Карпатські математичні публікації. Т.2, №2

Carpathian Mathematical Publications. V.2, №2

УДК 515.12+512.58

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CHARACTERIZATION OF THE MACRO-CANTOR SET UP TO COARSE EQUIVALENCE

Zarichnyi I.M. *Characterization of the macro-Cantor set up to coarse equivalence*, Carpathian Mathematical Publications, **2**, 2 (2010), 39–47.

We characterize metric spaces that are coarsely equivalent to the macro-Cantor set $2^{<\mathbb{N}}$.

The well-known Cantor set

$$\Big\{\sum_{i=1}^{\infty} k_i \cdot 3^{-i} \mid (k_i)_{i=1}^{\infty} \in \{0,2\}^{\mathbb{N}}\Big\} \subset \mathbb{R}$$

has a macro analog

$$\Big\{\sum_{i=1}^{m} k_i \cdot 3^{i} \mid m \in \mathbb{N},\ \ (k_i)_{i=1}^{m} \in \{0,2\}^{m}\Big\} \subset \mathbb{R},$$

called the *macro-Cantor set* (see, e. g., [1]).

The macro-Cantor set plays the same role in the zero-dimensional asymptotic geometry as the Cantor set does in the zero-dimensional topology. It is well known that every zero-dimensional compact metric space without isolated points is homeomorphic to the Cantor set (see e. g. [3]). The main result of this paper is a characterization of metric spaces that are coarsely equivalent to the macro-Cantor set.

It is convenient to introduce the notion of coarse equivalence with help of multi-valued maps. By definition, a *multi-valued map* between sets X, Y is any function f that assigns to each point $x \in X$ a (possibly empty) subset $f(x) \subset Y$. Such a function f assigns to a subset $A \subset X$ the subset $f(A) = \bigcup_{a\in A} f(a)$ of Y.

The *oscillation* of a multi-map $\Phi : X \to Y$ between metric spaces is the function $\omega_\Phi : [0,\infty) \to [0,\infty]$ assigning to each $\delta \geq 0$ the (finite or infinite) number

$$\omega_\Phi(\delta) = \sup\{\operatorname{diam}(\Phi(A)) \mid A \subset X,\ \operatorname{diam}(A) \leq \delta\}.$$

2000 *Mathematics Subject Classification:* 54E15, 54E35.
Key words and phrases: asymptotic dimension zero, macro-Cantor set, coarse equivalence.

Definition 1. *A multi-valued map $\Phi : X \to Y$ between metric spaces X, Y is called*

- macro-uniform, *if $\omega_\Phi(\delta)$ is finite for each $\delta < \infty$;*
- *a* coarse equivalence, *if $\Phi(X) = Y$, $\Phi^{-1}(Y) = X$ and both multi-valued maps Φ and Φ^{-1} are macro-uniform.*

Two metric spaces X, Y are called *coarsely equivalent* if there is a coarse equivalence $f : X \to Y$. In particular, the macro-Cantor set is coarsely equivalent to the macro-Cantor cube

$$2^{<\mathbb{N}} = \{(x_i)_{i\in\omega} \in \{0,1\}^\omega \mid \exists n \in \omega\ \forall n \geq m\ \ (x_n = 0)\}$$

endowed with the metric

$$d((x_i),(y_i)) = \min\{i \in \omega \mid x_j = y_j,\ \text{for all } j > i\}.$$

In the sequel, for a metric space (Y, ρ) and a subset $C \subset Y$ by $U_\varepsilon(C)$ we denote the ε-neighborhood of C in Y. For any nonempty sets $A, B \subset Y$ we put

$$\operatorname{dist}(A,B) = \inf\{\rho(a,b) \mid a \in A,\ b \in B\}.$$

The following is a characterization theorem for the macro-Cantor set.

Theorem 1. *A metric space (Y, ρ) is coarsely equivalent to the macro-Cantor set if and only if there exist numbers $a > 0$, $n \in \mathbb{N}$ and monotonically increasing divergent sequences $(a_i)_{i\in\mathbb{N}}$, $(n_i)_{i\in\mathbb{N}}$ of real and natural numbers respectively, such that the following holds: for every i the set Y can be written as the disjoint union of a countable family of sets $\{Y_j\}_{j\in\mathbb{N}}$, such that for every $j, k \in \mathbb{N}$ $\operatorname{diam}(Y_j) \leq a_i$, $\operatorname{dist}(Y_j, Y_k) > a_{i-1}$ and the set Y_j can be covered by 2^{n_i+n} sets and cannot be covered by less than 2^{n_i} sets of diameter not exceeding a.*

Proof. Without loss of generality we can assume that $n_{i-1} - n_{i-2} - n > 2$ for every i.

Necessity. Let a metric space (Y, ρ) be coarsely equivalent to the macro-Cantor set X. Then consider a multi-valued map $f\colon X \to Y$ from the definition of coarse equivalence and define sequences $\{a_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$ in the following way.

Put $b_1 = 1$. Suppose that we have defined $b_1, \dots b_i$ and $a_1, \dots, a_{i-1}$. From the definition of coarse equivalence for f there exist natural numbers $a_i > \omega_f(b_i)$ and $b_{i+1} > \omega_{f^{-1}}(a_i)$.

Let $i \in \mathbb{N}$. We can represent X as the union $X = \bigcup\limits_{j\in\mathbb{N}} X^i_j$, where $\operatorname{diam}(X^i_j) = b_i$, $U_{b_i}(X^i_j) = X^i_j$ for all $j \in \mathbb{N}$. Then for all $i, j \in \mathbb{N}$ define $Y^i_j = f(X^i_j)$. It is easy to see that $Y = \bigcup\limits_{j\in\mathbb{N}} Y^i_j$, $\operatorname{diam}(Y^i_j) = a_i$ for all $j \in \mathbb{N}$, $i \in \mathbb{N}$.

Since for all $i > 1$ $\operatorname{dist}(X^i_j, X^i_k) > b_i$, $i, j \in \mathbb{N}$, we easily obtain that $\operatorname{dist}(Y^i_j, Y^i_k) > a_{i-1}$, $j, k \in \mathbb{N}$.

Then we can see that for any $i, j, k \in \mathbb{N}$, $i < j$, there exists a unique $l \in \mathbb{N}$ such that $Y^i_k \subset Y^j_l$.

Note, that for all $i > 1, j \in \mathbb{N}$, the set Y^i_j can be written as the union $Y^i_j = \bigcup\limits_{k=1}^{2^{b_i-b_1}} Y^1_{l_k}$. The set Y^i_j can be covered by at most $2^{b_i-b_1}$ sets of diameter a_1.

Similarly, $Y^i_j = \bigcup_{k=1}^{2^{b_i-b_2}} Y^2_{l_k}$. Since the distance between the sets Y^2_p and Y^2_q is greater then a_1, the set Y^i_j cannot be covered by less than $2^{b_i-b_2}$ sets of diameter not exceeding a_1.

The necessity is proved.

Sufficiency. Let Y be a metric space, numbers $a > 0$, $n \in \mathbb{N}$, and monotonically increasing sequences $(a_i)_{i\in\mathbb{N}}$, $(n_i)_{i\in\mathbb{N}}$, of real and natural numbers respectively are from the conditions of the theorem.

Let X denotes the macro-Cantor set and

$$X^i_j = \{x = (x_1, x_2, x_3, \dots) \mid x_i = \eta_1, x_{i+1} = \eta_2, \dots, \},$$

where

$$j = 1 + \eta_1 + 2\cdot\eta_2 + 2^2\cdot\eta_3 + \dots + a_p\cdot 2^{p-1} + \dots,$$

$\eta_k \in \{0,1\}$. It is easy to see that $\operatorname{diam}(X^i_j) < 2^i$, and for every $a < b, c, d$, either $X^a_c \subset X^b_d$ or $X^a_c \cap X^b_d = \varnothing$.

We will use that $\Theta(A)$ is the minimal natural number k such that A can be written as the disjoint union of k balls of diameter not exceeding a.

For every natural i let $Y = Y^i_1 \cup Y^i_2 \cup \dots$ be a decomposition such that for every natural j, k, $\operatorname{diam}(Y^i_j) \le a_i$, $\operatorname{dist}(Y^i_j, Y^i_k) > a_{i-1}$ and $2^{n_j} \le \Theta(Y^j_k) \le 2^{n_j+n}$ for every natural j, k. Let $\Theta^j_{\max} = \max_k \Theta(Y^j_k)$.

We assume that

$$Y^k_1 = Y^t_1 \cup Y^t_2 \cup \dots \cup Y^t_{r_1},$$
$$Y^k_2 = Y^t_{r_1+1} \cup Y^t_{r_1+2} \cup \dots \cup Y^t_{r_2},$$
$$\dots$$

for every natural k, t, $k > t$.

Step a). Consider a sequence of real numbers (α_k) such that $1 < \alpha_k < 2$, $\prod_{k\in\mathbb{N}} \alpha_k = 2$, $\alpha_0 = 1$.

Let us construct sequences of natural numbers $(c_i)_{i\in\mathbb{N}}, (d_i)_{i\in\mathbb{N}}$ by induction. Let $d_1 = 1$, and let $c_i > d_i$ be such that

$$1 + \frac{2^{n_{d_i}+n}\cdot 2^{n_{d_1}+3n+2}\cdot 8}{2^{n_{c_i}}} \le \alpha_{2i+1}, \qquad 1 - \frac{2^{n_{d_i}}\cdot 2^{n_{d_1}+3n+2}}{2^{n_{c_i}+n}\cdot 8} \ge \frac{1}{\alpha_{2i+1}}, \tag{a(2i+1)}$$

$d_i > c_{i-1}$ and for any $t > 2^{n_{d_i}}$,

$$t + \Theta^{c_{i-1}}_{\max} \le t\cdot\alpha_{2i}, \qquad t - \Theta^{c_{i-1}}_{\max} \ge t\cdot\frac{1}{\alpha_{2i}}. \tag{a(2i)}$$

Let us consider the following conditions for a multi-valued function $f : A \to B$:

$$\omega_f(a_{c_{i-1}}) \le n_{d_i}, \quad \omega_{f^{-1}}(n_{d_i}) \le a_{c_i}. \tag{f(i)}$$

Let $p' = 2^{n_{d_1}+3n+2}$.

Step b). During this step for every natural i we have to construct a multi-valued surjective function $f_i(A,B)(x) : A \to B$, which maps the set $A \subset Y$ into $B \subset X$. Here $A = Y_{l_1}^{d_i} \cup \dots \cup Y_{l_p}^{d_i}$, $B = X_k^{n_{d_i}}$, $1 \le p \le p'$. Also the function f_i must satisfy conditions $(f(1))$, $(f(2))$, ..., $(f(i))$.

Fix $i \in \mathbb{N}$, and let A, B be sets. Let us construct $f_i(A,B)$ by induction. Without loss of generality we can assume that

$$A = Y_1^{d_i} \cup \dots \cup Y_p^{d_i}, \quad B = X_1^{n_{d_i}}.$$

It is easy to see that

$$p \cdot 2^{n_{d_i}} \le \Theta(A) \le p \cdot 2^{n_{d_i}+n}.$$

Base of induction. Let $A_1^0 = A$.

Step j, $j \in \{1, \dots, i-1\}$. We see that the set Y is written as a disjoint union of sets

$$Y = A_1^{j-1} \cup \dots \cup A_{2^{n_{d_i}-n_{d_{i-j+1}}}}^{j-1}$$

such that for any $k \in \{1, \dots, 2^{n_{d_i}-n_{d_{i-j+1}}}\}$

$$\frac{\Theta(A)}{2^{n_{d_i}-n_{d_{i-j+1}}}} \cdot \prod_{t=0}^{2(j-1)} \frac{1}{\alpha_t} \le \Theta(A_k^{j-1}) \le \frac{\Theta(A)}{2^{n_{d_i}-n_{d_{i-j+1}}}} \cdot \prod_{t=0}^{2(j-1)} \alpha_t,$$

the set A_k^{j-1} is the union of sets from the family $\{Y_q^{c_{i-j}}\}$, and it is assumed that

$$f_i(A,B)(A_k^{j-1}) = X_k^{n_{d_{i-j+1}}}, \quad f_i^{-1}(A,B)(X_k^{n_{d_{i-j+1}}}) = A_k^{j-1}.$$

Consider the set A_k^{j-1}. We can represent it as the union

$$A_1^{j-1} = Y_{k_1}^{c_{i-j}} \cup \dots \cup Y_{k_s}^{c_{i-j}}.$$

Now write the set $X_k^{n_{d_{i-j+1}}}$ as the disjoint union of $2^{n_{d_{i-j+1}}-n_{d_{i-j}}} = \xi$ sets, $X_k^{n_{d_{i-j+1}}} = X_{e_1}^{n_{d_{i-j}}} \cup \dots \cup X_{e_f}^{n_{d_{i-j}}}$. We have to divide these sets between the sets $Y_{k_r}^{c_{i-j}}$.

Let $\beta' : \{e_1, \dots, e_f\} \to \{k_1, \dots, k_s\}$ be a surjective function. Note that $\xi > s$. Let $\beta(k_r) = |\{e_t | \beta'(e_t) = k_r\}|$. We have to find β' that minimizes the difference of $\frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)}$.

We see that $\sum\limits_{l=1}^{s} \Theta(Y_{k_l}^{c_{i-j}}) = \Theta(A_k^{j-1})$. Thus there is β' such that, for every $l \in \{1, \dots, s\}$,

$$\frac{\Theta(A_k^{j-1})}{\xi}(\beta(k_r) - 2) \le \Theta(Y_{k_r}^{c_{i-j}}) \le \frac{\Theta(A_k^{j-1})}{\xi}(\beta(k_r) + 2),$$

$$\frac{\Theta(A_k^{j-1})}{\xi}(1 - \frac{2}{\beta(k_r)}) \le \frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} \le \frac{\Theta(A_k^{j-1})}{\xi}(1 + \frac{2}{\beta(k_r)}).$$

Now let us look at $\beta(k_r)$:

$$\frac{\Theta(Y_{k_r}^{c_{i-j}}) \cdot f}{\Theta(A_k^{j-1})} \cdot \frac{1}{4} \le \beta(k_r),$$

$$\beta(k_r) \geq \frac{\Theta(Y_{k_r}^{c_{i-j}}) \cdot \xi}{\Theta(A_k^{j-1}) \cdot 2} \geq \frac{2^{n_{c_{i-j}}} \cdot 2^{n_{d_{i-j+1}}} \cdot 2^{n_{d_i}}}{2^{n_{d_{i-j}}} \cdot p \cdot 2^{n_{d_i}+n} \cdot 2^{n_{d_{i-j+1}}} \cdot 4} \geq \frac{2^{n_{c_{i-j}}}}{2^{n_{d_{i-j}}+n} \cdot 4p}.$$

Then by condition $(a(2(i-j)+1))$

$$(1 + \frac{2}{\beta(k_r)}) \leq 1 + \frac{2^{n_{d_{i-j}}+n} \cdot 4p}{2^{n_{c_{i-j}}}} \leq \alpha_{2(i-j)+1},$$

$$(1 - \frac{2}{\beta(k_r)}) \geq 1 - \frac{2^{n_{d_{i-j}}} \cdot p}{2^{n_{c_{i-j}}+n} \cdot 4} \geq \frac{1}{\alpha_{2(i-j)+1}}.$$

We see that

$$\frac{\Theta(A)}{2^{n_{d_i}-n_{d_{i-j}}}} \cdot \prod_{t=2(i-j)+1}^{2i-1} \frac{1}{\alpha_t} \leq \frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} \leq \frac{\Theta(A)}{2^{n_{d_i}-n_{d_{i-j}}}} \cdot \prod_{t=2(i-j)+1}^{2i-1} \alpha_t.$$

Now consider the set $Y_{k_r}^{c_{i-j}}$ and $\beta(k_r)$. The sets $X_{o_1}^{n_{d_{i-j}}}, \ldots, X_{o_{\beta(k_r)}}^{n_{d_{i-j}}}$ are mapped to the set $Y_{k_r}^{c_{i-j}}$. Represent the set $Y_{k_r}^{c_{i-j}}$ as the disjoint union of sets

$$Y_{q_1}^{c_{i-j-1}} \cup \cdots \cup Y_{q_l}^{c_{i-j-1}}.$$

Now map them into the sets $X_{o_1}^{n_{d_{i-j}}}, \ldots, X_{o_{\beta(k_r)}}^{n_{d_{i-j}}}$ to minimize the difference of $\Theta(A_k^j)$, where $A_k^j = f^{-1}(X_k^{n_{d_{i-j}}})$.

This can be done so that

$$\frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} - \Theta_{\max}^{c_{i-j-1}} \leq \Theta(A_k^j) \leq \frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} + \Theta_{\max}^{c_{i-j-1}}.$$

We see that

$$\frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} \geq p \cdot 2^{n_{d_{i-j}}} \cdot \frac{1}{2} \geq 2^{n_{d_{i-j}}-1},$$

then by condition $(a(2(i-j)))$

$$\frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} + \Theta_{\max}^{c_{i-j-1}} \leq \frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} \cdot \alpha_{2(i-j)},$$

$$\frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} \cdot \frac{1}{\alpha_{2(i-j)}} \leq \frac{\Theta(Y_{k_r}^{c_{i-j}})}{\beta(k_r)} - \Theta_{\max}^{c_{i-j-1}}.$$

Thus,

$$\frac{\Theta(A)}{2^{n_{d_i}-n_{d_{i-j}}}} \cdot \prod_{t=2(i-j)}^{2i-1} \frac{1}{\alpha_t} \leq \Theta(A_k^j) \leq \frac{\Theta(A)}{2^{n_{d_i}-n_{d_{i-j}}}} \cdot \prod_{t=2(i-j)}^{2i-1} \alpha_t.$$

Assume that $f(A_k^j) = X_k^{d_{i-j}}$, $f^{-1}(X_k^{d_{i-j}}) = A_k^j$. As a result the following condition $(f(i-j))$ holds:

$$\omega_f(a_{c_{i-j-1}}) \leq n_{d_{i-j}} \quad \omega_{f^{-1}}(n_{d_{i-j}}) \leq a_{c_{i-j}}.$$

After step $(i-1)$ we obtain our function. The set A is written as the union of the family of sets $\{A_k^{i-1}\}$. For every k

$$0 < p \cdot 2^{n_{d_1}} \le \frac{\Theta(A)}{2^{n_{d_i}-n_{d_1}}} \cdot \prod_{t=2}^{2i-1} \frac{1}{\alpha_t} \le \Theta(A_k^{i-1}),$$

therefore $A_k^{(i-1)}$ is nonempty. For every k and for every $x \in A_k^{(i-1)}$ let $f(A,B)(x) = X_k^{n_{d_1}}$.

Note, that the constructed function satisfies conditions $(f(1))$–$(f(i))$.

Step c). Now we have a sequence (f_i) of functions. We have to construct function from Y to X. We can write Y as the union of the sets

$$Y_1 = Y_1^{d_1}, \quad Y_i = (Y_1^{d_i} \cup \dots \cup Y_{2^n}^{d_i}) \setminus Y_{i-1}.$$

Let $X = X_1 \cup X_2 \cup \dots$, where $X_1 = X_1^{n_{d_1}}$, and $X_i = X_1^{n_{d_i}} \setminus X_1^{n_{d_{i-1}}}$ for $i \in \mathbb{N} \setminus \{1\}$. For every i we have to map $(Y_1^{d_i} \setminus Y_1^{d_{i-1}})$ into X_i.

Consider step i. We see that $Y_i = Y_{l_1}^{d_{i-1}} \cup \dots \cup Y_{l_t}^{d_{i-1}}$. Also $2^{n_{d_i}+n} \le \Theta(Y_i) \le 2^{n_{d_i}+2n}$ and $2^{n_{d_{i-1}}} \le \Theta(Y_j^{d_{i-1}}) \le 2^{n_{d_{i-1}}+n}$. Then $2^{n_{d_i}-n_{d_{i-1}}} \le t \le 2^{n_{d_i}-n_{d_{i-1}}+2n}$.

We have $X_i = X_1^{n_{d_i}} \setminus X_1^{n_{d_{i-1}}} = X_{m_1}^{n_{d_{i-1}}} \cup \dots \cup X_{m_u}^{n_{d_{i-1}}}$. It is easy to see that $u = 2^{n_{d_i}-n_{d_{i-1}}} - 1$. Also $u < t < up'$.

Now we can write Y_i as the disjoint union of sets $Y_i = Y_{(i,1)} \cup \dots \cup Y_{(i,u)}$, where every $Y_{(i,k)}$ is the union of w sets of the family $(Y_{l_r}^{d_{i-1}})$, $1 \le w \le p'$.

Now for every $k \in \{1, \dots, u\}$ using the function $f_i(Y_{(i,k)}, X_{m_k}^{n_{d_{i-1}}})$ we shall map the set $Y_{(i,k)}$ into the set $X_{m_k}^{n_{d_{i-1}}}$. □

The last theorem can be reformulated.

Theorem 2. *A metric space (Y,ρ) is coarsely equivalent to the macro-Cantor set if and only if there exist numbers $a > 0$, $n \in \mathbb{N}$ and monotonically increasing divergent sequences $(a_i)_{i\in\mathbb{N}}$, $(n_i)_{i\in\mathbb{N}}$ of real and natural numbers respectively, such that the following holds: for every i the set Y can be written as the disjoint union of a countable family of sets $\{Y_j\}_{j\in\mathbb{N}}$, such that for every $j,k \in \mathbb{N}$, $\mathrm{diam}(Y_j) \le a_i$, $\mathrm{dist}(Y_j, Y_k) > a_{i-1}$ and the set Y_j can be covered by $n \cdot n_i$ sets and cannot be covered by less than n_i sets of diameter not exceeding a.*

Now using Theorem 2 we can prove its more general version.

Theorem 3. *A metric space (Y,ρ) is coarsely equivalent to the macro-Cantor set if and only if there exist monotonically increasing divergent sequences $(a_i)_{i\in\mathbb{N}\cup\{0\}}$ of reals, $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ of naturals, such that the following holds: for every i the set Y can be written as the disjoint union of a countable family of sets $\{Y_j\}_{j\in\mathbb{N}}$, such that for every $j,k \in \mathbb{N}$ $\mathrm{diam}(Y_j) \le a_i$, $\mathrm{dist}(Y_j,Y_k) > a_{i-1}$ and the set Y_j can be covered by m_i sets and cannot be covered by less than n_i sets of diameter not exceeding a_0.*

Proof. To prove this theorem we will show that for the space Y the conditions from Theorem 2 hold true. We will construct monotonically increasing sequences $(b_i)_{i\in\mathbb{N}}$ and (k_i) of real and natural numbers respectively, such that for all $i \in \mathbb{N}$ the set Y can be written as disjoint union of a countable family of sets $\{Z_j^i\}_{j\in\mathbb{N}}$, such that for all $j, l \in \mathbb{N}$ $\operatorname{diam}(Z_j^i) \le b_i, \operatorname{dist}(Z_j^i, Z_l^i) > b_{i-1}$ and the set Z_j^i can be covered by k_i sets and cannot be covered by less than $k \cdot k_i$ sets of diameter not exceeding $b_0 = a_0$.

By the formulation of the theorem, for all $i \in \mathbb{N}$ set Y can be written as disjoint union of a countable family of sets which we will denote by $\{Y_j^i\}_{j\in\mathbb{N}}$.

Define $k = \max\{3, [\frac{m_1}{n_1}] + 1\}$.

Base of induction. Put $b_1 = a_1$, $k_1 = n_1$, $t_1 = 1$. For all $j \in \mathbb{N}$, let $Z_j^1 = Y_j^1$. It is easy to see that, for the family $\{Z_j^1\}_{j\in\mathbb{N}}$, all conditions hold.

i-th step of induction, $i > 1$. We have a natural number t_{i-1} and a real number b_{i-1} such that $b_{i-1} = a_{t_{i-1}}$. We have to find numbers $b_i > b_{i-1}$ and k_i, and write Y as disjoint union of a countable family of sets $\{Z_j^i\}_{j\in\mathbb{N}}$, such that for all $j, l \in \mathbb{N}$ $\operatorname{diam}(Z_j^i) \le b_i, \operatorname{dist}(Z_j^i, Z_l^i) > b_{i-1}$ and the set Z_j^i can be covered by $k \cdot k_i$ sets and cannot be covered by less than k_i sets of diameter b_0.

Consider the family of sets $\{Y_j^{t_{i-1}+1}\}_{j\in\mathbb{N}}$. The mutual distances between the distinct elements of this family are at least $a_{t_{i-1}}$. Every of these sets can be covered by $n_{t_{i-1}+1}$ sets and cannot be covered by less than $m_{t_{i-1}+1}$ sets of diameter not exceeding b_0. Put $k_i = m_{t_{i-1}+1}$.

Take a number t_i, such that $n_{t_i} > m_{t_{i-1}+1}$. Consider the family of sets $\{Y_j^{t_i}\}_{j\in\mathbb{N}}$. The diameter of each of them is less than a_{t_i}. Put $b_i = a_{t_i}$. Each of them can be covered by m_{t_i} and cannot be covered by less than n_{t_i} of sets of diameter b_0.

For all $u \in \mathbb{N}$ consider the set $Y_u^{t_i}$. This set can be represented as disjoint union of a finite number of sets from the family $\{Y_j^{t_{i-1}+1}\}_{j\in\mathbb{N}}$. Without loss of generality we can write $Y_u^{t_i} = Y_1^{t_{i-1}+1} \cup Y_2^{t_{i-1}+1} \cup \cdots \cup Y_v^{t_{i-1}+1}$. Each of the sets $Y_p^{t_{i-1}+1}$, $p \in \{1, \ldots, v\}$, can be covered by $m_{t_{i-1}+1}$ sets of diameter b_0. The set $Y_u^{t_i}$ cannot be covered by less than $n_{t_i} > m_{t_{i-1}+1}$ sets of diameter b_0.

Put $p_0 = 0$. There exist numbers $p_1, p_2, \ldots, p_q$, $0 = p_0 < p_1 < p_2 < \cdots < p_q < v$, such that the sets $Y_{p_{l-1}+1}^{t_{i-1}+1} \cup Y_{p_{l-1}+2}^{t_{i-1}+1} \cup \cdots \cup Y_{p_l}^{t_{i-1}+1}$ can be covered by $2 \cdot m_{t_{i-1}+1}$ and cannot be covered by less than $m_{t_{i-1}+1}$ sets, and the set $Y_{p_q+1}^{t_{i-1}+1} \cup Y_{p_q+2}^{t_{i-1}+1} \cup \cdots \cup Y_v^{t_{i-1}+1}$ can be covered by $m_{t_{i-1}+1}$ sets of diameter b_0. Then define

$$Z_{u1}^i = Y_1^{t_{i-1}+1} \cup Y_2^{t_{i-1}+1} \cup \cdots \cup Y_{p_1}^{t_{i-1}+1},$$

$$Z_{u2}^i = Y_{p_1+1}^{t_{i-1}+1} \cup Y_{p_1+2}^{t_{i-1}+1} \cup \cdots \cup Y_{p_2}^{t_{i-1}+1},$$

$$\ldots$$

$$Z_{u,(q-1)}^i = Y_{p_{q-2}+1}^{t_{i-1}+1} \cup Y_{p_{q-2}+2}^{t_{i-1}+1} \cup \cdots \cup Y_{p_{q-1}}^{t_{i-1}+1},$$

$$Z_{u,(q)}^i = Y_{p_{q-1}+1}^{t_{i-1}+1} \cup Y_{p_{q-1}+2}^{t_{i-1}+1} \cup \cdots \cup Y_{p_q}^{t_{i-1}+1} \cup Y_{p_q+1}^{t_{i-1}+1} \cup \cdots \cup Y_u^{t_{i-1}+1}.$$

Put $u' = q$.

It is easy to see that the sets Z_{ur}^i, $r \in \{1, \ldots, q-1\}$, can be covered by $2 \cdot m_{t_{i-1}+1} \le k \cdot k_i$ sets and cannot be covered by less than $m_{t_{i-1}+1} = k_i$ sets of diameter b_0. The set Z_{uq}^i cannot be covered by less than $m_{t_{i-1}+1} = k_i$, can be covered by $3 \cdot m_{t_{i-1}+1} \le k \cdot k_i$ sets of diameter b_0. Also the diameters of these sets are less than $a_{t_i} = b_i$, and their pairwise distances are less than $a_{t_{i-1}} = b_{i-1}$.

So we represent the set Y as disjoint union of a countable family of sets, $Y = \bigcup\{Z^i_{p,q} | p \in \mathbb{N}, q = 1, \dots, p'\}$, for which all conditions are true. Now we can enumerate these sets by naturals and we shall represent Y as the disjoint union of the family $\{Z^i_j\}_{j \in \mathbb{N}}$. □

Definition 2. *A metric space (X, d) is called* asymptotically zero-dimensional *if for all $a > 0$ there exists a uniformly bounded a-disjoint cover of X.*

A cover $\mathcal{U}$ of metric space X is called

- *uniformly bounded* if its mesh $\sup\{\operatorname{diam} U : U \in \mathcal{U}\}$ is finite.
- *a-disjoint* if $\operatorname{dist}(A, B) > a$ for every $A, B \in \mathcal{U}$.

Theorem 4. *A metric asymptotically zero-dimensional space (X, ρ) is coarsely equivalent to macro-Cantor set if and only if there exists number $a > 0$, and the following conditions are true:*

1) for every $n \in \mathbb{N}$ there exists $r \in \mathbb{N}$, such that for any $x \in X$ the r-ball $U_r(x)$ cannot be covered by less than n balls of radius a,

2) for every $r \in \mathbb{N}$ there exists $m \in \mathbb{N}$, such that each r-ball $U_r(x)$ can be covered by m balls of radius a.

Proof. Necessity. By the Theorem 3 there exist monotonically increasing sequences $(a_i)_{i \in \mathbb{N} \cup \{0\}}$ of reals, $(n_i)_{i \in \mathbb{N}}$ and $(m_i)_{i \in \mathbb{N}}$ of natural numbers. Put $a = a_0$. We will show that conditions 1) and 2) are true.

a) Consider an arbitrary natural n. Then there exists j, such that $n_j > n$. Put $d = a_{j+1}$. It is easy to see that condition 1) is true.

b) Consider an arbitrary number d. Then there exists such j, that $a_j > d$. Put $m = m_{j+1}$. Easy to see that condition 2) is true.

Sufficiency. Suppose that (X, ρ) is a space and conditions 1) and 2) are true. We shall construct by induction monotonically increasing sequences $(a_i)_{i \in \mathbb{N} \cup \{0\}}$ of real, $(n_i)_{i \in \mathbb{N}}$ and $(m_i)_{i \in \mathbb{N}}$ of natural numbers to satisfy conditions of the Theorem 3.

Base of induction. Put $a_0 = a$, $m_0 = 1$.

i-th step of induction, $i \in \mathbb{N}$. Put $n_i = m_{i-1} + 1$. By condition 1) for number n_i there exists d. By definition of asymptotic dimension zero, for the space X there exists a totally bounded a_{i-1}-disjoint cover. Let b be the mesh of this cover. Put $a_i = \max\{b, d, a_{i-1} + 1\}$.

It is easy to see that this sequence satisfies the conditions of the Theorem 3. □

It is well known that every zero-dimensional compact metric space without isolated points is homeomorphic to the Cantor set. In our characterization the first condition is an analogue of "space without isolated points" in metric geometry.

Applying Characterization Theorem 4 one can easily prove the next corollary.

Corollary 1. *For every $n \in \mathbb{N}$ the hyperspace $\exp_n(2^{<\mathbb{N}})$ is coarsely equivalent to $2^{<\mathbb{N}}$.*

Here for a metric space Y by $\exp_n(Y)$ we denote the space of all at most n-element non-empty subsets of Y endowed with the Hausdorff distance

$$\rho_H(A, B) = \inf\{\varepsilon > 0 : A \subset U_\varepsilon(B), B \subset U_\varepsilon(A)\}.$$

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Received 22.10.2010

Зарічний І.М. *Характеризація макро-канторової множини з точністю до грубої еквівалентності* // Карпатські математичні публікації. — 2010. — Т.2, №2. — С. 39–47.

Наведено характеризацію метричних просторів, грубо еквівалентних до макроканторової множини $2^{<\mathbb{N}}$.

Заричный И.М. *Характеризацыя макро-канторового множества с точностью до грубой еквивалентности* // Карпатские математические публикации. — 2010. — Т.2, №2. — С. 39–47.

Приводится характеризиризацыя метрических пространств, грубо еквивалентных макроканторову множеству $2^{<\mathbb{N}}$.