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RELATED FIXED POINT RESULTS VIA C_*-CLASS FUNCTIONS ON C^*-ALGEBRA-VALUED G_b-METRIC SPACES

We initiate the concept of C^*-algebra-valued G_b-metric spaces. We study some basic properties of such spaces and then prove some fixed point theorems for Banach and Kannan types via C_*-*class* functions. Also, some nontrivial examples are presented to ensure the effectiveness and applicability of the obtained results.

Key words and phrases: fixed point, C_*-*class* function, C^*-algebra-valued G_b-metric space.

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1 INTRODUCTION

One of the main directions to obtain possible generalizations of fixed point results is the introduction of new types of spaces. For instance, Ma Z., Jiang, L. and Sun, H. in [21] initiated the notion of C^*-algebra-valued metric spaces, where the set of nonnegative reals was replaced by the set of positive elements of a unital C^*-algebra. Going in the same direction, many papers appeared. See, for example, [16,17,20,22,23,34,36].

In [19], the concept of a C^*-algebra-valued modular space has been introduced. It generalized the concept of a modular space. Now, let $T : X_\rho \to X_\rho$ be a self-mapping on a complete C^*-algebra-valued modular space such that there are $c \in \mathbb{A}$ and $\lambda, \sigma \in \mathbb{R}^+$ with $\|c\| < 1$ and $\lambda > \sigma$ so that

$$\rho(\lambda(T\mu - T\nu)) \preceq c^*\rho(\sigma(\mu - \nu))c, \qquad \forall \mu, \nu \in \mathcal{X}_\rho.$$

Then T admits a unique fixed point in X_ρ ([19]).

Bakhtin [10] considered the class of b-metric spaces. Later, many works such as [5–7, 11, 12, 30] have been provided. In [9], the concept of complex valued metric spaces was initiated. Rao et al. [32] initiated the concept of complex valued b-metric spaces. Mustafa and Sims [24] considered the class of G-metric spaces, where the considered metric depends on three variables. For other related papers, see [1,2,8,18,27,29,31,35].

The notion of G_b-metric spaces was presented by Aghajani et al. [3] (see also [25]). Later, Ege [14] introduced the notion of complex valued G_b-metric spaces and proved the related Banach and Kannan type fixed point theorems. In [15], Ege proved a common fixed point theorem via α-series. For other results on G_b-metric spaces, see [26,28,33].

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Very recently, Ansari et al. [4] defined the concept of complex valued *C-class* functions. Also, Moeini et al. [22] presented the notion of C_*-class functions.

In this presented work, we introduce the C^*-algebra-valued G_b-metric spaces which generalize the complex valued G_b-metric spaces. By using C_*-*class* functions, we establish Banach and Kannan type fixed point theorems in C^*-algebra-valued G_b-metric spaces. To support our results, some nontrivial examples are also given.

Definition 1 ([3]). *Let E be a nonempty set and $s \geq 1$. If the function $G : E \times E \times E \to \mathbb{R}_+$ verifies:*

(G_b1) $G(\mu, \eta, \xi) = 0$ if $\mu = \eta = \xi$;

(G_b2) $0 < G(\mu, \mu, \eta)$ for all $\mu, \eta \in E$ with $\mu \neq \eta$;

(G_b3) $G(\mu, \mu, \eta) \leq G(\mu, \eta, \xi)$ for all $\mu, \eta, \xi \in E$ with $\eta \neq \xi$;

(G_b4) $G(\mu, \eta, \xi) = G(p\{\mu, \eta, \xi\})$, where p is a permutation of μ, η, ξ;

(G_b5) $G(\mu, \eta, \xi) \leq s(G(\mu, a, a) + G(a, \eta, \xi))$ for all $\mu, \eta, \xi, a \in E$,

then G is said to be a G_b-metric and (E, G) is called a G_b-metric space.

Mention that any G-metric space is a G_b-metric space with $s = 1$.

Proposition 1 ([3]). *Let (E, G) be a G_b-metric space. For any $\mu, \eta, \xi, a \in E$, we have*

(i) if $G(\mu, \eta, \xi) = 0$, then $\mu = \eta = \xi$;

(ii) $G(\mu, \eta, \xi) \leq s(G(\mu, \mu, \eta) + G(\mu, \mu, \xi))$;

(iii) $G(\mu, \eta, \eta) \leq 2sG(\eta, \mu, \mu)$;

(iv) $G(\mu, \eta, \xi) \leq s(G(\mu, a, \xi) + G(a, \eta, \xi))$.

Definition 2 ([3]). *Let (E, G) be a G_b-metric space and $\{\mu_n\}$ be a sequence in E.*

(i) $\{\mu_n\}$ is G_b-convergent to μ if for each $\varepsilon > 0$, there is $p_0 \in \mathbb{N}$ so that $G(\mu, \mu_p, \mu_q) < \varepsilon$, $p, q \geq p_0$.

(ii) $\{\mu_n\}$ is said to be G_b-Cauchy if for every $\varepsilon > 0$, there is $p_0 \in \mathbb{N}$ so that $G(\mu_p, \mu_q, \mu_i) < \varepsilon$, $p, q, i \geq p_0$.

(iii) If each G_b-Cauchy sequence G_b-converges in (E, G), then (E, G) is called G_b-complete.

Proposition 2 ([3]). *Let E be a G_b-metric space. We have the following equivalences:*

(1) $\{\mu_n\}$ G_b-converges to μ;

(2) $G(\mu_p, \mu_p, \mu) \to 0$ as $p \to \infty$;

(3) $G(\mu_p, \mu, \mu) \to 0$ as $p \to \infty$.

A Banach algebra $\mathbb{A}$ (over the field of complex numbers $\mathbb{C}$) is called a C^*-algebra if there exists an involution $*$ in $\mathbb{A}$ (i.e., an operator $* : \mathbb{A} \to \mathbb{A}$ verifying $a^{**} = a$ for every $a \in \mathbb{A}$) so that, for all $c, d \in \mathbb{A}$ and $\eta, \nu \in \mathbb{C}$, we have:

(i) $(\eta c + \nu d)^* = \bar{\eta} c^* + \bar{\nu} d^*$;

(ii) $(cd)^* = d^* c^*$;

(iii) $\|c^* c\| = \|c\|^2$.

By (iii), we have $\|c\| = \|c^*\|$ for each $c \in \mathbb{A}$. also, $(\mathbb{A}, *)$ is said to be a unital $*$-algebra if the identity element $1_{\mathbb{A}}$ is contained in $\mathbb{A}$. An element $c \in \mathbb{A}$ is called positive if $c^* = c$ and its spectrum $\sigma(c) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathbb{A}} - c \text{ is noninvertible}\} \subset \mathbb{R}_+$. Denote by $\mathbb{A}_+$ the family of positive elements in $\mathbb{A}$. Define the partial order $'\succeq'$ on $\mathbb{A}$ as

$$d \succeq c \text{ iff } d - c \in \mathbb{A}_+.$$

If $c \in \mathbb{A}$ is positive, we write $c \succeq 0_{\mathbb{A}}$, where $0_{\mathbb{A}}$ is the zero element of $\mathbb{A}$. Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root. Denote by $\mathbb{A}$ a unital C^*-algebra with identity element $1_{\mathbb{A}}$. Moreover, $\mathbb{A}_+ = \{c \in \mathbb{A} : c \succeq 0_{\mathbb{A}}\}$ and $(c^* c)^{\frac{1}{2}} = |c|$.

Lemma 1 ([13])**.** *Let $\mathbb{A}$ be a unital C^*-algebra ($1_{\mathbb{A}}$ is its unit).*

(1) For each $z \in \mathbb{A}_+$, $z \preceq 1_{\mathbb{A}}$ iff $\|z\| \leq 1$.

(2) If $c \in \mathbb{A}_+$ with $\|c\| < \frac{1}{2}$, then $1_{\mathbb{A}} - c$ is invertible and $\|c(1_{\mathbb{A}} - c)^{-1}\| < 1$.

(3) Let $c, d \in \mathbb{A}$ so that $c, d \succeq 0_{\mathbb{A}}$ and $cd = dc$. We have $cd \succeq 0_{\mathbb{A}}$.

(4) Put $\mathbb{A}' = \{c \in \mathbb{A} : cd = dc, \forall d \in \mathbb{A}\}$. Let $c \in \mathbb{A}'$, $d, e \in \mathbb{A}$ with $d \succeq e \succeq 0_{\mathbb{A}}$ and $1_{\mathbb{A}} - c \in \mathbb{A}'$ is an invertible operator. We have

$$(1_{\mathbb{A}} - c)^{-1} d \succeq (1_{\mathbb{A}} - c)^{-1} e.$$

Note that if $0_{\mathbb{A}} \preceq c, d$, we have not $0_{\mathbb{A}} \preceq cd$ in a C^*-algebra. Indeed, take the C^*-algebra $\mathbb{M}_2(\mathbb{C})$ with $c = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, $d = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$, then $cd = \begin{pmatrix} -1 & 2 \\ -4 & 8 \end{pmatrix}$. Clearly c, d are in $\mathbb{M}_2(\mathbb{C})_+$, while cd is not.

The notion of complex C-class functions has been initiated by Ansari et al. [4].

Definition 3. *Define $S = \{z \in \mathbb{C} : z \succeq 0\}$. Let $F : S^2 \to \mathbb{C}$ be a continuous function. Such F is said to be a complex C-class function if for all $p, q \in S$*

(1) $F(p, q) \preceq p$;

(2) $F(p, q) = p$ implies that either $p = 0$ or $q = 0$.

For examples of these functions, see [4].

2 MAIN RESULTS

First, we initiate the concept of C^*-algebra-valued G_b-metric spaces.

Definition 4. *Let $\mathbb{A}$ be a unital C^*-algebra and E be a nonempty set. Let $s \in \mathbb{A}$ be such that $\|s\| \geq 1$. A mapping $G : E \times E \times E \to \mathbb{A}_+$ is said to be a C^*-algebra-valued G_b-metric on E if*

(CG_b1) $G(\mu, \eta, \xi) = 0_{\mathbb{A}}$ *if* $\mu = \eta = \xi$;

(CG_b2) $0_{\mathbb{A}} \prec G(\mu, \mu, \eta)$ *for all* $\mu, \eta \in E$ *with* $\mu \neq \eta$;

(CG_b3) $G(\mu, \mu, \eta) \preceq G(\mu, \eta, \xi)$ *for all* $\mu, \eta, \xi \in E$ *with* $\eta \neq \xi$;

(CG_b4) $G(\mu, \eta, \xi) = G(p\{\mu, \eta, \xi\})$, *where p is a permutation of* μ, η, ξ;

(CG_b5) $G(\mu, \eta, \xi) \preceq s(G(\mu, a, a) + G(a, \eta, \xi))$ *for all* $\mu, \eta, \xi, a \in E$.

The triplet $(E, \mathbb{A}, G)$ is called a C^*-algebra-valued G_b-metric space.

Remark 1. *By taking $\mathbb{A} = \mathbb{R}$, a C^*-algebra-valued G_b-metric space is a (real) G_b-metric space.*

As in Proposition 1, we have the following.

Proposition 3. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space. For all $\mu, \eta, \xi \in E$, we have*

(i) $G(\mu, \eta, \xi) \preceq s(G(\mu, \mu, \eta) + G(\mu, \mu, \xi))$;

(ii) $G(\mu, \eta, \eta) \preceq 2sG(\eta, \mu, \mu)$.

Definition 5. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space and $\{\mu_n\}$ be a sequence in E.*

(i) $\{\mu_n\}$ is G_b-convergent to $x \in E$ with respect to the algebra $\mathbb{A}$ iff for each $a \in \mathbb{A}$ with $0_{\mathbb{A}} \prec a$, there is $k \in \mathbb{N}$ so that $G(x, \mu_p, \mu_q) \prec a$ for all $p, q \geq k$.

(ii) $\{\mu_n\}$ is called G_b-Cauchy with respect to $\mathbb{A}$ if for each $a \in \mathbb{A}$ with $0_{\mathbb{A}} \prec a$, there is $k \in \mathbb{N}$ so that $G(\mu_p, \mu_q, \mu_i) \prec a$, $p, q, i \geq k$.

(iii) If each G_b-Cauchy sequence with respect to $\mathbb{A}$ G_b-converges with respect to $\mathbb{A}$, then $(E, \mathbb{A}, G)$ is said to be complete.

Proposition 4. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space and $\{\mu_n\}$ be a sequence in E. Then $\{\mu_n\}$ is G_b-convergent to μ with respect to $\mathbb{A}$ iff $\|G(\mu, \mu_n, \mu_m)\| \to 0$ as $n, m \to \infty$.*

Proof. ($\Rightarrow$) Let $\{\mu_n\}$ be G_b-convergent to μ with respect to $\mathbb{A}$ and let $a = \varepsilon.1_{\mathbb{A}}$ (where $\varepsilon > 0$). Then $0_{\mathbb{A}} \prec a \in \mathbb{A}$ and there is an integer k so that $G(\mu, \mu_n, \mu_m) \prec a$ for all $n, m \geq k$. Thus, $\|G(\mu, \mu_n, \mu_m)\| < \|a\| = \varepsilon$ and so $\|G(\mu, \mu_n, \mu_m)\| \to 0$ as $n, m \to \infty$.

($\Leftarrow$) Suppose that $\|G(\mu, \mu_n, \mu_m)\| \to 0$ as $n, m \to \infty$. For $a \in \mathbb{A}$ with $0_{\mathbb{A}} \prec a$, there is $\delta > 0$ so that for $z \in \mathbb{A}$,

$$\|z\| < \delta \quad \Rightarrow \quad z \prec a.$$

For such a $\delta > 0$, there is an integer k so that $\|G(x, \mu_n, \mu_m)\| < \delta$, i.e., $G(\mu, \mu_n, \mu_m) \prec a$ for all $n, m \geq k$, i.e., $\{\mu_n\}$ is G_b-convergent to μ with respect to $\mathbb{A}$. ☐

From Proposition 3 and Proposition 4, we state the following.

Theorem 1. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space. Let $\{\mu_n\}$ be a sequence in E and $\mu \in E$. We have equivalence of the following:*

(1) $\{\mu_n\}$ is G_b-convergent to μ with respect to $\mathbb{A}$;

(2) $\|G(\mu_p, \mu_p, \mu)\| \to 0$ when $p \to \infty$;

(3) $\|G(\mu_p, \mu, \mu)\| \to 0$ when $p \to \infty$;

(4) $\|G(\mu_q, \mu_p, \mu)\| \to 0$ when $p, q \to \infty$.

Proof. $(1) \Rightarrow (2)$. It follows from Proposition 4.
$(2) \Rightarrow (3)$. From Proposition 3, one writes

$$G(\mu_n, \mu, \mu) \preceq s(G(\mu_n, \mu_n, \mu) + G(\mu_n, \mu_n, \mu)).$$

Using (2), we get

$$\|G(\mu_n, \mu, \mu)\| \to 0 \text{ as } n \to \infty.$$

$(3) \Rightarrow (4)$. Using (CG_b4) and Proposition 3,

$$G(\mu_m, \mu_n, \mu) = G(\mu_n, \mu, \mu_m) \preceq s(G(\mu_m, \mu, \mu) + G(\mu, \mu_n, \mu)) = s(G(\mu_n, \mu, \mu) + G(\mu, \mu, \mu_m)).$$

Then $\|G(\mu_m, \mu_n, \mu)\| \to 0$ as $m, n \to \infty$.
$(4) \Rightarrow (1)$. By (CG_b3) and (CG_b4), we have

$$\begin{aligned} G(\mu, \mu_n, \mu_n) = G(\mu_n, \mu, \mu_n) &\preceq s(G(\mu_n, \mu, \mu_m) + G(\mu_m, \mu_m, \mu_n)) \\ &\preceq sG(\mu_n, \mu, \mu_m) + 2s^2 G(\mu_m, \mu_n, \mu)). \end{aligned}$$

Using the equivalence in Proposition 4, $\|G(\mu_m, \mu_n, \mu)\| \to 0$ as $m, n \to \infty$. Therefore, $\|G(\mu, \mu_n, \mu_n)\| \to 0$ as $n \to \infty$. □

Theorem 2. *Let $(E, \mathbb{A}, G)$ be a C^*-algebra-valued G_b-metric space and $\{\mu_n\}$ be a sequence in E. Then $\{\mu_n\}$ is G_b-Cauchy with respect to $\mathbb{A}$ if and only if $\|G(\mu_n, \mu_m, \mu_p)\| \to 0$ as $n, m, p \to \infty$.*

Proof. $(\Rightarrow)$ Let $b = \varepsilon \cdot 1_{\mathbb{A}}$ and $\varepsilon > 0$ be a real number. Then $0_{\mathbb{A}} \prec b \in \mathbb{A}$ and so there is an integer k such that $G(\mu_n, \mu_m, \mu_l) \prec b$ for all $n, m, l \geq k$. Thus, $\|G(\mu_n, \mu_m, \mu_l)\| < \|b\| = \varepsilon$ for all $n, m, l \geq k$.

$(\Leftarrow)$ Assume that $\|G(\mu_n, \mu_m, \mu_l)\| \to 0$ as $n, m, l \to \infty$. For $b \in \mathbb{A}$ with $0_{\mathbb{A}} \prec a$, there is $\gamma > 0$ so that for $z \in \mathbb{A}$

$$\|z\| < \gamma \text{ implies } z \prec b.$$

For such a γ, there is an integer k so that $\|G(\mu_n, \mu_m, \mu_l)\| < \gamma$ for all $n, m, l \geq k$. That is, $G(\mu_n, \mu_m, \mu_l) \prec b$ for all $n, m \geq k$. Then $\{\mu_n\}$ is G_b-Cauchy with respect to $\mathbb{A}$. □

Example 1. *Let $E = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$ the set of all 2×2 matrices. Consider the usual operations: scalar multiplication, addition and matrix multiplication. For $A \in \mathbb{A}$, consider*

$\|A\| = \left(\sum_{i,j=1}^{2} |a_{ij}|^2\right)^{\frac{1}{2}}$. *The operator* $* : \mathbb{A} \to \mathbb{A}$ *given as* $A^* = A$, *is a convolution on* $\mathbb{A}$. *Thus* $\mathbb{A}$ *becomes a unital* C^**-algebra. For*

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in \mathbb{A} = M_2(\mathbb{R}),$$

consider $a \preceq b$ *iff* $(a_{ij} - b_{ij}) \leq 0$, *for all* $i, j = 1, 2$.

Define $d(x,y) = diag(|x-y|, |x-y|)$ *with "diag" is a diagonal matrix and* $x, y \in \mathbb{R}$. *Suppose* $D_m(d)(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$ *for all* $x, y, z \in E$. *Define* $G : E \times E \times E \to \mathbb{A}_+$ *by*

$$G(x,y,z) = \Big(D_m(d)(x,y,z)\Big)^p,$$

where $p > 1$ *is an integer. It can be proved that* $(E, \mathbb{A}, G)$ *is a* C^**-algebra-valued* G_b*-metric space with* $s = 2^{p-1}.1_{\mathbb{A}}$.

To define the set of C_*-class functions (which contains complex valued C-class functions of [4]), it suffices to use the family of elements of a unital C^*-algebra instead of the set of complex numbers.

Definition 6 ([22]). *Let* $\mathbb{A}$ *be a unital* C^**-algebra and* $F : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ *be a continuous function. Such* F *is said to be a* C_**-class function if for all* $c, d \in \mathbb{A}_+$:

(1) $F(c,d) \preceq c$;

(2) $F(c,d) = c$ *implies that either* $c = 0_{\mathbb{A}}$ *or* $d = 0_{\mathbb{A}}$.

Let $\mathcal{C}_*$ be the set of C_*-class functions.

Remark 2. *If we replace* $\mathbb{A}$ *by* $\mathbb{C}$ *in Definition 6, the class* $\mathcal{C}_*$ *corresponds to the set of complex* C*-class functions.*

Example 2. *Consider* $\mathbb{A} = M_2(\mathbb{R})$ *as defined in Example 1.*

(1) Given $F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ *as* $F_*(c,d) = c - d$, *that is,*

$$F_*\left(c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}\right) = \begin{pmatrix} c_{11} - d_{11} & c_{12} - d_{12} \\ c_{21} - d_{21} & c_{22} - d_{22} \end{pmatrix}$$

for all $c_{p,q}, d_{p,q} \in \mathbb{R}_+$, $(p, q \in \{1,2\})$. *Then* $F_* \in \mathcal{C}_*$.

(2) Given $F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}$ *as*

$$F_*\left(\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}\right) = \lambda \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

for all $c_{p,q}, d_{p,q} \in \mathbb{R}_+$ *with* $(p, q \in \{1,2\})$, *where* $\lambda \in (0,1)$. *Then* $F_* \in \mathcal{C}_*$.

Example 3. *Let* $E = L^\infty(M)$ *and* $U = L^2(M)$, *where* M *is a Lebesgue measurable set. Let* $\mathcal{B}(U)$ *be the family of bounded linear operators on a Hilbert space* H. *Note that* $\mathcal{B}(U)$ *is a* C^**-algebra (with the usual operator norm). Given* $F_* : \mathcal{B}(U)_+ \times \mathcal{B}(U)_+ \to \mathcal{B}(U)$ *as*

$$F_*(P,Q) = P - \psi(P),$$

where $\psi : \mathcal{B}(U)_+ \to \mathcal{B}(U)_+$ *is continuous so that* $\psi(P) = 0_{\mathcal{B}(U)}$ *iff* $P = 0_{\mathcal{B}(U)}$. *Then* $F_* \in \mathcal{C}_*$.

Let Σ be the set of the functions $\sigma : \mathbb{A}_+ \to \mathbb{A}_+$ so that:

(a) σ is continuous;

(b) $\sigma(t) \succ 0_{\mathbb{A}}$ iff $t \succ 0_{\mathbb{A}}$ and $\sigma(0_{\mathbb{A}}) = 0_{\mathbb{A}}$.

Our first result is as follows.

Theorem 3. *Let $(E, \mathbb{A}, G)$ be a complete C^*-algebra-valued G_b-metric space with $s = (b.1_{\mathbb{A}}) \succ 1_{\mathbb{A}}$ and $T : E \to E$ be so that*

$$\sigma((b^{\varepsilon}.1_{\mathbb{A}})G(T\mu, T\eta, T\xi)) \preceq F_*\Big(\sigma(G(\mu,\eta,\xi)), \vartheta(G(\mu,\eta,\xi))\Big), \tag{1}$$

for all $\mu, \eta, \xi \in E$, where $F_* \in \mathcal{C}_*, \sigma, \vartheta \in \Sigma$ and $\varepsilon \in (1, \infty)$. Then T possesses a unique fixed point.

Proof. Let T verify (1). Consider $\mu_0 \in E$ and define $\mu_n = T^n \mu_0$. By (1), one writes

$$\sigma((b^{\varepsilon}.1_{\mathbb{A}})G(\mu_n, \mu_{n+1}, \mu_{n+1})) \preceq F_*\Big(\sigma(G(\mu_{n-1}, \mu_n, \mu_n)), \vartheta(G(\mu_{n-1}, \mu_n, \mu_n))\Big).$$

We have

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq (b^{\varepsilon}.1_{\mathbb{A}})^{-1} G(\mu_{n-1}, \mu_n, \mu_n), \quad \text{for all } n \geq 1. \tag{2}$$

The inequality (2) implies that

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq (b^{\varepsilon}.1_{\mathbb{A}})^{-2} G(\mu_{n-2}, \mu_{n-1}, \mu_{n-1}), \quad \text{for all } n \geq 2.$$

If the same process is continued, we get

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq (b^{\varepsilon}.1_{\mathbb{A}})^{-n} G(\mu_0, \mu_1, \mu_1), \quad \text{for all } n \geq 0. \tag{3}$$

Using (CG_b5) together with (3) ($n, m \in \mathbb{N}$ with $n < m$),

$$\begin{aligned}
&G(\mu_n, \mu_m, \mu_m) \preceq (b.1_{\mathbb{A}})[G(\mu_n, \mu_{n+1}, \mu_{n+1}) + G(\mu_{n+1}, \mu_m, \mu_m)] \\
&\preceq (b.1_{\mathbb{A}})[G(\mu_n, \mu_{n+1}, \mu_{n+1})] + (b.1_{\mathbb{A}})^2[G(\mu_{n+1}, \mu_{n+2}, \mu_{n+2}) + G(\mu_{n+2}, \mu_m, \mu_m)] \\
&\preceq (b.1_{\mathbb{A}})[G(\mu_n, \mu_{n+1}, \mu_{n+1})] + (b.1_{\mathbb{A}})^2[G(\mu_{n+1}, \mu_{n+2}, \mu_{n+2})] + \ldots \\
&+ (b.1_{\mathbb{A}})^{m-n}[G(\mu_{m-1}, \mu_m, \mu_m)] \\
&\preceq (b.1_{\mathbb{A}})((b.1_{\mathbb{A}})^{\varepsilon})^{-n} G(\mu_0, \mu_1, \mu_1) + (b.1_{\mathbb{A}})^2((b.1_{\mathbb{A}})^{\varepsilon})^{-n-1} G(\mu_0, \mu_1, \mu_1) + \ldots \\
&+ (b.1_{\mathbb{A}})^{m-n}((b.1_{\mathbb{A}})^{\varepsilon})^{-m+1} G(\mu_0, \mu_1, \mu_1) \\
&\preceq [(b.1_{\mathbb{A}})((b.1_{\mathbb{A}})^{\varepsilon})^{-n} + (b.1_{\mathbb{A}})^2((b.1_{\mathbb{A}})^{\varepsilon})^{-n-1} + \ldots + (b.1_{\mathbb{A}})^{m-n}((b.1_{\mathbb{A}})^{\varepsilon})^{-m+1}] G(\mu_0, \mu_1, \mu_1) \\
&\preceq (b.1_{\mathbb{A}})((b.1_{\mathbb{A}})^{\varepsilon})^{-n}[1_{\mathbb{A}} + (((b.1_{\mathbb{A}})^{\varepsilon-1})^{-1})^1 + \ldots + (((b.1_{\mathbb{A}})^{\varepsilon-1})^{-1})^{m-n-1}] G(\mu_0, \mu_1, \mu_1) \\
&= (b.1_{\mathbb{A}})((b.1_{\mathbb{A}})^{\varepsilon})^{-n} G(\mu_0, \mu_1, \mu_1) \sum_{k=1}^{m-n} ((s^{\varepsilon-1})^{-1})^{k-1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|G(\mu_n, \mu_m, \mu_m)\| &\leq \|b\| \|(b^{\varepsilon})^{-n}\| \|G(\mu_0, \mu_1, \mu_1)\| \sum_{k=1}^{m-n} \|((b^{\varepsilon-1})^{-1})\|^{k-1} \\
&\leq \|b\| \left(\frac{1}{\|b^{\varepsilon}\|}\right)^n \|G(\mu_0, \mu_1, \mu_1)\| \frac{\|b^{\varepsilon-1}\|}{\|b^{\varepsilon-1}\| - 1}.
\end{aligned}$$

If we take $n \to \infty$, then

$$\|b\| \left(\frac{1}{\|b^\varepsilon\|}\right)^n \|G(\mu_0, \mu_1, \mu_1)\| \frac{\|b^{\varepsilon-1}\|}{\|b^{\varepsilon-1}\| - 1} \to 0,$$

because $\varepsilon \in (1, +\infty)$ and $\|b\| > 1$. We deduce that

$$\lim_{n,m\to\infty} \|G(\mu_n, \mu_m, \mu_m)\| = 0. \tag{4}$$

From Proposition 3, we have

$$G(\mu_n, \mu_m, \mu_l) \preceq G(\mu_n, \mu_m, \mu_m) + G(\mu_m, \mu_m, \mu_l),$$

for $n, m, l \in \mathbb{N}$. Consequently,

$$\|G(\mu_n, \mu_m, \mu_l)\| \leq \|G(\mu_n, \mu_m, \mu_m)\| + \|G(\mu_m, \mu_m, \mu_l)\|.$$

By (4), we conclude that $\|G(\mu_n, \mu_m, \mu_l)\| \to 0$ as $n, m, l \to \infty$. Thus, $\{\mu_n\}$ is G_b-Cauchy with respect to $\mathbb{A}$. The completeness of $(E, \mathbb{A}, G)$ implies that there is some $u \in E$ so that $\{\mu_n\}$ is G_b-convergent to u with respect to $\mathbb{A}$.

We claim that $Tu = u$. Assume on the contrary $u \neq Tu$. By (1), we have

$$\sigma((b^\varepsilon.1_{\mathbb{A}})G(\mu_{n+1}, Tu, Tu)) \preceq F_*\Big(\sigma(G(\mu_n, u, u)), \vartheta(G(\mu_n, u, u))\Big), \quad \text{for all } n \geq 0.$$

Therefore,

$$G(\mu_{n+1}, Tu, Tu) \preceq (b^\varepsilon.1_{\mathbb{A}})^{-1} G(\mu_n, u, u), \quad \text{for all } n \geq 2,$$

and so

$$\|G(\mu_{n+1}, Tu, Tu)\| \leq \frac{1}{\|b^\varepsilon\|} \|G(\mu_n, u, u)\|.$$

Taking $n \to \infty$, we obtain $\lim_{n\to\infty} \|G(\mu_{n+1}, Tu, Tu)\| = 0$. Thus, $\{\mu_n\}$ G_b-converges to Tu. By uniqueness of limit $u = Tu$. Let $\zeta \neq u$ be another fixed point of T. From (1),

$$\sigma((b^\varepsilon.1_{\mathbb{A}})G(u, \zeta, \zeta)) = \sigma((b^\varepsilon.1_{\mathbb{A}})G(Tu, T\zeta, T\zeta)) \preceq F_*\Big(\sigma(G(u, \zeta, \zeta)), \vartheta(G(u, \zeta, \zeta))\Big),$$

so

$$G(u, \zeta, \zeta) \preceq (b^\varepsilon.1_{\mathbb{A}})^{-1} G(u, \zeta, \zeta).$$

Thus,

$$\|G(u, \zeta, \zeta)\| \leq \frac{1}{\|b^\varepsilon\|} \|G(u, \zeta, \zeta)\|.$$

We conclude that $\|G(u, \zeta, \zeta)\| \leq 0$ because $\frac{1}{\|b^\varepsilon\|} \in [0, \frac{1}{\|b\|}) \subset [0, 1)$. Therefore, u is the unique fixed point of T. $\square$

Taking $F_*(s, t) = k^* s k$ (where $k \in \mathbb{A}$ with $\|k\| < 1$ and $s \in \mathbb{A}_+$) in Theorem 3, we have the following.

Corollary 1. *Let $(E, \mathbb{A}, G)$ be a complete C^*-algebra-valued G_b-metric space with $s = (b.1_{\mathbb{A}}) \succ 1_{\mathbb{A}}$. Let $T : E \to E$ be so that*

$$\sigma((b^\varepsilon.1_{\mathbb{A}})G(T\mu, T\eta, T\xi)) \preceq k^* \sigma(G(\mu, \eta, \xi))k, \tag{5}$$

for all $\mu, \eta, \xi \in E$, where $k \in \mathbb{A}$ with $\|k\| < 1$, $\sigma \in \Sigma, \varepsilon \in (1, +\infty)$. Then T admits a unique fixed point.

Example 4. *Let $E = \mathbb{R}$. Consider $\mathbb{A} = M_2(\mathbb{R})$ as defined in Example 1. Let $G : E \times E \times E \to M_2(\mathbb{R})$ be defined as*

$$G(\mu,\eta,\xi) = diag\Big(\frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2, \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2\Big)$$

$$= \begin{pmatrix} \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 & 0 \\ 0 & \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 \end{pmatrix}$$

for all $\mu,\eta,\xi \in E$. Then $(E, M_2(\mathbb{R}), G)$ is a complete C^*-algebra-valued G_b-metric space with coefficient

$$s = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (2.1_{\mathbb{A}}).$$

Given $T : E \to E$ as $T\mu = \frac{\mu}{3^\varepsilon}$ for all $\mu \in E$, where $\varepsilon \in (1,+\infty)$. Take $\sigma : \mathbb{A}_+ \to \mathbb{A}_+$ as $\sigma(t) = t$. For all $\mu,\eta,\xi \in E$,

$$\sigma((2^\varepsilon.1_{\mathbb{A}})G(T\mu,T\eta,T\xi)) = \sigma\left((2^\varepsilon.1_{\mathbb{A}})G\left(\frac{\mu}{3^\varepsilon},\frac{\eta}{3^\varepsilon},\frac{\xi}{3^\varepsilon}\right)\right) = \sigma\left(\left(\frac{2^\varepsilon}{9^\varepsilon}.1_{\mathbb{A}}\right)G(\mu,\eta,\xi)\right)$$

$$= \sigma\left(\begin{pmatrix} \frac{2^\varepsilon}{9^\varepsilon} & 0 \\ 0 & \frac{2^\varepsilon}{9^\varepsilon} \end{pmatrix}\begin{pmatrix} \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 & 0 \\ 0 & \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \frac{2^\varepsilon}{9^\varepsilon} & 0 \\ 0 & \frac{2^\varepsilon}{9^\varepsilon} \end{pmatrix}\begin{pmatrix} \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 & 0 \\ 0 & \frac{1}{9}(|\mu-\eta|+|\eta-\xi|+|\xi-\mu|)^2 \end{pmatrix}$$

$$\preceq k^*\sigma(G(\mu,\eta,\xi))k,$$

where $k = \begin{pmatrix} \sqrt{\frac{2^\varepsilon}{9^\varepsilon}} & 0 \\ 0 & \sqrt{\frac{2^\varepsilon}{9^\varepsilon}} \end{pmatrix}$, $\|k\| < 1$, $\varepsilon \in (1,+\infty)$. The inequality (5) *holds. From Corollary 1, $\mu = 0$ is the unique fixed point of T in E.*

A related Kannan type fixed point theorem is stated as follows.

Theorem 4. *Let $(E,\mathbb{A},G)$ be a complete C^*-algebra-valued G_b-metric space. Let $T : E \to E$ verifies for all $\mu,\eta \in E$,*

$$\sigma(G(T\mu,T\eta,T\eta)) \preceq F_*\Big(\sigma(m(\mu,\eta)),\vartheta(m(\mu,\eta))\Big), \tag{6}$$

where $F_* \in \mathcal{C}_*, \sigma, \vartheta \in \Sigma$, and

$$m(\mu,\eta) = b(G(\mu,T\mu,T\mu) + G(\eta,T\eta,T\eta)),$$

where $b \in \mathbb{A}'_+$ and $\|b\| < \frac{1}{2}$. Then T possesses a unique fixed point.

Proof. Assume that $b \neq 0_{\mathbb{A}}$. Then $b \in \mathbb{A}'_{+}$, and so $b(G(\mu, T\mu, T\mu) + G(\eta, T\eta, T\eta))$ is also a positive element. Let μ_0 be in E. Take $\mu_{n+1} = T\mu_n = T^{n+1}\mu_0$ for all $n \geq 0$. We claim that $\{\mu_n\}$ is a G_b-Cauchy sequence with respect to $\mathbb{A}$. In case of $\mu_n = \mu_{n+1}$ for some n, μ_n is a fixed point of T. Therefore, assume that $\mu_n \neq \mu_{n+1}$ for all $n \geq 0$. Choose $G(\mu_n, \mu_{n+1}, \mu_{n+1}) = G_n$. It follows from (6) that

$$\begin{aligned}\sigma(G(\mu_n, \mu_{n+1}, \mu_{n+1})) &= \sigma(G(T\mu_{n-1}, T\mu_n, T\mu_n)) \\ &\preceq F_*\Big(\sigma(b(G(\mu_{n-1}, T\mu_{n-1}, T\mu_{n-1}) + G(\mu_n, T\mu_n, T\mu_n))), \\ &\quad \vartheta(b(G(\mu_{n-1}, T\mu_{n-1}, T\mu_{n-1}) + G(\mu_n, T\mu_n, T\mu_n)))\Big) \\ &\preceq F_*\Big(\sigma(b(G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_n, \mu_{n+1}, \mu_{n+1}))), \\ &\quad \vartheta(b(G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_n, \mu_{n+1}, \mu_{n+1})))\Big) \\ &\preceq \sigma(b(G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_n, \mu_{n+1}, \mu_{n+1}))).\end{aligned}$$

Hence,

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq b(G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_n, \mu_{n+1}, \mu_{n+1})),$$

and thus

$$G(\mu_n, \mu_{n+1}, \mu_{n+1}) \preceq (1_{\mathbb{A}} - b)^{-1} bG(\mu_{n-1}, \mu_n, \mu_n) = tG(\mu_{n-1}, \mu_n, \mu_n),$$

where $t = (1_{\mathbb{A}} - b)^{-1}b$. Inductively, we conclude that

$$\begin{aligned}G(\mu_n, \mu_{n+1}, \mu_{n+1}) &\preceq tG(\mu_{n-1}, \mu_n, \mu_n) \preceq t^2 G(\mu_{n-2}, \mu_{n-1}, \mu_{n-1}) \preceq \ldots \preceq t^n G(\mu_0, \mu_1, \mu_1) \\ &= t^n G_0.\end{aligned}$$

Since $|b| < \frac{1}{2}$, we have $\|t\| < 1$. Hence

$$\lim_{n\to\infty} \|G(\mu_n, \mu_{n+1}, \mu_{n+1})\| = 0.$$

and so

$$\lim_{n\to\infty} G(\mu_n, \mu_{n+1}, \mu_{n+1}) = 0_{\mathbb{A}}.$$

Now,

$$\begin{aligned}\sigma(G(\mu_n, \mu_m, \mu_m)) &= \sigma(G(T\mu_{n-1}, T\mu_{m-1}, T\mu_{m-1})) \\ &\preceq F_*\left(\sigma\left(\frac{G(\mu_{n-1}, T\mu_{n-1}, T\mu_{n-1}) + G(\mu_{m-1}, T\mu_{m-1}, T\mu_{m-1})}{2}\right),\right. \\ &\quad \left.\vartheta\left(\frac{G(\mu_{n-1}, T\mu_{n-1}, T\mu_{n-1}) + G(\mu_{m-1}, T\mu_{m-1}, T\mu_{m-1})}{2}\right)\right) \\ &\preceq F_*\left(\sigma\left(\frac{G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_{m-1}, \mu_m, \mu_m)}{2}\right),\right. \\ &\quad \left.\vartheta\left(\frac{G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_{m-1}, \mu_m, \mu_m)}{2}\right)\right) \\ &\preceq \sigma\left(\frac{G(\mu_{n-1}, \mu_n, \mu_n) + G(\mu_{m-1}, \mu_m, \mu_m)}{2}\right) \to \sigma(0_{\mathbb{A}}).\end{aligned}$$

This shows that $\{\mu_n\}$ is G_b-Cauchy with respect to $\mathbb{A}$. Since E is complete, there is $u \in E$ so that $\mu_n \to u$. We have

$$\begin{aligned}\sigma(G(Tu,\mu_{n+1},\mu_{n+1})) &= \sigma(G(T\mu_{n-1},T\mu_n,T\mu_n)) \\ &\preceq F_*\left(\sigma\left(\frac{G(u,Tu,Tu)+G(\mu_n,T\mu_n,T\mu_n)}{2}\right),\right. \\ &\quad \left.\vartheta\left(\frac{G(u,Tu,Tu)+G(\mu_n,T\mu_n,T\mu_n)}{2}\right)\right) \\ &\preceq F_*\left(\sigma\left(\frac{G(u,Tu,Tu)+G(\mu_n,\mu_{n+1},\mu_{n+1})}{2}\right),\right. \\ &\quad \left.\vartheta\left(\frac{G(u,Tu,Tu)+G(\mu_n,\mu_{n+1},\mu_{n+1})}{2}\right)\right).\end{aligned}$$

Letting $n \to \infty$, we get

$$\sigma(G(Tu,u,u)) \preceq F_*\Big(\sigma(G(Tu,u,u)),\vartheta(G(Tu,u,u))\Big).$$

That is, $\sigma(G(Tu,u,u))$=$F_*\Big(\sigma(G(Tu,u,u)),\vartheta(G(Tu,u,u))\Big)$. So,

$$\sigma(G(Tu,u,u)) = 0_{\mathbb{A}} \quad \text{or} \quad \vartheta(G(Tu,u,u)) = 0_{\mathbb{A}}.$$

That is, $G(Tu,u,u) = 0_{\mathbb{A}}$, i.e., $u = Tu$.

Let v be in E so that $v = Tv$. We have

$$\begin{aligned}\sigma(G(v,u,u)) &= \sigma(G(Tv,Tu,Tu)) \\ &\preceq F_*\left(\sigma\left(\frac{G(v,Tv,Tv)+G(u,Tu,Tu)}{2}\right),\vartheta\left(\frac{G(v,Tv,Tv)+G(u,Tu,Tu)}{2}\right)\right) \\ &= F_*\left(\sigma\left(\frac{G(u,u,u)+G(v,v,v)}{2}\right),\vartheta\left(\frac{G(u,u,u)+G(v,v,v)}{2}\right)\right) \\ &= F_*\Big(\sigma(0_{\mathbb{A}}),\vartheta(0_{\mathbb{A}})\Big) \preceq \sigma(0_{\mathbb{A}}) = 0_{\mathbb{A}},\end{aligned}$$

which implies that $u = v$. □

If we consider $F_*(s,t) = s - t$ (for $s,t \in \mathbb{A}_+$) in Theorem 4, we get the following.

Corollary 2. *Let (E,G) be a complete C^*-algebra-valued G_b-metric space. Let $T : E \to E$ be so that*

$$\sigma(G(T\mu,T\eta,T\eta)) \preceq \sigma(m(\mu,\eta)) - \vartheta(m(\mu,\eta)),$$

for all $\mu,\eta \in E$, where $\sigma,\vartheta \in \Sigma$ and

$$m(\mu,\eta) = \frac{G(\mu,T\mu,T\mu)+G(\eta,T\eta,T\eta)}{2}.$$

Then T admits a unique fixed point.

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Моейні Б., Ішік Г., Айді Г. *Стосовно результатів про нерухому точку для функцій класу C_* на C^*-алгеброзначних G_b-метричних просторах* // Карпатські матем. публ. — 2020. — Т.12, №1. — C. 94–106.

Запропоновано концепцію C^*-алгеброзначних G_b-метричних просторів. Досліджено деякі основні властивості таких просторів і доведено деякі теореми про нерухому точку типу Банаха і Каннана для функцій класу C_*. Також наведено деякі нетривіальні приклади, щоб показати ефективність і застосовність отриманих результатів.

Ключові слова і фрази: нерухома точка, функція класу C_*, C^*-алгеброзначний G_b-метричний простір.