known ([13, p. 24]) that an entire function f of order $\rho \in (0,+\infty)$ may be represented in the form

$$f(z) = z^\lambda e^{Q(z)} \prod_{n=1}^{\infty} E\left(\frac{z}{\lambda_n}, p\right),$$

where λ_n are all nonzero roots of the function $f(z)$, $\lambda \in \mathbb{Z}_+$ is the multiplicity of the root at the origin, $Q(z) = \sum_{k=1}^{\nu} Q_k z^k$ is a polynomial of degree $\nu \le \rho$, $p \le \rho$ is the smallest integer for which $\sum_{n=1}^{\infty} |\lambda_n|^{-p-1} < +\infty$ and $E(w,p) = (1-w)\exp(w + w^2/2 + \dots + w^p/p)$ is the Weierstrass primary factor.

Let f be an entire function of order $\rho \in (0,+\infty)$. The function

$$h(\varphi) = h_f(\varphi) = \limsup_{r\to\infty} \frac{\log|f(re^{i\varphi})|}{r^\rho}, \quad \varphi \in [0,2\pi],$$

УДК 517.5
2010 *Mathematics Subject Classification:* 30D15, 30D20, 30D30.

is called the *indicator* of f ([13, p. 51]). The indicator is a continuous 2π-periodic ρ-trigonometrically convex function (see [13, pp. 53–54]). A set $C \subset \mathbb{C}$ is called a C^0-*set* ([13, p. 90]) if it can be covered by a system of disks $\{z : |z - a_k| < s_k\}$, $k \in \mathbb{N}$, satisfying $\sum_{|a_k|\le r} s_k = o(r)$ as $r \to +\infty$.

An entire function f of order $\rho \in (0, +\infty)$ with the indicator $h(\varphi)$ is said to be of *completely regular growth* in the sense of Levin and Pfluger ([13, p. 139]) if there exists a C^0-set such that $\log |f(re^{i\varphi})| = r^\rho h(\varphi) + o(r^\rho)$, $C^0 \not\ni re^{i\varphi} \to \infty$, uniformly in $\varphi \in [0, 2\pi)$. In the theory of entire functions of completely regular growth (see [13, pp. 139–167]) the following theorem is valid.

Theorem A ([13, p. 150]). *In order that an entire function f of order $\rho \in (0, +\infty)$ with the indicator $h(\varphi)$ be of completely regular growth, it is necessary and sufficient that uniformly in $\varphi \in [0, 2\pi]$ one of the following relations hold:*

$$J_f^r(\varphi) := \int_1^r \frac{\log |f(te^{i\varphi})|}{t}\, dt = \frac{r^\rho}{\rho} h(\varphi) + o(r^\rho), \quad r \to +\infty,$$
$$I_f^r(\varphi) := \int_1^r J_f^t(\varphi) \frac{dt}{t} = \frac{r^\rho}{\rho^2} h(\varphi) + o(r^\rho), \quad r \to +\infty.$$

Similar results for entire functions of ρ-regular growth were obtained by A. Grishin [2] and for meromorphic functions of completely regular growth of finite λ-type ([11, p. 75]) by A. Kondratyuk [11, p. 112] and Ya. Vasyl'kiv [14] (see also Yu. Lapenko [12]).

In [5, 16] the notion of entire function of improved regular growth was introduced, and a criterion for this regularity was obtained in terms of the distribution of zeros under the condition that they are located on a finite system of rays.

An entire function f is called a function of *improved regular growth* ([5, 16]) if for some $\rho \in (0, +\infty)$ and $\rho_1 \in (0, \rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi) \not\equiv -\infty$ there exists a set $U \subset \mathbb{C}$ contained in the union of disks with finite sum of radii and such that $\log |f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$. If an entire function f is of improved regular growth, then it has the order ρ and indicator $h(\varphi)$ ([16]). In the case when zeros of an entire function f of improved regular growth are situated on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \ldots, m\}$, $0 \le \psi_1 < \psi_2 < \ldots < \psi_m < 2\pi$, the indicator h has the form (see [16])

$$h(\varphi) = \sum_{j=1}^m h_j(\varphi), \quad \rho \in (0, +\infty) \setminus \mathbb{N}, \tag{1}$$

where $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j, \psi_j + 2\pi)$

$$h_j(\varphi) = \frac{\pi \Delta_j}{\sin \pi\rho} \cos \rho(\varphi - \psi_j - \pi), \quad \Delta_j \in [0, +\infty).$$

In the case $\rho \in \mathbb{N}$, the indicator h is defined by the formula ([5])

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \sum_{j=1}^m h_j(\varphi), & p = \rho, \\ Q_\rho \cos \rho\varphi, & p = \rho - 1, \end{cases} \tag{2}$$

where $\delta_f \in \mathbb{C}$, $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$ and $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j, \psi_j + 2\pi)$

$$h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j) \sin \rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho} \cos \rho(\varphi - \psi_j).$$

At present, many different conditions are known that are necessary and sufficient for the improved regular growth of entire functions (see [1,3–10,15–17]). In view of this, it is natural to establish an analog of Theorem A for the class of entire functions of improved regular growth. In this direction, the following results were obtained in [6,8].

Theorem B ([8]). *If an entire function f of order $\rho \in (0,+\infty)$ is of improved regular growth, then for some $\rho_2 \in (0,\rho)$, one has*

$$I_f^r(\varphi) = \frac{r^\rho}{\rho^2} h(\varphi) + O(r^{\rho_2}), \quad r \to +\infty,$$

uniformly in $\varphi \in [0, 2\pi]$.

Theorem C ([6]). *If an entire function f of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is of improved regular growth, then for some $\rho_3 \in (0,\rho)$ the relation*

$$J_f^r(\varphi) = \frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_3}), \quad r \to +\infty, \tag{3}$$

holds uniformly in $\varphi \in [0, 2\pi]$, where $h(\varphi)$ be defined by (1) *and* (2).

However, the problem of finding the converse of Theorems B and C remained open. The aim of the present paper is to prove the converse of Theorem C. Our principal result is the following theorem.

Theorem 1. *Let f be an entire function of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$ and $h(\varphi)$ be its indicator. If for some $\rho_3 \in (0,\rho)$ the relation* (3) *holds uniformly in $\varphi \in [0,2\pi]$ with $h(\varphi)$ defined by* (1) *and* (2)*, then f is a function of improved regular growth.*

2 PRELIMINARIES

Let f be an entire function with $f(0) = 1$ and $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of its zeros. For $k \in \mathbb{Z}$ and $r > 0$, we set

$$n_k(r,f) := \sum_{|\lambda_n|\le r} e^{-ik\arg\lambda_n}, \qquad N_k(r,f) := \int_0^r \frac{n_k(t,f)}{t}\,dt,$$

$$N_k^*(r,f) := \int_0^r \frac{N_k(t,f)}{t}\,dt, \qquad n(r,\psi;f) := \sum_{|\lambda_n|\le r,\ \ \arg\lambda_n=\psi} 1,$$

$$N(r,\psi;f) := \int_0^r \frac{n(t,\psi;f)}{t}\,dt, \qquad N^*(r,\psi;f) := \int_0^r \frac{N(t,\psi;f)}{t}\,dt,$$

$$c_k(r,\log|f|) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi}\log|f(re^{i\varphi})|\,d\varphi, \qquad c_k(r,J_f^r) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi} J_f^r(\varphi)\,d\varphi.$$

In the proof of Theorem 1, we use the following auxiliary statements.

Lemma 1 ([5,16]). *An entire function f of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if for some $\rho_4 \in (0,\rho)$ and each $j \in \{1,\dots,m\}$*

$$n(t,\psi_j;f) = \Delta_j t^\rho + o(t^{\rho_4}), \quad t \to +\infty, \quad \Delta_j \in [0,+\infty), \tag{4}$$

and, in addition, for $\rho \in \mathbb{N}$ and some $\rho_5 \in (0, \rho)$ and $\delta_f \in \mathbb{C}$, one has

$$\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} = \delta_f + o(r^{\rho_5-\rho}), \quad r \to +\infty. \tag{5}$$

In this case, the indicator $h(\varphi)$ be defined by formulas (1) *and* (2).

We remark that, for $\rho = p+1$ equality (4) holds with $\Delta_j = 0$, because $\sum_{n\in\mathbb{N}} |\lambda_n|^{-p-1} < +\infty$ (see [5, p. 19]).

Lemma 2. *If an entire function f of order $\rho \in (0, +\infty)$ satisfies the conditions of Theorem 1, then for some $\rho_3 \in (0, \rho)$ and each $k \in \mathbb{Z}$, one has*

$$c_k(r, J_f^r) = c_k \frac{r^\rho}{\rho} + o(r^{\rho_3}), \quad r \to +\infty, \tag{6}$$

$$N_k^*(r, f) = c_k \left(1 - \frac{k^2}{\rho^2}\right) \frac{r^\rho}{\rho} + o(r^{\rho_3}), \quad r \to +\infty, \tag{7}$$

where

$$c_k := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} h(\varphi)\, d\varphi = \frac{\rho}{\rho^2 - k^2} \sum_{j=1}^{m} \Delta_j e^{-ik\psi_j}, \quad \Delta_j \in [0, +\infty), \tag{8}$$

if $\rho \in (0, +\infty) \setminus \mathbb{N}$, and

$$c_k = \begin{cases} \dfrac{\rho}{\rho^2 - k^2} \displaystyle\sum_{j=1}^{m} \Delta_j e^{-ik\psi_j}, & |k| \ne \rho = p, \\ \dfrac{\tau_f e^{i\theta_f}}{2} - \dfrac{1}{4\rho} \displaystyle\sum_{j=1}^{m} \Delta_j e^{-i\rho\psi_j}, & k = \rho = p, \\ 0, & |k| \ne \rho = p+1, \\ \dfrac{Q_\rho}{2}, & k = \rho = p+1, \end{cases} \tag{9}$$

if $\rho \in \mathbb{N}$.

Proof. Under the conditions of the lemma, by using (3), for some $\rho_3 \in (0, \rho)$ and each $k \in \mathbb{Z}$, we get

$$c_k(r, J_f^r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} \left(\frac{r^\rho}{\rho} h(\varphi) + o(r^{\rho_3})\right) d\varphi = c_k \frac{r^\rho}{\rho} + o(r^{\rho_3}), \quad r \to +\infty,$$

where c_k is defined by formulas (8) and (9) (see [6, 7, 9, 10]). Thus, relation (6) holds. Let us prove relation (7). Using relations (see [14, pp. 39, 43], [11, pp. 107, 112], [6, p. 13])

$$c_k(r, J_f^r) = \int_0^r \frac{c_k(t, \log|f|)}{t}\, dt,$$

$$N_k(r, f) = c_k(r, \log|f|) - k^2 \int_0^r \frac{dt}{t} \int_0^t \frac{c_k(u, \log|f|)}{u}\, du, \quad k \in \mathbb{Z}, \quad r > 0,$$

we obtain

$$N_k^*(r, f) = \int_0^r \frac{N_k(t, f)}{t}\, dt = c_k(r, J_f^r) - k^2 \int_0^r \frac{dt}{t} \int_0^t \frac{c_k(u, J_f^u)}{u}\, du, \quad k \in \mathbb{Z}, \quad r > 0.$$

Then, using (6) and passing to the limit as $r \to +\infty$, we get

$$N_k^*(r, f) = c_k \frac{r^\rho}{\rho} + o(r^{\rho_3}) - k^2 \int_0^r \frac{dt}{t} \int_0^t \left(c_k \frac{u^{\rho-1}}{\rho} + o(u^{\rho_3-1})\right) du = c_k \left(1 - \frac{k^2}{\rho^2}\right) \frac{r^\rho}{\rho} + o(r^{\rho_3}).$$

Lemma 2 is proved. □

Lemma 3. *Let f be an entire function of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$. In order that the equality*

$$N^*(r, \psi_j; f) = \frac{\Delta_j}{\rho^2} r^\rho + o(r^{\rho_3}), \quad r \to +\infty, \quad \Delta_j \in [0, +\infty), \tag{10}$$

holds for some $\rho_3 \in (0, \rho)$ and each $j \in \{1, \dots, m\}$, it is necessary and sufficient that, for some $\rho_3 \in (0, \rho)$ and $k_0 \in \mathbb{Z}$ and each $k \in \{k_0, k_0 + 1, \dots, k_0 + m - 1\}$, relation (7) *with c_k, defined by* (8) *and* (9) *be true. Besides, we have $\sum_{j=1}^{m} \Delta_j e^{-i\rho\psi_j} = 0$, if $\rho \in \mathbb{N}$.*

Proof. Necessity. Since (see [11, p. 127])

$$n_k(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} n(r, \psi_j; f), \quad k \in \mathbb{Z},$$

then

$$N_k(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} \int_0^r \frac{n(t, \psi_j; f)}{t} \, dt = \sum_{j=1}^{m} e^{-ik\psi_j} N(r, \psi_j; f),$$

$$N_k^*(r, f) = \sum_{j=1}^{m} e^{-ik\psi_j} N^*(r, \psi_j; f), \quad k \in \mathbb{Z}.$$

Using (10), for some $\rho_3 \in (0, \rho)$ and each $k \in \mathbb{Z}$ we obtain relation (7) with c_k, defined by (8) and (9). In this case, $\sum_{j=1}^{m} \Delta_j e^{-i\rho\psi_j} = 0$, if $\rho \in \mathbb{N}$.

Let us prove the *sufficiency*. Without loss of generality, we can assume that $k_0 = 0$. Then, by analogy with [7, p. 1957] (see also [10, p. 118], [11, p. 127]), for $k \in \{0, 1, \dots, m-1\}$ we get

$$\begin{aligned}
N_0^*(r, f) &= N^*(r, \psi_1; f) + N^*(r, \psi_2; f) + \dots + N^*(r, \psi_m; f), \\
N_1^*(r, f) &= e^{-i\psi_1} N^*(r, \psi_1; f) + e^{-i\psi_2} N^*(r, \psi_2; f) + \dots + e^{-i\psi_m} N^*(r, \psi_m; f), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
N_{m-1}^*(r, f) &= e^{-i(m-1)\psi_1} N^*(r, \psi_1; f) + e^{-i(m-1)\psi_2} N^*(r, \psi_2; f) + \dots + e^{-i(m-1)\psi_m} N^*(r, \psi_m; f).
\end{aligned}$$

This is a system of linear equations for the unknowns $N^*(r, \psi_j; f)$, $j \in \{1, \dots, m\}$. Its determinant is the nonzero Vandermonde determinant

$$D = \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{-i\psi_1} & e^{-i\psi_2} & \dots & e^{-i\psi_m} \\ \dots & \dots & \dots & \dots \\ e^{-i(m-1)\psi_1} & e^{-i(m-1)\psi_2} & \dots & e^{-i(m-1)\psi_m} \end{vmatrix} \ne 0.$$

Therefore, the functions $N^*(r, \psi_j; f)$, $j \in \{1, \dots, m\}$, can be represented as linear combinations of the functions $N_k^*(r, f)$, $k \in \{0, 1, \dots, m-1\}$. Using (7), we obtain relation (10), where by the Cramer's rule $\Delta_j = \rho^2 D_j / D$, $j \in \{1, \dots, m\}$, and D_j is the determinant formed from the determinant D by replacing the j-column with the corresponding column $(\widetilde{c}_0, \widetilde{c}_1, \dots, \widetilde{c}_{m-1})$, $\widetilde{c}_k := \frac{c_k}{\rho}\left(1 - \frac{k^2}{\rho^2}\right)$, $k \in \{0, 1, \dots, m-1\}$. Lemma 3 is proved. ∎

Remark 1. *Let* $\rho \in (0,+\infty) \setminus \mathbb{N}$, $\mu_n = \big(n + \frac{n}{\log n}\big)^{1/\rho}$, $\{\lambda_n : n \in \mathbb{N}\setminus\{1\}\} := \bigcup_{j=1}^{m}\{\mu_n e^{i\frac{2\pi(j-1)}{m}} : n \in \mathbb{N}\setminus\{1\}\}$, *$m \in \mathbb{N} \setminus \{1\}$ and ([7, p. 1958])*

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\lambda_n}\right)\exp\left(\sum_{\zeta=1}^{p}\frac{1}{\zeta}\left(\frac{z}{\lambda_n}\right)^{\zeta}\right), \quad p = [\rho].$$

Then for each $j \in \{1,\dots,m\}$, we obtain (see [7, p. 1959])

$$N^*\left(r, \frac{2\pi(j-1)}{m}; f\right) = \frac{r^\rho}{\rho^2} + O\left(\frac{r^\rho}{\log r}\right), \quad r \to +\infty.$$

Therefore, relation (10) *is not true for any $\rho_3 \in (0,\rho)$. Furthermore,*

$$N_0^*(r,f) = \sum_{j=1}^{m} N^*\left(r, \frac{2\pi(j-1)}{m}; f\right) = \frac{m}{\rho^2}r^\rho + O\left(\frac{r^\rho}{\log r}\right), \quad r \to +\infty.$$

Thus, relation (7) *is not true for $k = 0$. Moreover, since*

$$\sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}} = \frac{1 - e^{-2\pi k i}}{1 - e^{-i\frac{2\pi k}{m}}} = 0, \quad k \in \{1,\dots,m-1\},$$

we conclude that

$$n_k(r,f) = \sum_{\mu_n \le r}\sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}} = 0,$$

for each $k \in \{1,\dots,m-1\}$ and all $r > 0$. Therefore, relation (7) *holds for any $\rho_3 \in (0,\rho)$ and each $k \in \{1,\dots,m-1\}$.*

Lemma 4. *Let f be an entire function of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$. In order that the equality* (4) *holds for some $\rho_4 \in (0,\rho)$ and each $j \in \{1,\dots,m\}$, it is necessary and sufficient that for some $\rho_3 \in (0,\rho)$ and each $j \in \{1,\dots,m\}$ relation* (10) *be true.*

Proof. Indeed, using Lemma 3 from [15, p. 143] twice, we obtain the required statement. ☐

3 PROOF OF THEOREM 1

Let the conditions of Theorem 1 be satisfied. Then, by Lemmas 2–4, the relations (6), (7) and (4) hold. Let us prove the equality (5) for $\rho \in \mathbb{N}$. Since (see the proof of Lemmas 2 and 3)

$$c_k(r, \log|f|) = N_k(r,f) + k^2\int_0^r \frac{c_k(t, J_f^t)}{t}\,dt, \quad N_k(r,f) = \sum_{j=1}^{m} e^{-ik\psi_j}N(r,\psi_j;f), \quad k \in \mathbb{Z},$$

and ([4, p. 101])

$$c_\rho(r, \log|f|) = \frac{1}{2}Q_\rho r^\rho + \frac{1}{2\rho}\sum_{0<|\lambda_n|\le r}\left(\left(\frac{r}{\lambda_n}\right)^\rho - \left(\frac{\overline{\lambda}_n}{r}\right)^\rho\right), \quad k = \rho = p \in \mathbb{N},$$

then, using formulas (4), (6), (7), (9) and the identity $\sum\limits_{j=1}^{m} \Delta_j e^{-i\rho\psi_j} = 0$, $\rho = p \in \mathbb{N}$, for some $\rho_5 \in (0,\rho)$ we get

$$\begin{aligned}
\sum_{0<|\lambda_n|\le r} \lambda_n^{-\rho} =& 2\rho r^{-\rho} c_\rho(r,\log|f|) - \rho Q_\rho + r^{-\rho} \sum_{0<|\lambda_n|\le r} \left(\frac{\overline{\lambda}_n}{r}\right)^{\rho} \\
=& 2\rho r^{-\rho}\left(N_\rho(r,f) + \rho^2 \int_0^r \frac{c_\rho(t, J_f^t)}{t}\,dt\right) - \rho Q_\rho + r^{-2\rho} \sum_{j=1}^{m} e^{-i\rho\psi_j} \int_0^r t^\rho\, dn(t,\psi_j;f) \\
=& 2\rho r^{-\rho}\left(\sum_{j=1}^{m} e^{-i\rho\psi_j} \int_0^r \frac{n(t,\psi_j;f)}{t}\,dt + \rho^2 \int_0^r \frac{c_\rho(t,J_f^t)}{t}\,dt\right) - \rho Q_\rho \\
&+ r^{-2\rho} \sum_{j=1}^{m} e^{-i\rho\psi_j}\left(r^\rho n(r,\psi_j;f) - \rho \int_0^r t^{\rho-1} n(t,\psi_j;f)\,dt\right) \\
=& 2\rho r^{-\rho}\left(\sum_{j=1}^{m} e^{-i\rho\psi_j} \int_0^r (\Delta_j t^{\rho-1} + o(t^{\rho_4-1}))\,dt + \rho^2 \int_0^r \left(\frac{c_\rho}{\rho} t^{\rho-1} + o(t^{\rho_3-1})\right) dt\right) \\
&- \rho Q_\rho + r^{-2\rho} \sum_{j=1}^{m} e^{-i\rho\psi_j}\left(\Delta_j r^{2\rho} + o(r^{\rho_4+\rho}) - \rho \int_0^r (\Delta_j t^{2\rho-1} + o(t^{\rho_4+\rho-1}))\,dt\right) \\
=& 2\rho r^{-\rho}\left(\frac{r^\rho}{\rho} \sum_{j=1}^{m} \Delta_j e^{-i\rho\psi_j} + c_\rho r^\rho + o(r^{\rho_4}) + o(r^{\rho_3})\right) - \rho Q_\rho \\
&+ r^{-2\rho} \sum_{j=1}^{m} e^{-i\rho\psi_j}\left(\frac{\Delta_j}{2} r^{2\rho} + o(r^{\rho_4+\rho})\right) \\
=& \rho(\tau_f e^{i\theta_f} - Q_\rho) + o(r^{\rho_4-\rho}) + o(r^{\rho_3-\rho}) = \delta_f + o(r^{\rho_5-\rho}), \quad r \to +\infty.
\end{aligned}$$

Hence, equality (5) holds for $\rho = p$ with $\delta_f = \rho(\tau_f e^{i\theta_f} - Q_\rho)$. In the case $\rho = p+1$, condition (5) follows from (4) (see [5, p. 23, Remark 2]). Thus, according to Lemma 1, the entire function f is a function of improved regular growth. This completes the proof of Theorem 1.

Combining Theorem 1 with Theorem C, we obtain the following theorem.

Theorem 2. *In order that an entire function f of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, be of improved regular growth with the indicator $h(\varphi)$ defined by* (1) *and* (2), *it is necessary and sufficient that for some $\rho_3 \in (0,\rho)$ the relation* (3) *holds uniformly in $\varphi \in [0,2\pi]$.*

Remark 2. *For each $m \in \mathbb{N} \setminus \{1;2\}$ there exists an entire function f of order $\rho \in (0,+\infty) \setminus \mathbb{N}$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $\psi_j := \frac{2\pi(j-1)}{m}$, $j \in \{1,\dots,m\}$, such that uniformly in $\varphi \in [0,2\pi]$ the relation* (3) *is not true for any $\rho_3 \in (0,\rho)$ and f is not a function of improved regular growth.*

Indeed, let f be an entire function of order $\rho \in (0,+\infty) \setminus \mathbb{N}$, defined as in Remark 1. Then (see [7, p. 1959])

$$n\left(t, \frac{2\pi(j-1)}{m}; f\right) = t^\rho - \frac{t^\rho}{\rho \log t} + o\left(\frac{t^\rho}{\log t}\right), \quad t \to +\infty,$$

for each $j \in \{1,\dots,m\}$. Thus, relation (4) *is not true for any $\rho_4 \in (0,\rho)$, and, according to Lemma 1, the entire function f is not a function of improved regular growth. Further, for each*

$j \in \{1, \ldots, m\}$, we obtain ([7, p. 1959])

$$c_0(r, \log|f|) = \sum_{j=1}^{m} N\left(r, \frac{2\pi(j-1)}{m}; f\right) = \frac{m}{\rho} r^{\rho} + O\left(\frac{r^{\rho}}{\log r}\right), \quad r \to +\infty.$$

Furthermore, (see [6, p. 11], [7, p. 1959])

$$c_k(r, \log|f|) = \overline{c_{-k}(r, \log|f|)}, \quad k \le -1,$$
$$c_k(r, \log|f|) = \frac{1}{2k} \sum_{\mu_n \le r} \left[\left(\frac{r}{\mu_n}\right)^k - \left(\frac{\mu_n}{r}\right)^k\right] \sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}}, \quad 1 \le k \le p,$$

and

$$c_k(r, \log|f|) = -\frac{1}{2k}\left\{\sum_{\mu_n > r}\left(\frac{r}{\mu_n}\right)^k + \sum_{\mu_n \le r}\left(\frac{\mu_n}{r}\right)^k\right\} \sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}}, \quad k \ge p+1,$$

where (see Remark 1)

$$\sum_{j=1}^{m} e^{-ik\frac{2\pi(j-1)}{m}} = \begin{cases} 0, & k \in \mathbb{N}, \quad k \ne m, \\ m, & k = m. \end{cases}$$

In view of this, since

$$c_k(r, J_f^r) = \int_0^r \frac{c_k(t, \log|f|)}{t}\, dt, \quad k \in \mathbb{Z}, \quad r > 0,$$
$$c_0(r, J_f^r) = \frac{m}{\rho^2} r^{\rho} + O\left(\frac{r^{\rho}}{\log r}\right), \quad r \to +\infty,$$
$$J_f^r(\varphi) = \sum_{k \in \mathbb{Z}} c_k(r, J_f^r) e^{ik\varphi} = c_0(r, J_f^r) + \sum_{k \in \mathbb{Z}\setminus\{0\}} c_k(r, J_f^r) e^{ik\varphi}, \quad \varphi \in [0, 2\pi],$$

we conclude that the relation (3) *is not true for any $\rho_3 \in (0, \rho)$.*

Acknowledgment. The author would like to thank anonymous referee for valuable comments and careful reading.

REFERENCES

[1] Chyzhykov I.E. *Pfluger-type theorem for functions of refined regular growth*. Mat. Stud. 2017, **47** (2), 169–178. doi:10.15330/ms.47.2.169-178

[2] Grishin A.F. Subharmonic functions of finite order. Doctor-Degree Thesis (Physics and Mathematics), Kharkov, 1992. (in Russian)

[3] Hirnyk M.O. *Subharmonic functions of improved regular growth*. Dopov. Nats. Akad. Nauk Ukr. 2009, **4**, 13–18. doi:10.1007/s11253-012-0624-2 (in Ukrainian)

[4] Khats' R.V. *On the Fourier coefficients of one class of entire functions*. Mat. Stud. 2005, **23** (1), 99–102. (in Ukrainian)

[5] Khats' R.V. *On entire functions of improved regular growth of integer order with zeros on a finite system of rays*. Mat. Stud. 2006, **26** (1), 17–24.

[6] Khats' R.V. *Averaging of entire functions of improved regular growth with zeros on a finite system of rays*. Visn. Nats. Univ. L'viv. Politekh., Fiz.-Mat. Nauky 2011, **718** (718), 10–14.

[7] Khats' R.V. *Regularity of growth of Fourier coefficients of entire functions of improved regular growth*. Ukrainian Math. J. 2012, **63** (12), 1953–1960. doi:10.1007/s11253-012-0624-2 (translation of Ukrain. Mat. Zh. 2011, **63** (12), 1717–1723. (in Ukrainian))

[8] Khats' R.V. *Asymptotic behavior of averaging of entire functions of improved regular growth*. Carpathian Math. Publ. 2013, **5** (1), 129–133. doi:10.15330/cmp.5.1.129-133

[9] Khats' R.V. *Asymptotic behavior of entire functions of improved regular growth in the metric of* $L^p[0,2\pi]$. Carpathian Math. Publ. 2013, **5** (2), 341–344. doi:10.15330/cmp.5.2.341-344

[10] Khats' R.V. *Regular growth of Fourier coefficients of the logarithmic derivative of entire functions of improved regular growth*. Bukovinian Math. J. 2019, **7** (1), 114–120. doi:10.31861/bmj2019.01.114

[11] Kondratyuk A.A. Fourier series and meromorphic functions. Vyshcha shkola, Lviv, 1988. (in Russian)

[12] Lapenko Yu.P. On entire and meromorphic functions of completely regular growth and their defects. Candidate-Degree Thesis (Physics and Mathematics), Rostov-on-Don, 1981. (in Russian)

[13] Levin B.Ya. Distribution of zeros of entire functions. In: Transl. Math. Monographs, vol. 5, Amer. Math. Soc., Providence, RI, 1980.

[14] Vasylkiv Ya.V. *Asymptotic behavior of logarithmic derivatives and logarithms of meromorphic functions of completely regular growth in the metric of* $L^p[0,2\pi]$. *I*. Mat. Stud. 1999, **12** (1), 37–58. (in Ukrainian)

[15] Vynnyts'kyi B.V., Khats' R.V. *On asymptotic behavior of entire functions of noninteger order*. Mat. Stud. 2004, **21** (2), 140–150. (in Ukrainian)

[16] Vynnyts'kyi B.V., Khats' R.V. *On growth regularity of entire function of noninteger order with zeros on a finite system of rays*. Mat. Stud. 2005, **24** (1), 31–38. (in Ukrainian)

[17] Vynnyts'kyi B.V., Khats' R.V. *On asymptotic properties of entire functions, similar to the entire functions of completely regular growth*. Visn. Nats. Univ. L'viv. Politekh., Fiz.-Mat. Nauky 2011, **718** (718), 5–9.

Received 19.11.2019

Хаць Р.В. *Достатні умови покращеного регулярного зростання цілих функцій в термінах їх усереднення* // Карпатські матем. публ. — 2020. — Т.12, №1. — С. 46–54.

Нехай f — ціла функція порядку $\rho \in (0,+\infty)$ з нулями на скінченній системі променів $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$ і $h(\varphi)$ — її індикатор. У 2011 році автор цієї статті довів, що якщо f є функцією покращеного регулярного зростання (ціла функція f називається функцією покращеного регулярного зростання, якщо для деяких $\rho \in (0,+\infty)$, $\rho_1 \in (0,\rho)$ і 2π-періодичної ρ-тригонометрично опуклої функції $h(\varphi) \not\equiv -\infty$ існує множина $U \subset \mathbb{C}$, яка міститься в об'єднанні кругів із скінченною сумою радіусів, така, що $\log|f(z)| = |z|^\rho h(\varphi) + o(|z|^{\rho_1})$, $U \not\ni z = re^{i\varphi} \to \infty$), то для деякого $\rho_3 \in (0,\rho)$ співвідношення

$$\int_1^r \frac{\log|f(te^{i\varphi})|}{t}\,dt = \frac{r^\rho}{\rho}h(\varphi) + o(r^{\rho_3}), \quad r \to +\infty,$$

виконується рівномірно за $\varphi \in [0,2\pi]$. В даній роботі, використовуючи метод коефіцієнтів Фур'є, ми встановлюємо обернене твердження, а саме, якщо для деякого $\rho_3 \in (0,\rho)$ останнє асимптотичне співвідношення виконується рівномірно за $\varphi \in [0,2\pi]$, то f є функцією покращеного регулярного зростання. Це доповнює аналогічні результати Б. Левіна, А. Гришина, А. Кондратюка, Я. Васильківа та Ю. Лапенка про функції цілком регулярного зростання.

Ключові слова і фрази: ціла функція цілком регулярного зростання, ціла функція покращеного регулярного зростання, індикатор, коефіцієнти Фур'є, усереднення, скінченна система променів.