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A NOTE ON PELL-PADOVAN NUMBERS AND THEIR CONNECTION WITH FIBONACCI NUMBERS

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In this paper, we find new relations involving the Pell-Padovan sequence which arise as determinants of certain families of Toeplitz-Hessenberg matrices. These determinant formulas may be rewritten as identities involving sums of products of Pell-Padovan numbers and multinomial coefficients. In particular, we establish four connection formulas between the Pell-Padovan and the Fibonacci sequences via Toeplitz-Hessenberg determinants.

Key words and phrases: Pell-Padovan sequence, Fibonacci sequence, Toeplitz-Hessenberg matrix, Trudi's formula.

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INTRODUCTION

The *Pell-Padovan sequence* $\{P_n\}_{n\ge 0}$ is defined by the third-order recurrence

$$P_n = 2P_{n-2} + P_{n-3}, \qquad n \ge 3, \tag{1}$$

with initial values $P_0 = P_1 = P_2 = 1$ (sequence A066983 in [18]). The Pell-Padovan numbers also can be expressed directly in terms of Fibonacci numbers (sequence A000045 in [18]) as follows

$$P_n = 2F_{n-1} - (-1)^n, \qquad n \ge 1. \tag{2}$$

The list of first 16 terms of the Fibonacci and Pell-Padovan sequences is given in Table 1.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987
P_n	1	1	1	3	3	7	9	17	25	43	67	111	177	289	465	755	1219

Table 1: Terms of F_n and P_n.

The Pell-Padovan numbers and their generalizations have been studied by some authors (see [2–4, 19–21] for more details). For instance, Deveci in [3] studied the Pell-Padovan sequence, the Pell-Padovan sequence modulo m and defined the Pell-Padovan orbit of a 2-generator group, then examined the lengths of the periods of these orbits. Deveci and Shannon in [4] investigated properties of recurrence sequences defined from circulant matrices obtained from the characteristic polynomial of the Pell-Padovan sequence. In [2], Atanassov et al. investigated a property of an extended form of the Pell-Padovan sequence in the form $P_1 = a$,

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$P_2 = b$, $P_3 = c$ and $P_{n+3} = pP_{n+1} + qP_n$, for $n \geq 1$, where a, b, c, p, q are positive integer numbers. Tasci [19] extended Pell-Padovan numbers to Gaussian Pell-Padovan numbers GP_n defined by the recurrence $GP_0 = 1$, $GP_1 = 1 + i$, $GP_2 = 1 + i$, $GP_n = GP_{n-2} + GP_{n-3}$, $n \geq 3$, and obtained Binet-like formula, generating function and some identities related with Gaussian Pell-Padovan numbers. In [20], Tasci studied Pell-Padovan quaternions, defined by $QP_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3}$, where i, j, k are the orthonormal basis in $\mathbb{R}^3$, and gave Binet-like formula, generating function, sums formulas, matrix representation of QP_n. Zuo et al. [21] showed the explicit determinants of the Ppoeplitz matrix and Ppankel matrix which are both involving Pell-Padovan sequences and gave the expressions of the entries of the inverse for these kinds of matrices.

The purpose of the present paper is to study some families of the Toeplitz-Hessenberg determinants whose entries are Pell-Padavan numbers with successive, odd or even subscripts. As a consequence, we obtain for these numbers new identities involving multinomial coefficients. In particular, we establish connection between Pell-Padovan numbers with Fibonacci numbers via Toeplitz-Hessenberg determinants. Similar results for Pell and Padovan numbers are derived in [6,7].

1 TOEPLITZ-HESSENBERG DETERMINANTS AND FORMULAS FOR THEIR EVALUATION

A square matrix of the order n having the form

$$M_n(a_0, a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix}, \tag{3}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$, is called a lower *Toeplitz-Hessenberg matrix*.

This class of matrix have been encountered in many scientific and engineering applications (see, among others, [1,13–16] and related references therein).

Expanding the *Toeplitz-Hessenberg determinant* $\det(M_n)$ along the first column gives the following recurrence

$$\det(M_0) := 1, \quad \det(M_n) = \sum_{k=1}^{n} (-a_0)^{k-1} a_k \det(M_{n-k}), \qquad n \geq 1. \tag{4}$$

The following result, which provides a multinomial expansion of $\det(M_n)$, is known as *Trudi's formula* [17, Ch. 7].

Lemma 1. *Let n be a positive integer. Then*

$$\det(M_n) = \sum_{\substack{s_1,\ldots,s_n \geq 0 \\ s_1+2s_2+\cdots+ns_n=n}} (-a_0)^{n-(s_1+\cdots+s_n)} \binom{s_1 + \cdots + s_n}{s_1, \ldots, s_n} a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}, \tag{5}$$

where $\binom{s_1+\cdots+s_n}{s_1,\ldots,s_n} = \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ is the multinomial coefficient, or, equivalently,

$$\det(M_n) = \sum_{k=1}^{n} (-a_0)^{n-k} \Big(\sum_{\substack{i_1,\ldots,i_k \geq 1 \\ i_1+i_2+\cdots+i_k=n}} a_{i_1} a_{i_2} \cdots a_{i_k} \Big).$$

Note, that the number of solutions $(s_1, s_2, \ldots, s_n)$ of the Diophantine equation

$$s_1 + 2s_2 + \cdots + ns_n = n, \qquad s_i \geq 0,$$

is equal to the number of partitions of n; see [16].

We will investigate the particular cases of the matrix (3), in which $a_0 = \pm 1$. For the sake of brevity, throughout the paper we will use the notation

$$\det(\pm 1; a_1, a_2, \ldots, a_n) = \det\left(M_n(\pm 1, a_1, a_2, \ldots, a_n)\right).$$

2 Fibonacci numbers via Toeplitz-Hessenberg determinants with Pell-Padovan numbers entries

In this and next sections, we consider families of some Toeplitz-Hessenberg determinants having various translates of the Pell-Padovan sequence (or of the respective half sequences) as their entries.

Our first result provides a connection between the Pell-Padovan and Fibonacci numbers via Toeplitz-Hessenberg determinants.

Theorem 1. *The following formulas hold:*

$$\begin{aligned}
F_{2n} &= \frac{1}{2}\det(1; P_3, P_4, \ldots, P_{n+2}), \quad n \geq 2,\\
F_{n-2} &= \frac{1}{2}\det(1; P_1, P_2, \ldots, P_n), \quad n \geq 2,\\
F_{n-4} &= \frac{(-1)^{n-1}}{2}\det(1; P_3, P_5, \ldots, P_{2n+1}), \quad n \geq 4,\\
F_{n-1} &= \frac{(-1)^{n}}{2}\det(1; P_0, P_2, \ldots, P_{2n-2}) + 2^{n-2}, \quad n \geq 1.
\end{aligned}$$

Proof. We will prove only the first formula using the principle of mathematical induction on n. The proof of others formulas follow similarly, so we omit them. Let

$$D_n = \det(1; P_3, P_4, \ldots, P_{n+2}).$$

One may verify, that the formula holds, when $n = 2$ and $n = 3$. Suppose it is true for all $k \leq n-1$, where $n \geq 4$. By recurrences (4) and (1), we have

$$\begin{aligned}
D_n &= \sum_{k=1}^{n}(-1)^{k-1}P_{k+2}D_{n-k} = \sum_{k=1}^{n}(-1)^{k-1}\left(2P_k + P_{k-1}\right)D_{n-k}\\
&= 2\sum_{k=1}^{n}(-1)^{k-1}P_kD_{n-k} + \sum_{k=1}^{n}(-1)^{k-1}P_{k-1}D_{n-k} = 2\Big(P_1D_{n-1} - P_2D_{n-2} + \sum_{k=3}^{n}(-1)^{k-1}P_kD_{n-k}\Big)\\
&\quad + P_0D_{n-1} - P_1D_{n-2} + P_2D_{n-3} + \sum_{k=4}^{n}(-1)^{k-1}P_{k-1}D_{n-k}\\
&= 2\Big(D_{n-1} - D_{n-2} + \sum_{k=1}^{n-2}(-1)^{k+1}P_{k+2}D_{n-k-2}\Big)\\
&\quad + D_{n-1} - D_{n-2} + D_{n-3} - \sum_{k=1}^{n-3}(-1)^{k+1}P_{k+2}D_{n-k-3}\\
&= 2\left(D_{n-1} - D_{n-2} + D_{n-2}\right) + D_{n-1} - D_{n-2} + D_{n-3} - D_{n-3} = 3D_{n-1} - D_{n-2}.
\end{aligned}$$

Using the induction hypothesis and formula $F_{2n} = 3F_{2n-2} - F_{2n-4}$ for $n \geq 2$, we conclude

$$D_n = 3 \cdot 2F_{2n-2} - 2F_{2n-4} = 2F_{2n}.$$

Consequently, the stated formula is true in the n case. Therefore, by induction, the formula works for all $n \geq 2$. $\square$

3 Some Toeplitz-Hessenberg determinants with Pell-Padovan numbers entries

Our objective in this section is to investigate several Toeplitz-Hessenberg determinants with special Pell-Padovan numbers entries.

Theorem 2. *For $n \geq 2$, the following formulas hold:*

$$\det(-1; P_1, P_3, \ldots, P_{2n-1}) = \frac{2}{25}\left(\left(\frac{5+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{5-\sqrt{5}}{2}\right)^{n+1}\right),$$

$$\det(1; P_0, P_1, \ldots, P_{n-1}) = \sum_{k=0}^{n-4}\left((-1)^{n-k-1} - 1\right)\sum_{j=0}^{n-4-k}\binom{k}{j}\binom{j+1-\frac{n-k}{2}}{j},$$

$$\det(-1; P_0, P_1, \ldots, P_{n-1}) = \frac{\sqrt{13}}{13}\left((3+\sqrt{13})\,\omega^{n-2} - (3-\sqrt{13})\left(-\frac{3}{\omega}\right)^{n-2}\right),$$

$$\det(1; P_2, P_3, \ldots, P_{n+1}) = (-1)^{\lfloor (n+1)/3 \rfloor + \lfloor (n+2)/3 \rfloor},$$

$$\det(1; P_1, P_3, \ldots, P_{2n-1}) = -\frac{2}{3}\left(\left(\frac{-1+\sqrt{3}i}{2}\right)^{n-2} + \left(\frac{-1-\sqrt{3}i}{2}\right)^{n-2}\right) - \frac{(-2)^n}{6},$$

$$\det(1; P_2, P_4, \ldots, P_{2n}) = 2(-1)^{n-1}\sum_{j=0}^{\lfloor (n-1)/3 \rfloor}(-1)^j\binom{n-1}{3j+1},$$

$$\det(1; P_4, P_6, \ldots, P_{2n+2}) = (-1)^{\lfloor (2n-2)/3 \rfloor} + (-1)^{\lfloor (2n-1)/3 \rfloor},$$

where $\lfloor \alpha \rfloor$ is the floor of α, $\omega = \frac{1+\sqrt{13}}{2}$, and $i = \sqrt{-1}$.

Proof. We will prove only the first formula; the others can be proved in the same way. Let

$$D_n = \det(-1; P_1, P_3, \ldots, P_{2n-1}).$$

When $n = 1$, it is easy to see that $D_1 = P_1 = 1$. When $n = 2$ and $n = 3$, the formula is seen to hold. Assuming the formula holds for $n-1$, we prove it for $n \geq 3$.

Using (4), (2), the induction hypothesis and well-known Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \psi^n\right), \quad n \geq 0,$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1-\sqrt{5}}{2},$$

we then obtain

$$\begin{aligned}
D_n &= \sum_{k=1}^{n} P_{2k-1} D_{n-k} = \sum_{k=1}^{n-2} \left(2F_{2k-2} + (-1)^{2k-2}\right) D_{n-k} \\
&\quad + \left(2F_{2n-4} + (-1)^{2n-4}\right) D_1 + \left(2F_{2n-2} + (-1)^{2n-2}\right) D_0 \\
&= \sum_{k=1}^{n-2} \left(2F_{2k-2} + 1\right) D_{n-k} + 2F_{2n-4} + 2F_{2n-2} + 2 \\
&= \frac{2}{25} \sum_{k=1}^{n-2} \left(\frac{2}{\sqrt{5}} \left(\varphi^{2k-2} - \psi^{2k-2}\right) + 1 \right) \left(\left(\sqrt{5}\varphi\right)^{n+1-k} + \left(-\sqrt{5}\psi\right)^{n+1-k} \right) \\
&\quad + 2F_{2n-4} + 2F_{2n-2} + 2 \\
&= \frac{4(\sqrt{5}-2)}{25} \left(-\sqrt{5}\psi\right)^{n} \sum_{k=1}^{n-2} \left(\frac{5+2\sqrt{5}}{5} \right)^{k} - \frac{4(\sqrt{5}+2)}{25} \left(\sqrt{5}\varphi\right)^{n} \sum_{k=1}^{n-2} \left(\frac{5-2\sqrt{5}}{5} \right)^{k} \\
&\quad + \left(\frac{4\sqrt{5}}{25} \left(\sqrt{5}\varphi\right)^{n-1} + \frac{2}{25} \left(-\sqrt{5}\psi\right)^{n+1} \right) \sum_{k=1}^{n-2} \left(\frac{\sqrt{5}\varphi}{5} \right)^{k} \\
&\quad + \left(\frac{2}{25} \left(\sqrt{5}\varphi\right)^{n+1} - \frac{4\sqrt{5}}{25} \left(-\sqrt{5}\psi\right)^{n-1} \right) \sum_{k=1}^{n-2} \left(-\frac{\sqrt{5}\psi}{5} \right)^{k} \\
&\quad + 2F_{2n-4} + 2F_{2n-2} + 2.
\end{aligned} \tag{6}$$

Using the geometric series, it is seen that the following sums hold

$$\sum_{k=1}^{n-2} \left(\frac{5 \pm 2\sqrt{5}}{5} \right)^{k} = \frac{2 \pm \sqrt{5}}{2} \left(\left(\frac{5 \pm 2\sqrt{5}}{2} \right)^{n-2} - 1 \right),$$

$$\sum_{k=1}^{n-2} \left(\frac{5 \pm \sqrt{5}}{10} \right)^{k} = \frac{3 \pm \sqrt{5}}{2} \left(1 - \left(\frac{5 \pm \sqrt{5}}{10} \right)^{n-2} \right).$$

From (6), using the formulas above we obtain

$$\begin{aligned}
D_n &= \frac{3-\sqrt{5}}{5} \left(-\sqrt{5}\psi\right)^{n-1} + \frac{3+\sqrt{5}}{5} \left(\sqrt{5}\varphi\right)^{n-1} \\
&\quad + (\sqrt{5}-1) \left(\frac{3-\sqrt{5}}{2} \right)^{n-2} - (\sqrt{5}+1) \left(\frac{3+\sqrt{5}}{2} \right)^{n-2} + 2F_{2n-4} + 2F_{2n-2} \\
&= \frac{2}{25} \left(\left(\sqrt{5}\varphi\right)^{n+1} + \left(-\sqrt{5}\psi\right)^{n+1} \right) + (2\sqrt{5}+4)\psi^{2n} - (2\sqrt{5}-4)\varphi^{2n} + 2F_{2n-4} + 2F_{2n-2} \\
&= \frac{2}{25} \left(\left(\sqrt{5}\varphi\right)^{n+1} + \left(-\sqrt{5}\psi\right)^{n+1} \right) - 10F_{2n} + 4\left(\varphi^{2n} + \psi^{2n}\right) + 2F_{2n-4} + 2F_{2n-2} \\
&= \frac{2}{25} \left(\left(\sqrt{5}\varphi\right)^{n+1} + \left(-\sqrt{5}\psi\right)^{n+1} \right) - 10F_{2n} + 4(F_{2n-1} + F_{2n+1}) + 2F_{2n-4} + 2F_{2n-2} \\
&= \frac{2}{25} \left(\left(\sqrt{5}\varphi\right)^{n+1} + \left(-\sqrt{5}\psi\right)^{n+1} \right) + 2(F_{2n-3} + F_{2n-4} - F_{2n-2}) \\
&= \frac{2}{25} \left(\left(\sqrt{5}\varphi\right)^{n+1} + \left(-\sqrt{5}\psi\right)^{n+1} \right).
\end{aligned}$$

Since the stated formula holds for n, it follows by induction that it is true for all integers $n \geq 2$. $\square$

4 Applications of Trudi's formula

For any n-tuple $s = (s_1, s_2, \ldots, s_n)$ of integers $s_i \geq 0$ we will use the following notations $|s_n| = s_1 + \cdots + s_n$ and $m_n(s) = \binom{s_1+\cdots+s_n}{s_1,\ldots,s_n}$.

In this section, we use Trudi's formula (5) to obtain different combinatorial identities for Pell-Padovan and Fibonacci numbers. For example, Trudi's formula, taken together with Theorem 1, yields the following result.

Corollary 1. *We have*

$$F_{n-2} = \frac{(-1)^n}{2} \sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_1^{s_1} P_2^{s_2} \cdots P_n^{s_n}, \quad n \geq 2, \tag{7}$$

$$F_{2n} = \frac{(-1)^n}{2} \sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_3^{s_1} P_4^{s_2} \cdots P_{n+2}^{s_n}, \quad n \geq 2, \tag{8}$$

$$F_{n-4} = \frac{1}{2} \sum_{\sigma_n = n} (-1)^{|s_n|+1} m_n(s) P_3^{s_1} P_5^{s_2} \cdots P_{2n+1}^{s_n}, \quad n \geq 4, \tag{9}$$

$$F_{n-1} = \frac{1}{2} \sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_0^{s_1} P_2^{s_2} \cdots P_{2n-2}^{s_n} + 2^{n-2}, \quad n \geq 2,$$

where the summations are over all n-tuples $s = (s_1, s_2, \ldots, s_n)$ of integers $s_i \geq 0$ satisfying Diophantine equation $\sigma_n := s_1 + 2s_2 + \cdots + ns_n = n$.

Example 1. *When $n = 4$, formula (7) yields*

$$F_2 = \frac{1}{2} \sum_{\sigma_4 = 4} (-1)^{|s_4|} \frac{(s_1 + s_2 + s_3 + s_4)!}{s_1! s_2! s_3! s_4!} P_1^{s_1} P_2^{s_2} P_3^{s_3} P_4^{s_4}$$

or

$$P_1^4 - 3P_1^2 P_2 + 2P_1 P_3 + P_2^2 - P_4 = 2F_2.$$

Similarly, it follows from (8) and (9) respectively that

$$P_3^5 - 4P_3^3 P_4 + 3P_3^2 P_5 + 3P_3 P_4^2 - 2P_3 P_6 - 2P_4 P_5 + P_7 = 2F_0,$$
$$P_3^6 - 5P_3^4 P_5 + 4P_3^3 P_7 + 6P_3^2 P_5^2 - 3P_3^2 P_9 - 3P_3 P_5 P_7 + 2P_3 P_{11} - P_5^3 + 2P_5 P_9 + P_7^2 - P_{13} = -F_2.$$

In view of Corollary 1 and formula (2) we obtain the following Pell-Padovan identities.

Corollary 2. *We have*

$$P_{n-1} = (-1)^n \sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_1^{s_1} P_2^{s_2} \cdots P_n^{s_n} + (-1)^n, \quad n \geq 2,$$

$$P_{2n+1} = (-1)^n \sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_3^{s_1} P_4^{s_2} \cdots P_{n+2}^{s_n} + 1, \quad n \geq 2,$$

$$P_{n-3} = \sum_{\sigma_n = n} (-1)^{|s_n|+1} m_n(s) P_3^{s_1} P_5^{s_2} \cdots P_{2n+1}^{s_n} + (-1)^n, \quad n \geq 4,$$

$$P_n = \sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_0^{s_1} P_2^{s_2} \cdots P_{2n-2}^{s_n} + 2^{n-1} - (-1)^n, \quad n \geq 2,$$

where the summations are over integers $s_i \geq 0$ satisfying $\sigma_n = n$.

Using (2), now we obtain from Corollary 1 the following multinomial Fibonacci identities (many identities with Fibonacci numbers and their generalizations are given in [5,8–12]).

Corollary 3. *Let $n \geq 2$, except when noted otherwise. Then*

$$F_{n-2} = \frac{(-1)^n}{2} \sum_{\sigma_n = n} (-2)^{|s_n|} m_n(s) \left(F_0 + \frac{1}{2}\right)^{s_1} \left(F_1 - \frac{1}{2}\right)^{s_2} \cdots \left(F_{n-1} - \frac{(-1)^n}{2}\right)^{s_n},$$

$$F_{2n} = \frac{(-1)^n}{2} \sum_{\sigma_n = n} (-2)^{|s_n|} m_n(s) \left(F_2 + \frac{1}{2}\right)^{s_1} \left(F_3 - \frac{1}{2}\right)^{s_2} \cdots \left(F_{n+1} - \frac{(-1)^n}{2}\right)^{s_n},$$

$$F_{n-4} = -\frac{1}{2} \sum_{\sigma_n = n} (-2)^{|s_n|} m_n(s) \left(F_2 + \frac{1}{2}\right)^{s_1} \left(F_4 + \frac{1}{2}\right)^{s_2} \cdots \left(F_{2n} + \frac{1}{2}\right)^{s_n}, \quad n \geq 4,$$

$$F_{n-1} = \sum_{\sigma_n = n} \frac{(-2)^{|s_n|}}{2} m_n(s) \left(\frac{1}{2}\right)^{s_1} \left(F_1 - \frac{1}{2}\right)^{s_2} \left(F_3 - \frac{1}{2}\right)^{s_3} \cdots \left(F_{2n-3} - \frac{1}{2}\right)^{s_n} + \frac{2^n}{4},$$

where the summations are over integers $s_i \geq 0$ satisfying $\sigma_n = n$.

The determinant identities from Theorem 2 may be rewritten in terms of Trudi's formula as follows.

Corollary 4. *For $n \geq 2$, the following identities hold*

$$\sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_0^{s_1} P_1^{s_2} \cdots P_{n-1}^{s_n} = \sum_{k=0}^{n-4} \left((-1)^{k+1} - (-1)^n\right) \sum_{j=0}^{n-k-4} \binom{k}{j} \binom{j+1-\frac{n-k}{2}}{j},$$

$$\sum_{\sigma_n = n} m_n(s) P_0^{s_1} P_1^{s_2} \cdots P_{n-1}^{s_n}$$
$$= \frac{\sqrt{13}}{13} \left((3+\sqrt{13}) \left(\frac{1+\sqrt{13}}{2}\right)^{n-2} - (3-\sqrt{13}) \left(\frac{1-\sqrt{13}}{2}\right)^{n-2} \right),$$

$$\sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_2^{s_1} P_3^{s_2} \cdots P_{n+1}^{s_n} = (-1)^{\lfloor (4n+1)/3 \rfloor} + (-1)^{\lfloor (4n+2)/3 \rfloor}, \tag{10}$$

$$\sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_1^{s_1} P_3^{s_2} \cdots P_{2n-1}^{s_n}$$
$$= -\frac{2}{3} \left(\left(\frac{1-\sqrt{3}i}{2}\right)^{n-2} + \left(\frac{1+\sqrt{3}i}{2}\right)^{n-2} + 2^{n-2} \right),$$

$$\sum_{\sigma_n = n} m_n(s) P_1^{s_1} P_3^{s_2} \cdots P_{2n-1}^{s_n} = \frac{2}{25} \left(\left(\frac{5+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{5-\sqrt{5}}{2}\right)^{n+1} \right), \tag{11}$$

$$\sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_2^{s_1} P_4^{s_2} \cdots P_{2n}^{s_n} = 2 \sum_{j=0}^{\lfloor (n-1)/3 \rfloor} (-1)^{j+1} \binom{n-1}{3j+1},$$

$$\sum_{\sigma_n = n} (-1)^{|s_n|} m_n(s) P_4^{s_1} P_6^{s_2} \cdots P_{2n+2}^{s_n} = (-1)^{\lfloor (5n-2)/3 \rfloor} + (-1)^{\lfloor (5n-1)/3 \rfloor}. \tag{12}$$

Example 2. *It follows from* (10), (11), *and* (12) *that*

$$\begin{aligned}
&P_2^4 - 3P_2^2P_3 + 2P_2P_4 + P_3^2 - P_5 = 0,\\
&P_1^6 + 5P_1^4P_3 + 4P_1^3P_5 + 6P_1^2P_3^2 + 3P_1^2P_7\\
&\qquad\qquad + 3P_1P_3P_5 + 2P_1P_9 + P_3^3 + 2P_3P_7 + P_5^2 + P_{11} = 650,\\
&P_4^5 - 4P_4^3P_6 + 3P_4^2P_8 + 3P_4P_6^2 - 2P_4P_{10} - 2P_6P_8 + P_{12} = 0,
\end{aligned}$$

respectively.

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Гой Т., Шарин С. *Про числа Пелля-Падована та їхній зв'язок з числами Фібоначчі* // Карпатські матем. публ. — 2020. — Т.12, №2. — С. 280–288.

У цій роботі за допомогою визначників певних сімей матриць Тепліца-Гессенберга ми встановили деякі комбінаторні формули для чисел Пелля-Падована. Ці формули ми записуємо як тотожності, що включають суми добутків чисел Пелля-Падована та мультиноміальні коефіцієнти. Зокрема, це дозволило нам довести чотири формули, які встановлюють зв'язок між послідовностями Пелля-Падована та Фібоначчі.

Ключові слова і фрази: послідовність Пелля-Падована, послідовність Фібоначчі, матриця Тепліца-Гессенберга, формула Труді.