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ATOMIZED MEASURES AND SEMICONVEX COMPACTA

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A class of atomized measures on compacta, which are generalizations of regular real-valued measures, is introduced. It has also been shown that the space of normalized (weakly) atomized measures on a compactum is a free object over this compactum in the category of (strongly) semiconvex compacta.

Introduction

It has been known for a long that the space PX of probability measures on a compact Hausdorff space X with the weak* topology is a *convex compactum*, i.e. it can be embedded into a locally convex topological vector space as a compact convex set. Moreover, it is a *free convex compactum* [3] *over* X, i.e. it contains X as a closed subspace so that each continuous mapping from X to a convex compactum K can be uniquely extended to an *affine* continuous mapping from PX to K.

Some applications require the class of convex compacta to be extended to the class of so-called *semiconvex compacta* [8]. The goal of this work is to show that free semiconvex compacta can also be obtained as spaces of special measures, which we call *atomized.*

We use the following terminology and denotations : $I = [0,1]$ is *a unit segment*, $\mathbb{R}_+ = [0;+\infty)$, $\mathbb{R}^+ = (0;+\infty)$, $\mathbb{Q}^+ = \mathbb{Q} \cap (0;+\infty)$. A *compactum* is a (not necessarily metrizable) (bi)compact Hausdorff topological space.

For basic definitions and facts of the category theory cf. [7]. The *category of Tychonoff spaces* $\mathcal{T}$ych and the *category of compacta* $\mathcal{C}$omp consist of all Tychonoff spaces and all compact Hausdorff spaces, respectively, and their continuous maps.

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1 Atomized measures

In the sequel X is a compactum, $\operatorname{Exp} X$ is the collection of all closed subsets of X, and $\exp X = \operatorname{Exp} X \setminus \{\varnothing\}$. Each regular real-valued additive measure on X is uniquely determined by its values on closed subsets of X, hence the following is equivalent to the usual definition:

Definition. *A function $m : \operatorname{Exp} X \to \mathbb{R}$ is a* regular additive measure *on X if, for all $A, B \in \operatorname{Exp} X$:*

(1) $m(\varnothing) = 0$;

*(2) $A \subset B$ implies $m(A) \leqslant m(B)$ (*monotonicity*);*

(3) $m(A \cup B) + m(A \cap B) = m(A) + m(B)$ (a property which is equivalent to additivity*);*

*(4) for each filtered subcollection $\mathcal{A} \subset \operatorname{Exp} X$, the equality $m(\bigcap \mathcal{A}) = \inf_{A \in \mathcal{A}} m(A)$ (τ-*smoothness*, which is equivalent to outer regularity) is valid.*

Obviously (1), (2) imply $m(\operatorname{Exp} X) \subset \mathbb{R}_+$. From now on all measures are considered regular. For a closed subset $X_0 \subset X$ and a measure m on X, the restriction of m to $\operatorname{Exp} X_0$ is a measure as well, and it is denoted by $m|_{X_0}$ for brevity.

We denote $|m| = m(X)$. If $|m| = 1$ ($|m| \leqslant 1$), then the measure m is *normalized* or a *probability measure* (resp. a *subnormalized measure*). We denote by $\overline{P}X$ the set of all measures on X, and PX and $\underline{P}X$ are its subsets that consist of all normalized and all subnormalized measures respectively. The three sets $PX \subset \underline{P}X \subset \overline{P}X$ are considered with the weak* topologies [3]. Recall that PX and $\underline{P}X$ are compacta, and $\overline{P}X$ is not compact, but it is a Tychonoff space.

For each continuous mapping of compacta $f : X \to Y$ and a measure m on X, the set function $m' : \operatorname{Exp} Y \to \mathbb{R}$, $m'(B) = m(f^{-1}(B))$, for all $B \underset{\mathrm{cl}}{\subset} Y$, is a measure on Y as well. We denote m' by $\overline{P}f(m)$ and obtain a mapping $\overline{P}f : \overline{P}X \to \overline{P}Y$, which is continuous. Thus a functor [7] $\overline{P} : \mathcal{C}\mathrm{omp} \to \mathcal{T}\mathrm{ych}$ is obtained. Since m' is normalized (subnormalized) whenever m belongs to the respective class, we have subfunctors $P : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$, $\underline{P} : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$ of $\overline{P}$. The functor P is the famous *probability measure functor* [3].

A measure m on a compactum X is *purely atomic* if there is a finite or countable sequence $(x_i, p_i)_{i \in \mathcal{I}}$ in $X \times (0, +\infty)$ such that, for each $A \in \operatorname{Exp} X$, $m(A)$ is equal to $\sum\{p_i \mid i \in \mathcal{I}, x_i \in A\}$. This implies that $\sum_{i \in \mathcal{I}} p_i$ is finite, and it is equal to 1 (is not greater that 1) if and only if m is normalized (resp. subnormalized). The points x_i are called *atoms* of the measure m, and p_i are their *masses*. Recall that the *Dirac measure* δ_x concentrated in $x \in X$ is the set function

$$\delta_x(A) = \begin{cases} 1, x \in A, \\ 0, x \notin A, \end{cases} \qquad A \in \operatorname{Exp} X.$$

Then we can write $m = \sum_{i \in \mathcal{I}} p_i \delta_{x_i}$. The latter definitions of atoms and purely atomic measures differ from the usual ones for arbitrary measure spaces, but agree with them for regular measures on compacta.

Now assume that we want to count all atoms of a purely atomic measure m separately, i.e. the measure of a closed set $A \subset X$ is a finite or countable *list* (determined up to permutation) of the masses of all atoms that are in A. We even allow for a finite or countable set of atoms to coexist in one point, provided the sum of their masses is finite, i.e. atoms of m can be split. The obtained set function is called a *purely atomized* measure.

The subsets of all purely atomic measures are not closed in either of the three spaces PX, $\underline{P}X$, $\overline{P}X$, if X is infinite. Moreover, it is easy to show that they are dense. Therefore in this case natural attempts to determine a compact Hausdorff topology on the sets of all normalized or subnormalized purely atomized measures fail. These sets are to be enriched by "missing" limits of nets. A simpler approach is to assume that our measure can have a purely atomized part and a non-atomized part which is an ordinary regular measure on X. The latter one can have its own atoms, but they are not counted separately, just go into a "common sum". Thus we obtain what we call a *weakly atomized measure*. Another (more complicated) way is to consider non-atomized part in more detail (to say, "atomize" it a little as well), and it leads to *atomized measures*.

To properly define our measures, we first define sets, which will be their codomains. Let us start with measures with finite numbers of atoms. We denote by $\overline{\mathcal{S}}$ the quotient set of the disjoint union $\bigcup_{n=0}^{\infty}(0,+\infty)^n$ w.r.t. the equivalence relation that identifies finite sequences of positive numbers if they coincide up to permutation. The equivalence class of $(\lambda_1,\lambda_2,\dots,\lambda_n)$ is denoted by $[\lambda_1,\lambda_2,\dots,\lambda_n]$, and $\big|[\lambda_1,\lambda_2,\dots,\lambda_n]\big| = \lambda_1+\lambda_2+\dots+\lambda_n$. Observe that $\overline{\mathcal{S}}$ is an Abelian monoid with a unit $[1]$, if a multiplication is defined as follows:

$$[\lambda_1,\lambda_2,\dots,\lambda_m]\cdot[\mu_1,\mu_2,\dots,\mu_n] = [\lambda_1\mu_1,\lambda_1\mu_2,\dots,\lambda_m\mu_n]$$

(all mn pairwise products at the right side). Since $|\lambda\cdot\mu| = |\lambda|\cdot|\mu|$ for all $\lambda,\mu\in\overline{\mathcal{S}}$, the sets

$$\mathcal{S} = \{\lambda\in\overline{\mathcal{S}} \mid |\lambda| = 1\} \text{ and } \underline{\mathcal{S}} = \{\lambda\in\overline{\mathcal{S}} \mid |\lambda|\leqslant 1\}$$

are submonoids of $\overline{\mathcal{S}}$.

We also define an addition on $\overline{\mathcal{S}}$ by the formula:

$$[\lambda_1,\lambda_2,\dots,\lambda_m]+[\mu_1,\mu_2,\dots,\mu_n] = [\lambda_1,\lambda_2,\dots,\lambda_m,\mu_1,\mu_2,\dots,\mu_n],$$

making it an Abelian monoid with a unit $[\,]$ (the equivalence class of the empty sequence). Since "$\cdot$" distributes over "$+$", $(\overline{\mathcal{S}},+,\cdot)$ is an Abelian semiring [5].

Three partial orders naturally arise on $\overline{\mathcal{S}}$ (and hence on $\mathcal{S}$ and $\underline{\mathcal{S}}$):

(1) for $\lambda,\mu\in\overline{\mathcal{S}}$, $\lambda\leqslant\mu$ if $\lambda=[\mu_1,\mu_2,\dots,\mu_m]$, $\mu=[\mu_1,\mu_2,\dots,\mu_m,\mu_{m+1},\dots,\mu_n]$, $m\leqslant n$;

(2) for $\lambda,\mu\in\overline{\mathcal{S}}$, $\lambda|\mu$ (λ *divides* μ) if $\mu=\lambda\cdot\nu$ for some $\nu\in\overline{\mathcal{S}}$;

(3) for $\lambda,\mu\in\overline{\mathcal{S}}$, $\lambda\prec\mu$ (μ is a *refinement* of λ) if $\lambda=[\lambda_1,\lambda_2,\dots,\lambda_m]$, $\mu=[\mu_1,\mu_2,\dots,\mu_n]$, $m\leqslant n$, and there is a surjection $\sigma:\{1,2,\dots,n\}\to\{1,2,\dots,m\}$ such that $\lambda_i=\sum_{\sigma(j)=i}\mu_j$ for all $1\leqslant i\leqslant m$.

Observe that $(\overline{\mathcal{S}},\leqslant)$ is a lattice and a complete lower semilattice, and a function $m:\operatorname{Exp}X\to\overline{\mathcal{S}}$ such that:

(1) $m(\varnothing)=[\,]$;

(2) $A\subset B$ implies $m(A)\leqslant m(B)$;

(3) $m(A\cup B)+m(A\cap B)=m(A)+m(B)$;

(4) $m(\bigcap\mathcal{A})=\inf_{A\in\mathcal{A}}m(A)$ for each filtered subcollection $\mathcal{A}\subset\operatorname{Exp}X$;

is precisely a purely atomized measure with a finite number of atoms. We call m normalized (subnormalized) if $|m(X)| = 1$ ($|m(X)| \leqslant 1$ respectively).

We denote by $\overline{\mathcal{S}}^a$ the set of all measures on I such that their restrictions to $[c, 1]$ are purely atomic for all $0 < c \leqslant 1$, and the masses of each such c are multiples of c (probably are equal to 0). It is an easy exercise to show that $\overline{\mathcal{S}}^a$ is closed in $\overline{P}I$. Let elements $\bar\lambda, \bar\mu$ of $\overline{P}I$ be multiplied as follows: $\bar\lambda \cdot \bar\mu = \overline{P}(\cdot)(\bar\lambda \otimes \bar\mu)$, where $\bar\lambda \otimes \bar\mu \in \overline{P}(I \times I)$ is the product measure [11], and $\cdot : I \times I \to I$ is the multiplication of reals. Then $\overline{P}I$ is an Abelian Tychonoff topological monoid, PI and $\underline{P}I$ are its compact Hausdorff submonoids [11]. Assume $\bar\lambda, \bar\mu \in \overline{\mathcal{S}}^a$, then

$$\bar\lambda = \sum_{i\in\mathcal{I}} \lambda_i \delta_{\lambda_i} + \alpha\delta_0, \quad \lambda_i > 0 \text{ for all } i \in \mathcal{I}, \quad \sum_{i\in\mathcal{I}} \lambda_i < \infty,$$

$$\text{and} \qquad \bar\mu = \sum_{j\in\mathcal{J}} \mu_j \delta_{\mu_j} + \beta\delta_0, \quad \mu_j > 0 \text{ for all } j \in \mathcal{J}, \quad \sum_{j\in\mathcal{J}} \mu_j < \infty.$$

It is straightforward to verify that

$$\bar\lambda \cdot \bar\mu = \sum_{i\in\mathcal{I}, j\in\mathcal{J}} \lambda_i\mu_j\delta_{\lambda_i\mu_j} + (\alpha\beta + \sum_{i\in\mathcal{I}} \lambda_i \cdot \beta + \alpha \cdot \sum_{j\in\mathcal{J}} \mu_j)\delta_0 \in \overline{\mathcal{S}}^a,$$

therefore $\overline{\mathcal{S}}^a \subset \overline{P}I$ is a closed submonoid.

The intersections

$$\mathcal{S}^a = \overline{\mathcal{S}}^a \cap PI = \{\bar\lambda \in \overline{\mathcal{S}}^a \mid |\bar\lambda| = 1\} \ \text{ and } \ \underline{\mathcal{S}}^a = \overline{\mathcal{S}}^a \cap \underline{P}I = \{\bar\lambda \in \overline{\mathcal{S}}^a \mid |\bar\lambda| \leqslant 1\}$$

are compact Hausdorff topological monoids.

It is easy to see that $\overline{\mathcal{S}}^a$ is closed under the "argumentwise" addition of measures, thus $(\overline{\mathcal{S}}^a, +, \cdot)$ is an Abelian Tychonoff topological semiring. It is also a lattice and a complete lower semilattice w.r.t. obvious comparison.

For all $\lambda = [\lambda_1, \dots, \lambda_m] \in \overline{\mathcal{S}}$, the measure $i^a(\lambda) = \sum_{i=1}^m \lambda_i \delta_{\lambda_i}$ is in $\overline{\mathcal{S}}^a$, and the mapping $i^a : \overline{\mathcal{S}} \to \overline{\mathcal{S}}^a$ preserves multiplication, zero and unit, $|\dots|$, pairwise suprema and arbitrary infima. It restrictions provide similar embeddings $\mathcal{S} \to \mathcal{S}^a$ and $\underline{\mathcal{S}} \to \underline{\mathcal{S}}^a$. Thus we consider $\overline{\mathcal{S}}$, $\underline{\mathcal{S}}$, and $\mathcal{S}$ as submonoids of $\overline{\mathcal{S}}^a$, $\underline{\mathcal{S}}^a$, and $\mathcal{S}^a$ respectively.

Thus we arrive at a required

Definition. *A function $m : \operatorname{Exp} X \to \overline{\mathcal{S}}^a$ such that:*

(1) $m(\varnothing) = 0$;

(2) $A \subset B$ implies $m(A) \leqslant m(B)$;

(3) $m(A \cup B) + m(A \cap B) = m(A) + m(B)$;

(4) $m(\bigcap \mathcal{A}) = \inf_{A\in\mathcal{A}} m(A)$ for each filtered subcollection $\mathcal{A} \subset \operatorname{Exp} X$;

is called a weakly atomized measure. *If $m(X)(0) = 0$ (hence $m(A)(0) = 0$ for all $A \in \operatorname{Exp} X$), then we call m* purely atomized. *A function m is* normalized *(*subnormalized*) if $|m(X)| = 1$ ($|m(X)| \leqslant 1$ respectively). The correspondence $m_n : A \mapsto m(A)(0)$ is called the* non-atomized part *of m, and $m_a : A \mapsto m(A) - m_n(A)\delta_0$ is the* purely atomized part *of m.*

Such m is of the following form: there is a real-valued measure m_0 on X and a finite or countable sequence $(x_i, p_i)_{i\in\mathcal{I}}$ in $X \times \mathbb{R}^+$ such that, for each $A \in \operatorname{Exp} X$, $m(A)$ is equal to $\sum\{p_i\delta_{p_i} \mid i \in \mathcal{I}, x_i \in A\} + m_0(A)\delta_0$. Then m_a send each A to $\sum\{p_i\delta_{p_i} \mid i \in \mathcal{I}, x_i \in A\}$ (and is purely atomized indeed), and $m_n = m_0$, hence the non-atomized part is a regular real-valued measure.

We identify such a weakly atomized measure $m : \operatorname{Exp} X \to \overline{\mathcal{S}}^a$ with the following real-valued measure $\hat{m} \in P(X \times I)$:

$$\hat{m}(B) = \sum\{p_i \mid i \in \mathcal{I}, (x_i, p_i) \in B\} + m_0\bigl(\mathrm{pr}_1(B \cap (X \times \{0\}))\bigr), \quad B \in \operatorname{Exp}(X \times I).$$

It is a unique real-valued measure $m' \in P(X \times I)$ such that $m'(A \times F) = m(A)(F)$ for all $A \in \operatorname{Exp} X$, $F \in \operatorname{Exp} I$. The set $\overline{P}^a X$ of measures $\hat{m}$ for all weakly atomized measures m on X consists of all real-valued measures on $X \times I$ which are purely atomic outside of $X \times \{0\}$, and masses in all $(x, p) \in X \times (0; 1]$ are multiples of p. The subset $\overline{P}^a X \subset \overline{P}(X \times I)$ is closed, as well as the analogous subsets $\underline{P}^a X \subset \underline{P}(X \times I)$ and $P^a X \subset P(X \times I)$ that correspond to subnormalized and normalized atomized measures.

By observing that, for each continuous mapping of compacta $f : X \to Y$, the inclusions $\overline{P}(f \times \mathbf{1}_I)(\overline{P}^a X) \subset \overline{P}^a Y$, $\underline{P}(f \times \mathbf{1}_I)(\underline{P}^a X) \subset \underline{P}^a Y$, and $P(f \times \mathbf{1}_I)(P^a X) \subset P^a Y$ are valid, we obtain subfunctors $\overline{P}^a : \mathcal{C}\mathrm{omp} \to \mathcal{T}\mathrm{ych}$, $\underline{P}^a : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$, and $P^a : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$, of the functors $\overline{P}(- \times I) : \mathcal{C}\mathrm{omp} \to \mathcal{T}\mathrm{ych}$, $\underline{P}(- \times I) : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$, and $P(- \times I) : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$. We call them the *functor of weakly atomized measures*, the *functor of subnormalized weakly atomized measures*, and the *functor of normalized weakly atomized measures*, respectively.

To proceed, we recall that the *Bohr compactification* [6] of the multiplicative group of positive reals is a compact Hausdorff Abelian topological group $(\mathit{bohr}\,\mathbb{R}^+, \cdot)$, together with a continuous homomorphism $b_{\mathbb{R}^+} : \mathbb{R}^+ \to \mathit{bohr}\,\mathbb{R}^+$, such that, for each continuous homomorphism $f : \mathbb{R}^+ \to G$ into a compact Hausdorff Abelian topological group, there is a unique continuous homomorphism $\mathit{bohr}\,f : \mathit{bohr}\,\mathbb{R}^+ \to G$ such that $\mathit{bohr}\,f \circ b_{\mathbb{R}^+} = f$. The image $b_{\mathbb{R}^+}(\mathbb{R}^+)$ is dense in $\mathit{bohr}\,\mathbb{R}^+$. Moreover, it is not difficult to prove the following:

Lemma 1.1. *For all $g \in \mathit{bohr}\,\mathbb{R}^+$, there is a net (n_β) in $\mathbb{N}$ such that $(n_\beta) \to +\infty$, $b_{\mathbb{R}^+}(n_\beta) \to g$.*

By I^b we denote the subset

$$\{(t, b_{\mathbb{R}^+}(t)) \mid t \in (0; 1]\} \cup (\{0\} \times \mathit{bohr}\,\mathbb{R}^+)$$

of the compact Hausdorff Abelian monoid $I \times \mathit{bohr}\,\mathbb{R}^+$. Obviously I^b is closed, hence is a compact Hausdorff Abelian monoid as well. Similarly to the above, we make $\overline{P}(I^b)$ a Tychonoff Abelian monoid by putting

$$\bar{\lambda} \cdot \bar{\mu} = \overline{P}(\cdot)(\bar{\lambda} \otimes \bar{\mu}), \text{ for } \bar{\lambda}, \bar{\mu} \in \overline{P}(I^b).$$

Let $\overline{\mathcal{S}}^b$ be the set of all real-valued measures on I^b that are purely atomic outside of $\{0\} \times \mathit{bohr}\,\mathbb{R}^+$, and masses of all (t, g), $t \in (0; 1]$, are multiples of t. Then $\overline{\mathcal{S}}^b \subset \overline{P}(I^b)$ is a closed submonoid, and the intersections $\underline{\mathcal{S}}^b = \overline{\mathcal{S}}^b \cap \underline{P}(I^b)$, $\mathcal{S}^b = \overline{\mathcal{S}}^b \cap P(I^b)$ are compact Hausdorff Abelian monoids. An embedding $i^b : \overline{\mathcal{S}} \to \overline{\mathcal{S}}^b$ is determined by the formula: for all $\lambda = [\lambda_1, \ldots, \lambda_m] \in \overline{\mathcal{S}}$, $i^b(\lambda) = \sum_{i=1}^m \lambda_i \delta_{(\lambda_i, b_{\mathbb{R}^+}(\lambda_i))}$. Its restrictions provide embeddings $\underline{\mathcal{S}} \to \underline{\mathcal{S}}^b$ and $\mathcal{S} \to \mathcal{S}^b$. The restriction p^b_a of $P\,\mathrm{pr}_1 : P(I \times \mathit{bohr}\,\mathbb{R}^+) \to PI$ to $\overline{\mathcal{S}}^b$ is a surjective homomorphism onto $\overline{\mathcal{S}}^a$, and its restrictions map $\underline{\mathcal{S}}^b$ onto $\underline{\mathcal{S}}^a$ and $\mathcal{S}^b$ onto $\mathcal{S}^a$.

Lemma 1.2. *The set $\mathcal{S}$ is dense both in $\mathcal{S}^a$ and $\mathcal{S}^b$.*

Proof is straightforward.

Since $\overline{\mathcal{S}}^b$ is an Abelian semiring, a lattice and a complete lower semilattice, we use it the same way as $\overline{\mathcal{S}}^a$ and suggest:

Definition. *A function $m : \operatorname{Exp} X \to \overline{\mathcal{S}}^b$ such that:*

(1) $m(\varnothing) = 0$;

(2) $A \subset B$ implies $m(A) \leqslant m(B)$;

(3) $m(A \cup B) + m(A \cap B) = m(A) + m(B)$;

(4) $m(\bigcap \mathcal{A}) = \inf_{A \in \mathcal{A}} m(A)$ for each filtered subcollection $\mathcal{A} \subset \operatorname{Exp} X$;

is called an atomized measure. *If $m(X)(\{0\} \times \mathit{bohr}\, \mathbb{R}^+) = 0$ (hence $m(A)(\{0\} \times \mathit{bohr}\, \mathbb{R}^+) = 0$ for all $A \in \operatorname{Exp} X$), then we call m* purely atomized. *A function m is* normalized *(*subnormalized*) if $|m(X)| = 1$ ($|m(X)| \leqslant 1$ respectively). The correspondence $m_n : A \mapsto m(A)|_{(\{0\} \times \mathit{bohr}\, \mathbb{R}^+)}$ is called the* non-atomized part *of m, and $m_a = m - m_n$ is the* purely atomized part *of m.*

Note that the non-atomized part m_n is not a real-valued measure, but a measure with values in the space of real-valued measures on a compact Hausdorff *group*, namely on $\{0\} \times \mathit{bohr}\, \mathbb{R}^+$.

We identify again each atomized measure m on a compactum X with a unique real-valued measure $\hat{m}$ on $X \times I^b$ such that $\hat{m}(A \times F) = m(A)(F)$ for all $A \in \operatorname{Exp} X$, $F \in \operatorname{Exp} I^b$. The set $\overline{P}^b X$ of all such $\hat{m}$ consists of all measures on $X \times I^b \subset X \times I \times \mathit{bohr}\, \mathbb{R}^+$ that are purely atomic outside of $X \times \{0\} \times \mathit{bohr}\, \mathbb{R}^+$, and masses of all (x, t, g), $t > 0$, are multiples of t. The compact Hausdorff subspaces $\underline{P}^b X$ of all subnormalized atomized measures and $P^b X$ of all normalized atomized measures, as well as the *functors* $\overline{P}^b : \mathcal{C}\mathrm{omp} \to \mathcal{T}\mathrm{ych}$ *of atomized measures*, $\underline{P}^b : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$ *of subnormalized atomized measures*, and $P^b : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$ *of normalized atomized measures* are defined obviously.

Reasons to introduce such a complicated notion of atomized measure (comparing to the definition of weakly atomized measure) will be clarified in the next sections.

2 SEMICONVEX COMPACTA

First recall some definitions and facts from [8].

Let X be a *convex compactum* (i.e. a convex compact set in a locally convex topological vector space) and $c(x, y, \lambda) = \lambda x + (1 - \lambda) y$, for all $x, y \in X$, $\lambda \in I$, i.e. c is a pairwise convex combination. For the sake of brevity we will write $\lambda(x, y)$ instead of $c(x, y, \lambda)$. The ternary operation $c : X \times X \times I \to X$ satisfies the following properties:

(1) for all $x, y \in X$, $\lambda \in I :\ \lambda(x, y) = (1 - \lambda)(y, x)$ (*commutative law*);

(2) for all $x, y, z \in X$, $\lambda, \mu, \nu \in I$, $\lambda + \mu + \nu = 1$, $\mu \neq 0 :$

$$\lambda(x, \frac{\mu}{\mu + \lambda}(y, z)) = (\lambda + \mu)(\frac{\lambda}{\lambda + \mu}(x, y), z)$$

(*associative law*);

(3) for all $x, y \in X:\ 1(x, y) = x$;

(4) each neighborhood U of the diagonal $\Delta_X = \{(x,x) \mid x \in X\}$ in $X \times X$ contains a neighborhood B of Δ_X such that $(x,y),(z,t) \in B,\ \lambda \in I$ implies $(\lambda(x,z), \lambda(y,t)) \in B$;

(5) $\lambda(x,x) = x$ for all $x \in X$ and $\lambda \in I$ (*absence of loops*).

In the presence of (1)–(3), the property (4) provides *local convexity* and is equivalent to the following :

(4') the topology on X is generated by a family of pseudometrics $(d_\alpha)_{\alpha\in\mathcal{A}}$ such that $x,y,z,t \in X,\ \varepsilon > 0, \alpha \in \mathcal{A},\ d_\alpha(x,y) < \varepsilon,\ d_\alpha(z,t) < \varepsilon,\ \lambda \in I$ implies $d_\alpha(\lambda(x,z), \lambda(y,t)) < \varepsilon$.

Remark. *If pseudometrics d_α, d_β satisfy $(4)'$, then the expression $\max\{d_\alpha(x,y), d_\beta(x,y)\}$ is also a pseudometric, which satisfies (4'). Therefore we can assume that the family $(d_\alpha)_{\alpha\in\mathcal{A}}$ is directed and even saturated [2].*

Results of Świrszcz [10] imply that any compactum X with an operation c that satisfies (1)–(5) can be embedded as a convex compact set into a locally convex topological vector space so that c is a pairwise convex combination. In particular, the hyperspace $\operatorname{cc} K$, of all non-empty convex closed subsets of a convex compactum K with the Vietoris topology [11], with the operation c defined as $c(A,B,\lambda) = \{\lambda a + (1-\lambda) b \mid a \in A, b \in B\}$, for all $A, B \in \operatorname{cc} K$, $\lambda \in I$, satisfies (1)–(5) and is a convex compactum as well.

Unfortunately, if we use the latter formula to define combinations of elements of the hyperspace $\exp K$ of *all* closed non-empty subsets of a convex compactum K, only properties (1)–(4), but not (5), are valid, hence $\exp K$ does not become a convex compactum this way. There are a lot of similar examples, involving, e.g., convolutions of measures, such that (5) fails. Therefore we will relax the requirements to cover such structures.

A semiconvex compactum is a compactum X with a continuous ternary operation $c : X \times X \times I \to X$ (we usually call it *semiconvex combination* and write $\lambda(x,y)$ instead of $c(x,y,\lambda)$) such that (1)–(4) are valid. In the sequel we assume that a family of pseudometrics $(d_\alpha)_{\alpha\in\mathcal{A}}$ on X, whose existence is assured by (4'), is fixed and saturated for each particular X.

Extend the notion of semiconvex combination onto finite number of elements of X. Let $\lambda_1, \dots, \lambda_n \geqslant 0$, $\lambda_1 + \dots + \lambda_n = 1$ and $x_1, \dots, x_n \in X$, then

$$(\lambda_1, \dots, \lambda_n)(x_1, \dots, x_n) = \begin{cases} x_1, & \text{if } \lambda_1 = 1; \\ \lambda_1(x_1, (\frac{\lambda_2}{1-\lambda_1}, \dots, \frac{\lambda_n}{1-\lambda_1})(x_2, \dots, x_n)), & \text{if } \lambda_1 \neq 1. \end{cases}$$

If *arguments* $x_1, \dots, x_n$ of semiconvex combination are permutted simultaneously with the respective *coefficients* $\lambda_1, \dots, \lambda_n$, the value of semiconvex combination does not change. We can also drop arguments that correspond to zero coefficients. We call a subset of X *semiconvex* if it is closed under semiconvex combinations.

By continuity semiconvex combinations are naturally defined also for *countable* numbers of elements. Let $x_i \in X$, $\lambda_i \in I$, $i = 1, 2, \dots$, be such that $\sum_{i=1}^{\infty} \lambda_i = 1$. Then the sequence $(\lambda_1, \dots, \lambda_{n-1}, 1 - \lambda_1 - \dots - \lambda_{n-1})(x_1, \dots, x_n)$, $n \in \mathbb{N}$, has a limit, which we regard as the value of $(\lambda_1, \lambda_2, \dots)(x_1, x_2, \dots)$. This value is continuous w.r.t. $(x_1, x_2, \dots) \in X^{\mathbb{N}}$ for a fixed $(\lambda_1, \lambda_2, \dots)$.

The constructed in the previous section monoid $\mathcal{S}$ naturally acts on X:

$$[\lambda_1, \dots, \lambda_n]x = (\lambda_1, \dots, \lambda_n)(x, \dots, x), \quad x \in X, [\lambda_1, \dots, \lambda_n] \in \mathcal{S},$$

and all correspondences $x \mapsto sx$, for $s \in \mathcal{S}$, are non-expanding w.r.t. all pseudometrics d_α.

For $A \in \exp X$ and $s = [\lambda_1, \dots, \lambda_n] \in \mathcal{S}$, we write $sA = \{sx \mid x \in A\}$ and $s * A = \{(\lambda_1, \dots, \lambda_n)(x_1, \dots, x_n) \mid x_1, \dots, x_n \in A\}$, and obtain two actions of $\mathcal{S}$ on $\exp X$. If A is semiconvex, then so are sA and $s * A$.

For any subset $A \subset X$ the set

$$\mathcal{S}A = \{s * A \mid s \in \mathcal{S}\} = \{(\lambda_1, \dots, \lambda_n)(x_1, \dots, x_n) \mid x_1, \dots, x_n \in A, \\ n \in \mathbb{N}, \lambda_1, \dots, \lambda_n \in I, \lambda_1 + \dots + \lambda_n = 1\}$$

is a least semiconvex subset in X that contains A. It is called the *semiconvex hull* of A. Its closure is a least *closed* semiconvex subset in X that contains A, and therefore is called the *closed semiconvex hull* of A. In particular, $\mathrm{Cl}(\mathcal{S}\{a\})$ is a least closed semiconvex set that contains $a \in X$.

A mapping $f : X \to Y$ between semiconvex compacta is called *affine* if it preserves semiconvex combinations, i.e. $f(\lambda(x_1, x_2)) = \lambda(f(x_1), f(x_2))$ whenever $x_1, x_2 \in X$, $\lambda \in I$.

For the reader's convenience, until the end of this section we reproduce several statements from [8], in particular because of changes in notation.

Lemma 2.1. *Let $a \in X$, $A = \mathrm{Cl}(\mathcal{S}\{a\})$, then $\bigcap\limits_{s \in \mathcal{S}} sA$ is a unique minimal w.r.t. inclusion closed semiconvex subset $B \subset A$.*

Proof. By Zorn Lemma such a minimal subset B exists. Since, for any $s \in \mathcal{S}$, the set $sB \subset B$ is also closed and semiconvex, we obtain $sB = B$. Then $B \subset A$, $B = sB \subset sA$ implies $B \subset \bigcap\limits_{s \in \mathcal{S}} sA = A_0$, and the latter set is closed, semiconvex, and satisfies $A_0 = sA_0$ for all $s \in \mathcal{S}$.

Let $x \in A_0$, $\varepsilon > 0$, $\alpha \in \mathcal{A}$, $b \in B$. Since $B \subset \mathrm{Cl}(\mathcal{S}\{a\})$, there is $s' \in \mathcal{S}$ such that $d_\alpha(b, s'a) \leqslant \varepsilon/2$. Choose $y \in A_0$ such that $x = s'y$. There is $s'' \in \mathcal{S}$ such that $d_\alpha(y, s''a) \leqslant \varepsilon/2$. By non-expansion, obtain $d_\alpha(x, s's''a) = d_\alpha(s'y, s's''a) \leqslant \varepsilon/2$, $d_\alpha(s''b, s's''a) \leqslant \varepsilon/2$. Then $d_\alpha(x, s''b) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$, $s''b \in B$, therefore $d_\alpha(x, B) \leqslant \varepsilon$, which implies $x \in B$. Thus $B = \bigcap\limits_{s \in \mathcal{S}} sA$. $\square$

Due to the above lemma, for all $s \in \mathcal{S}$, the correspondence $B \to B$ that takes each b to sb is a non-expanding surjection w.r.t. all pseudometrics d_α, $\alpha \in \mathcal{A}$. Since any non-expanding surjection of a metric compactum onto itself is an isometry, for all $a, b \in X$, $s \in \mathcal{S}$, $\alpha \in \mathcal{A}$, we have $d_\alpha(sa, sb) = d_\alpha(a, b)$.

Lemma 2.2. *The set $B = \bigcap\limits_{s \in \mathcal{S}} sA$ consists of a single point.*

Proof. Putting $\lambda((x, y), (z, t)) = (\lambda(x, z), \lambda(y, t))$, for all $(x, y), (z, t) \in B \times B$, $\lambda \in I$, we turn $B \times B$ into a semiconvex compactum. For $x, y \in B$, let $(x \to y) = \mathrm{Cl}(\mathcal{S}\{(x, y)\}) \subset B \times B$. Since $\mathcal{S}\{x\}$ and $\mathcal{S}\{y\}$ are dense in the compactum B, $\mathrm{pr}_1((x \to y)) = \mathrm{pr}_2((x \to y)) = B$.

Let $(z_1,t_1),(z_2,t_2) \in \mathcal{S}\{(x,y)\}$, i.e. $z_1 = s_1x$, $t_1 = s_1y$, $z_2 = s_2x$, $t_2 = s_2y$ for some $s_1, s_2 \in \mathcal{S}$. For all $\varepsilon > 0$, $\alpha \in \mathcal{A}$ there is $s \in \mathcal{S}$ such that $d_\alpha(sx,y) < \varepsilon$. Then

$$d_\alpha(t_1,t_2) = d_\alpha(s_1y,s_2y) \leqslant d_\alpha(s_1y,s_1sx) + d_\alpha(s_1sx,s_2sx) + d_\alpha(s_2sx,s_2y) = \\ d_\alpha(y,sx) + d_\alpha(s_1x,s_2x) + d_\alpha(sx,y) = d_\alpha(z_1,z_2) + 2\varepsilon,$$

hence $d_\alpha(t_1,t_2) \leqslant d_\alpha(z_1,z_2)$. The reverse inequality is valid as well, thus $d_\alpha(t_1,t_2) = d_\alpha(z_1,z_2)$ for all elements $(z_1,t_1),(z_2,t_2)$ of $\mathcal{S}\{(x,y)\}$ and therefore of $\mathrm{Cl}(\mathcal{S}\{(x,y)\}) = (x \to y)$. It implies that $(x \to y)$ is the graph of a mapping $B \to B$ which is an isometry w.r.t. all d_α, $\alpha \in \mathcal{A}$, and $(y \to x)$ is the graph of an inverse isometry.

Let $(z,t_1) \in (x \to y_1)$, $(z,t_2) \in (x \to y_2)$. There exist a net of the form $(s_\beta x)$, $s_\beta \in \mathcal{S}$, such that $s_\beta x \to z$, then $s_\beta y_1 \to t_1$, $s_\beta y_2 \to t_2$. Since $d_\alpha(s_\beta y_1, s_\beta y_2) = d_\alpha(y_1,y_2)$, the equality $d_\alpha(y_1,y_2) = d_\alpha(t_1,t_2)$ holds.

Fix an arbitrary point $b \in B$. For all $x, y \in B$, there is a unique $x \in B$ such that $(x \to y) = (b \to z)$. Thus we can properly define an operation on B: for $z_1, z_2 \in B$, let $z = z_1 \cdot z_2$ be such that $(z_2 \to z) = (b \to z_1)$. By the above, this operation is an isometry w.r.t. each d_α in each argument separately, hence is a continuous mapping $B \times B \to B$.

Assume that $z_1 = s_1b$, $z_2 = s_2b$, then $(b \to z_1) = \{(x, s_1x) \mid x \in B\}$, therefore $z_1 \cdot z_2 = s_1s_2b = s_2s_1b = z_2 \cdot z_1$. Such z_1, z_2 are dense in B, and "$\cdot$" is continuous, hence it is commutative for all arguments. Similarly the associative law in $\mathcal{S}$ implies the associativity of "$\cdot$".

The inverse for $x \in B$ is a unique $y \in B$ such that $(y,b) \in (b \to x)$. Uniqueness of the inverse and the compactness of B imply the continuity of the inversion.

Consequently B is a contractible compact Abelian topological group. It is known [1, 4] that such a group is trivial, i.e. is a singleton. $\square$

The point $b \in B$ is unique in A such that $\lambda(b,b) = b$ for all $\lambda \in I$. Let $bX : X \to X$ be the map taking each $a \in X$ to such a point $b \in \mathrm{Cl}(\mathcal{S}\{a\})$. Then $bX(X)$ is a closed subset of X consisting of all points $b \in X$ such that $\lambda(b,b) = b$ for all $\lambda \in I$. This set is called the *center* of X and denoted $Ctr(X)$.

Theorem 1. *The net $(sx)_{s \in (\mathcal{S},|)}$ is uniformly convergent to $bX(x)$, for $x \in X$. The mapping bX is an affine and non-expanding w.r.t. all d_α, $\alpha \in \mathcal{A}$, retraction of the semiconvex compactum X onto its center $Ctr(X)$.*

Proof. Since X is a compactum and all mappings $s(-) : X \to X$ are non-expanding, it is sufficient to prove the pointwise convergence. Let $x \in X$, $A = \mathrm{Cl}(\mathcal{S}\{x\})$. For all $s, s' \in \mathcal{S}$, $s|s'$ we have $s'A \subset sA$. Since $\{bX(x)\} = \bigcap\limits_{s \in \mathcal{S}} sA$, for each neighborhood U of $bX(x)$ there is $s \in \mathcal{S}$ such that $sA \subset U$, hence for all $s' \in \mathcal{S}$ such that $s|s'$ we have $s'x \in s'A \subset U$.

All mappings $s(-) : X \to X$ are affine, therefore the same is valid for bX. Obviously $bX(x) = x$ if and only if $x \in Ctr(X)$. $\square$

This implies that (1)–(5) are valid for $Ctr(X)$, and the center is a convex compactum, a largest one of all convex compacta that are (algebraically and topologically) embedded into X.

Let A_i, $i \in \mathcal{I}$, be subsets of a compactum X, $\mathcal{F}$ a filter in the index set $\mathcal{I}$. By $\overline{\lim}$ we denote the upper limit:

$$\overline{\lim_{\mathcal{F}}} A_i = \{x \in X \mid \text{for all neighborhoods } U \ni x \text{ and } F \in \mathcal{F}$$
$$\text{there is } i \in F \text{ such that } A_i \cap U \neq \varnothing\}.$$

It is obvious that $\overline{\lim_{\mathcal{F}}} A_i \subset X$ is closed and nonempty whenever all A_i are nonempty. For all $s = [\lambda_1, \ldots, \lambda_n] \in \overline{\mathcal{S}}$ we denote $\max s = \max\{\lambda_1, \ldots, \lambda_n\}$.

Lemma 2.3. *For any subset $A \subset X$ the equality $\overline{\lim\limits_{s\in(\mathcal{S},|)}}\, s * A = \overline{\lim\limits_{\max s \to 0}}\, s * A$ holds.*

Proof. Let $b \in \overline{\lim\limits_{s\in(\mathcal{S},|)}}\, s * A$, Ub be a neighborhood of b, and $\varepsilon > 0$. Take an arbitrary $s_0 \in \mathcal{S}$ such that $\max s_0 < \varepsilon$. There is $s \in \mathcal{S}$ such that $s|s_0$, $s * A \cap Ua \neq \varnothing$, hence $\max s < \varepsilon$. Thus $b \in \overline{\lim\limits_{\max s \to 0}}\, s * A$.

Let $s = [\lambda_1, \ldots, \lambda_m] \in \mathcal{S}$. We can (non-uniquely) visualize s as a partition of the unit segment I into adjacent segments of lengths $\lambda_1, \ldots, \lambda_m$. Their ends form an increasing sequence s': $s'_0 = 0$, $s'_1 = \lambda_1$, $s'_2 = \lambda_1 + \lambda_2$, $\ldots$, $s'_m = \lambda_1 + \cdots + \lambda_m = 1$. If $t = [\mu_1, \ldots, \mu_n] \in \mathcal{S}$ is such that the respective sequence t', with $t'_0 = 0$, $t'_1 = \mu_1$, $t'_2 = \mu_1 + \mu_2$, $\ldots$, $t'_n = \mu_1 + \cdots + \mu_n = 1$, is a subsequence of s', then $t \prec s$, i.e. s is a refinement of t.

Now let $b \in \overline{\lim\limits_{\max s \to 0}}\, s * A$, $\varepsilon > 0$, $\alpha \in \mathcal{A}$. Since $c : X \times X \times I \to X$ is a continuous mapping of compacta, there is $\delta > 0$ such that, for all $x, y, z \in X$, $0 \leqslant \lambda < \delta$, the inequality $d_\alpha(\lambda(x, z), \lambda(y, z)) < \varepsilon/2$ holds. Choose arbitrary $s_0 = [\lambda_1, \ldots, \lambda_m] \in \mathcal{S}$. There are $t = [\mu_1, \ldots, \mu_n] \in \mathcal{S}$, $a_1, \ldots, a_n \in A$ such that $\max t < \delta/m$, $d_\alpha(a, b) < \varepsilon/2)$ for $a = (\mu_1, \ldots, \mu_n)(a_1, \ldots, a_n)$. Construct the described above sequences s'_0 and t' for s_0 and t, and let s' be the union of s'_0 and t' in ascending order. Then s' represents $s \in \mathcal{S}$ which is a refinement both of s_0 and of t. Let each segment $[t_{i-1}, t_i]$ of length μ_i is split by elements of s' into $k_i \geqslant 1$ parts of lengths $\mu_i^1, \ldots, \mu_i^{k_i}$. Calculate the point

$$c = (\mu_1^1, \ldots, \mu_1^{k_1}, \ldots, \mu_i^1, \ldots, \mu_i^{k_i}, \ldots, \mu_n^1, \ldots, \mu_1^{k_n})(\underbrace{a_1, \ldots, a_1}_{k_1 \text{ times}}, \ldots, \underbrace{a_i, \ldots, a_i}_{k_i \text{ times}}, \ldots, \underbrace{a_n, \ldots, a_n}_{k_n \text{ times}}).$$

At most m segments were split into $\geqslant 2$ parts, hence the combinations a and c differ (in obvious sense) only in arguments such that the sums of the respective coefficients are less than $m \cdot \delta/m = \delta$. By the choice of δ this implies $d_\alpha(a, c) < \varepsilon/2$, hence $d_\alpha(b, c) \leqslant d_\alpha(b, a) + d_\alpha(a, c) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Observe that $c \in s * A$, and by $s_0|s$ we obtain $b \in \overline{\lim\limits_{s\in(\mathcal{S},|)}}\, s * A$. $\square$

It is easy to see that a largest closed subset $A \subset X$, such that $[\lambda_1, \ldots, \lambda_n](-) : A^n \to A$ is surjective for any $[\lambda_1, \ldots, \lambda_n] \in \mathcal{S}$, is equal to $\bigcap\limits_{s\in\mathcal{S}} s * X = \overline{\lim\limits_{s\in(\mathcal{S},|)}}\, s * X$, hence is semiconvex. Similarly, for a particular $\lambda \in (0; 1)$, a largest closed subset $A \subset X$ such that $\lambda : A^2 \to A$ is surjective is equal to $\bigcap\limits_{n\in\mathbb{N}} [\lambda, 1-\lambda]^n * X$. Obviously

$$\bigcap_{n\in\mathbb{N}} [\lambda, 1-\lambda]^n * X \supset \bigcap_{s\in\mathcal{S}} s * X.$$

On the other hand, $\max[\lambda, 1-\lambda]^n \to 0$ as $n \to \infty$, therefore

$$\bigcap_{n\in\mathbb{N}} [\lambda, 1-\lambda]^n * X \subset \overline{\lim_{\max s\to 0}}\, s * X,$$

and by the latter lemma

$$\overline{\lim_{s\in(\mathcal{S},|)}}\, s * X = \overline{\lim_{\max s\to 0}}\, s * X.$$

Therefore all the constructed sets are equal. We call any of them the *weak center* of X and denote it by $WCtr(X)$. Since

$$WCtr(X) = \bigcap_{\substack{n\in\mathbb{N},\lambda_1,\dots,\lambda_n\in I,\\ \lambda_1+\dots+\lambda_n=1}} \{(\lambda_1,\dots,\lambda_n)(x_1,\dots,x_n) \mid x_1,\dots,x_n \in X\},$$

$$Ctr(X) = \bigcap_{\substack{n\in\mathbb{N},\lambda_1,\dots,\lambda_n\in I,\\ \lambda_1+\dots+\lambda_n=1}} \{(\lambda_1,\dots,\lambda_n)(x,\dots,x) \mid x \in X\},$$

we obtain $Ctr(X) \subset WCtr(X)$.

Recall that $X_0 = WCtr(X)$ is semiconvex and closed in X, therefore is a semiconvex compactum as well. Denote $[\frac{1}{n},\dots,\frac{1}{n}](x,\dots,x) = \langle\frac{1}{n}\rangle x$. For all $m,n \in \mathbb{N}$, the mapping $\langle\frac{1}{n}\rangle(-) : X_0 \to X_0$ is a non-expanding surjection, hence an isometry, w.r.t. all d_α, and $\langle\frac{1}{mn}\rangle x = \langle\frac{1}{m}\rangle \circ \langle\frac{1}{n}\rangle x = \langle\frac{1}{n}\rangle \circ \langle\frac{1}{m}\rangle x$ for all $x \in X$. Therefore $\langle\frac{1}{m}\rangle^{-1} \circ \langle\frac{1}{n}\rangle x = \langle\frac{1}{km}\rangle^{-1} \circ \langle\frac{1}{kn}\rangle x$ for all $m,n,k \in \mathbb{N}$, $x \in X_0$, and we use the latter expression as a definition of $\langle\frac{m}{n}\rangle x$. In particular, $\langle n\rangle x = \langle\frac{1}{n}\rangle^{-1} x$.

Thus an action of the multiplicative group $\mathbb{Q}^+$ on X_0 is obtained.

Lemma 2.4. *The obtained action $\mathbb{Q}^+ \times X_0 \to X_0$ is equicontinuous (with $x \in X_0$ as a parameter) w.r.t. all d_α and a standard metric on $\mathbb{Q}^+ \subset \mathbb{R}$.*

Proof. For each $\varepsilon > 0$, $\alpha \in \mathcal{A}$ there is $0 < \delta < 1$ such that $d_\alpha(\lambda(a,b),b) < \varepsilon$ whenever $a,b \in X$, $\lambda \in I$, $\lambda < \delta$. Let $x \in X_0$, $p,q \in \mathbb{Q}^+$, $p(1-\delta) < q < p/(1-\delta)$. If $p = q$, then $\langle p\rangle x = \langle q\rangle x$. If $q < p$, we can assume that $p = \frac{m}{n}$, $q = \frac{m}{n'}$, $n < n' < n/(1-\delta)$. Denote $y = \langle m\rangle x$, then $\langle\frac{m}{n}\rangle x = [\frac{1}{n},\dots,\frac{1}{n}]y,$, $\langle\frac{m}{n'}\rangle x = [\frac{1}{n'},\dots,\frac{1}{n'}]y$, and

$$[\frac{1}{n'},\dots,\frac{1}{n'}]y = \frac{n'-n}{n'}([\frac{1}{n'-n},\dots,\frac{1}{n'-n}]y, [\frac{1}{n},\dots,\frac{1}{n}]y).$$

Observe that $0 < \frac{n'-n}{n'} < \delta$, hence

$$d_\alpha(\langle p\rangle x, \langle q\rangle x) = d_\alpha([\frac{1}{n},\dots,\frac{1}{n}]y, [\frac{1}{n'},\dots,\frac{1}{n'}]y) < \varepsilon.$$

If $q > p$, then $q < p < q/(1-\delta)$, and we similarly prove that the previous inequality is valid. Thus we have constructed a neighborhood $Op = (p(1-\delta), p/(1-\delta)) \cap \mathbb{Q}^+$ of p such that $q \in Op$ implies $d_\alpha(\langle p\rangle x, \langle q\rangle x) < \varepsilon$ for all $x \in X_0$. □

Therefore an action $\langle\dots\rangle$ of $\mathbb{Q}^+$ on X_0 can be uniquely extended to a continuous action, of $\mathbb{R}^+$ on X_0, for which we preserve the same denotation $\langle\dots\rangle$. We define a new operation

$\diamond : X_0 \to X_0 \times I \to X_0$ by the formula $\lambda \diamond (x,y) = \lambda(\langle \frac{1}{\lambda}\rangle x, \langle \frac{1}{1-\lambda}\rangle y)$ if $0 < \lambda < 1$, $1 \diamond (x,y) = x$, $0 \diamond (x,y) = y$ for all $x, y \in X_0$. It is straightforward to verify that for "$\diamond$" the properties (1)–(3), (4'), (5) are valid, hence $(X_0, \diamond)$ is a convex compactum. We arrive at the following statement:

Theorem 2. *A semiconvex compactum X is a weak center of some semiconvex compactum if and only if there exist a continuous operation $\diamond : X \times X \times I \to X$ and a continuous action $\langle \dots \rangle$ of $(0; +\infty)$ on X such that $(X, \diamond)$ is a convex compactum, all mappings $\langle\lambda\rangle : (X, \diamond) \to (X, \diamond)$ are affine, and, for all $x, y \in X$, $\lambda \in (0;1)$, the equality $\lambda(x,y) = \lambda \diamond (\langle\lambda\rangle x, \langle 1-\lambda\rangle y)$ holds. These "$\diamond$" and "$\langle \dots \rangle$" are uniquely determined.*

Remark 2.1. *The center $Ctr(X)$ is a subset of $WCtr(X)$ that consists of all points x such that $\langle\lambda\rangle x = x$ for all $\lambda \in (0; +\infty)$ and the previously defined action $\langle \dots \rangle$.*

Definition. *If $Ctr(X) = WCtr(X)$, then we call X* a strongly semiconvex compactum.

Here is an alternative definition: a semiconvex compactum X is called strongly semiconvex if for any $x \in X$ the point $[\lambda_1, \dots, \lambda_n]x$ converges to a unique point $y \in X$ whenever $\lambda_1, \dots, \lambda_n \geqslant 0$, $\lambda_1 + \dots + \lambda_n = 1$, $\max\{\lambda_1, \dots, \lambda_n\} \to 0$. This implies that if $f : X \to Y$ is an affine surjective map of strongly semiconvex compacta, and X is strongly semiconvex, then Y is strongly semiconvex as well.

Many (but not all) "real-life" examples of semiconvex compacta belong to this class, e.g. the previously mentioned hyperspace $\exp K$ of closed non-empty subsets of a convex compactum K.

By the above, for all $x \in X$, we have $d_\alpha([\lambda_1, \dots, \lambda_n]x, WCtr(X)) \to 0$ as $\max[\lambda_1, \dots, \lambda_n] \to 0$. Now we are going to extend results of [8] and to clarify the behaviour if the expression $[\lambda_1, \dots, \lambda_n]x$ in a simpler case $\lambda_1 = \dots = \lambda_n = \frac{1}{n}$.

For all $x \in X$ and $n \in \mathbb{N}$, let a sequence $\bar{x}$ in X be defined by the formula $\bar{x}_n = \langle \frac{1}{n}\rangle x$. On the set $X^{\mathbb{N}}$ of all sequences in X we consider a pseudometric $\bar{d}_\alpha$, which is defined as follows:

$$\bar{d}_\alpha((x_n), (y_n)) = \overline{\lim_{n\to\infty}}\, d_\alpha(x_n, y_n), \quad (x_n), (y_n) \in X^{\mathbb{N}}.$$

Theorem 3. *For each $x \in X$, there is a unique $x_0 \in WCtr(X)$ such that $d_\alpha(\langle \frac{1}{n}\rangle x, \langle \frac{1}{n}\rangle x_0) \to 0$ as $n \to \infty$ for all $\alpha \in \mathcal{A}$. The mapping $wbX : X \to WCtr(X)$ that sends each x to the respective x_0 is an affine and non-expanding w.r.t. all d_α retraction of a semiconvex compactum X onto its weak center $WCtr(X)$, and satisfies the equality $bX \circ wbX = bX$.*

Proof. For all $\alpha \in \mathcal{A}$, the mapping $(X, d_\alpha) \to (X^{\mathbb{N}}, \bar{d}_\alpha)$ that sends each x to $\bar{x}$, is non-expanding, and its restriction to $WCtr(X)$ is an isometry because $d_\alpha(\langle \frac{1}{n}\rangle x_1, \langle \frac{1}{n}\rangle x_2) = d_\alpha(x_1, x_2)$ for all $x_1, x_2 \in WCtr(X)$. Thus the uniqueness is immediate.

Let $x \in X$, $\alpha \in \mathcal{A}$. The sequence $(d_\alpha(\langle \frac{1}{n}\rangle x, WCtr(X)))_{n\in\mathbb{N}}$ tends to zero, therefore there is a sequence (y_n) in $WCtr(X)$ such that $d_\alpha(\langle \frac{1}{n}\rangle x, y_n) \to 0$ as $n \to \infty$. For all $n \in \mathbb{N}$, choose $x_n \in WCtr(X)$ such that $\langle \frac{1}{n}\rangle x_n = y_n$. Since $WCtr(X)$ is a compactum, there is a subsequence (x_{n_i}) that converges to some $x_0 \in WCtr(X)$, hence $d_\alpha(\langle \frac{1}{n_i}\rangle x_0, y_{n_i}) \to 0$ as $i \to \infty$. By the triangle inequality for d_α, we obtain that $d_\alpha(\langle \frac{1}{n_i}\rangle x, \langle \frac{1}{n_i}\rangle x_0) \to 0$, $i \to \infty$.

For each $\varepsilon > 0$, there is $i \in \mathbb{N}$ such that $d_\alpha(\langle\frac{1}{n_i}\rangle x, \langle\frac{1}{n_i}\rangle x_0) < \varepsilon/3$. We can also choose $m \in \mathbb{N}$ such that, for all $x, y \in X$, $\lambda \in I$, the inequality $\lambda \leqslant \frac{1}{m}$ implies $d_\alpha(\lambda(x,y), y) < \varepsilon/3$. Let $n \geqslant n_0 = mn_i$, then $n = kn_i + l$, $k \geqslant m$, $0 \leqslant l < n_i$. We have:

$$d_\alpha(\langle\frac{1}{kn_i}\rangle x, \langle\frac{1}{kn_i}\rangle x_0) = d_\alpha(\langle\frac{1}{k}\rangle\langle\frac{1}{n_i}\rangle x, \langle\frac{1}{k}\rangle\langle\frac{1}{n_i}\rangle x_0) \leqslant d_\alpha(\langle\frac{1}{n_i}\rangle x, \langle\frac{1}{n_i}\rangle x_0) < \varepsilon/3.$$

If $l \neq 0$, then:

$$\begin{gathered} d_\alpha(\langle\frac{1}{n}\rangle x, \langle\frac{1}{n}\rangle x_0) = d_\alpha(\langle\frac{1}{kn_i+l}\rangle x, \langle\frac{1}{kn_i+l}\rangle x_0) = \\ d_\alpha(\frac{l}{kn_i+l}(\langle\frac{1}{l}\rangle x, \langle\frac{1}{kn_i}\rangle x), \frac{l}{kn_i+l}(\langle\frac{1}{l}\rangle x_0, \langle\frac{1}{kn_i}\rangle x_0)) \leqslant \\ d_\alpha(\frac{l}{kn_i+l}(\langle\frac{1}{l}\rangle x, \langle\frac{1}{kn_i}\rangle x), \langle\frac{1}{kn_i}\rangle x) + d_\alpha(\langle\frac{1}{kn_i}\rangle x, \langle\frac{1}{kn_i}\rangle x_0) + \\ d_\alpha(\langle\frac{1}{kn_i}\rangle x_0, \frac{l}{kn_i+l}(\langle\frac{1}{l}\rangle x_0, \langle\frac{1}{kn_i}\rangle x_0)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{gathered}$$

Hence $d_\alpha(\langle\frac{1}{n}\rangle x, \langle\frac{1}{n}\rangle x_0)$ for all $n \geqslant n_0$. Therefore $d_\alpha(\langle\frac{1}{n}\rangle x, \langle\frac{1}{n}\rangle x_0) \to 0$, $n \to \infty$, for at least one $x_0 \in WCtr(X)$. For a particular pseudometric d_α, such x_0 form a closed set, and the family of all such sets is filtered because the family $(d_\alpha)_{\alpha\in\mathcal{A}}$ is directed. Therefore there is $x_0 \in WCtr(X)$ such that $(d_\alpha(\langle\frac{1}{n}\rangle x, \langle\frac{1}{n}\rangle x_0)) \to 0$, $n \to \infty$, for all $\alpha \in \mathcal{A}$. Thus a required x_0 exists and is unique.

Observe that, for all $\alpha \in \mathcal{A}$, $x, y \in X$, and $x_0 = wbX(x)$, $y_0 = wbX(y)$, we have $d_\alpha(x_0, y_0) = \bar{d}_\alpha(\bar{x}, \bar{y}) \leqslant d_\alpha(x_0, y_0)$, hence wbX is non-expanding. The equality $wbX(x) = x$ for $x \in WCtr(X)$ and preservation of semiconvex combinations by wbX are obvious.

Let the set $\mathbb{N} \times \mathcal{S}$ be partially ordered as follows: $(m, s)|(n, t)$ if $m|n$ (i.e. m divides n), $s|t$. For $x \in X$, consider the net $(s[\frac{1}{n}, \ldots, \frac{1}{n}]x)_{(n,s)\in(\mathbb{N}\times\mathcal{S},|)}$. It is obvious that it converges to $bX(x)$. If $s \in \mathcal{S}$ is fixed, then $(s[\frac{1}{n}, \ldots, \frac{1}{n}]x)_{n\in(\mathbb{N},|)}$ uniformly w.r.t. s and x converges to $s(wbX(x))$, and $(s(wbX(x))_{s\in(\mathcal{S},|)}$ converges to $bX(wbX(x))$. Thus $bX(x) = bX(wbX(x))$ for all $x \in X$. □

3 Spaces of atomized measures as free semiconvex compacta

Let $\mathcal{SC}$onv, $\mathcal{SSC}$onv, and $\mathcal{WSC}$onv be the categories that consist of all semiconvex compacta, of all strongly semiconvex compacta, and of all semiconvex compacta that coincide with their weak centers, respectively, and of all affine continuous mappings of these spaces. There are obvious forgetful functors:

$$\begin{tikzcd}
& \mathcal{C}\mathrm{onv} \arrow[dl, "U_{ss}"'] \arrow[dr, "U_w"] \arrow[dd, "U_s"] & \\
\mathcal{SSC}\mathrm{onv} \arrow[dr, "U_s^{ss}"] \arrow[ddr, "U^{ss}"'] & & \mathcal{WSC}\mathrm{onv} \arrow[dl, "U_s^w"'] \arrow[ddl, "U^w"] \\
& \mathcal{SC}\mathrm{onv} \arrow[d, "U^s"] & \\
& \mathcal{C}\mathrm{omp} &
\end{tikzcd}$$

Our goal is to construct free objects, and hence left adjoints [7] to these functors. Recall that a left adjoint to the composition $U^s \circ U_s : \mathcal{C}\mathrm{onv} \to \mathcal{C}\mathrm{omp}$ is known (the *probability measure functor* P [3, 10]) and thoroughly invesigated.

The "upper part" is easier. Observe first that, for a morphism $f : X \to Y$ in $\mathcal{SC}\mathrm{onv}$, the inclusions $f(Ctr(X)) \subset Ctr(Y)$, $f(WCtr(X)) \subset WCtr(Y)$ and the equalities $bY \circ f = f \circ bX$, $wbY \circ f = f \circ wbX$ are valid. Thus we denote by $Ctr(f) : Ctr(X) \to Ctr(Y)$ and $WCtr(f) : WCtr(X) \to WCtr(Y)$ the restrictions of f. They are morphisms in $\mathcal{C}\mathrm{onv}$ and $\mathcal{WSC}\mathrm{onv}$, resp., hence we obtain functors $Ctr : \mathcal{SC}\mathrm{onv} \to \mathcal{C}\mathrm{onv}$ and $WCtr : \mathcal{SC}\mathrm{onv} \to \mathcal{WSC}\mathrm{onv}$.

Then the following statements, which extend Theorem 3 [8], are at hand.

Theorem 4. *The functor Ctr is a left adjoint to the embedding of categories $U_s : \mathcal{C}\mathrm{onv} \to \mathcal{SC}\mathrm{onv}$, $bX : X \to Ctr(X)$ is a component of a natural transformation $b : \mathbf{1}_{\mathcal{SC}\mathrm{onv}} \to Ctr$, which is a unit of the adjunction. The restrictions of Ctr to $\mathcal{SSC}\mathrm{onv}$ and $\mathcal{WSC}\mathrm{onv}$ are left adjoints to the embeddings $U_{ss} : \mathcal{C}\mathrm{onv} \to \mathcal{SSC}\mathrm{onv}$ and $U_w : \mathcal{C}\mathrm{onv} \to \mathcal{WSC}\mathrm{onv}$, respectively.*

Similarly:

Theorem 5. *The functor $WCtr$ is a left adjoint to the embedding of categories $U_s^w : \mathcal{WSC}\mathrm{onv} \to \mathcal{SC}\mathrm{onv}$, $wbX : X \to WCtr(X)$ is a component of a natural transformation $wb : \mathbf{1}_{\mathcal{SC}\mathrm{onv}} \to WCtr$, which is a unit of the adjunction.*

On other words, $\mathcal{C}\mathrm{onv} \subset \mathcal{SC}\mathrm{onv}$ and $\mathcal{WSC}\mathrm{onv} \subset \mathcal{SC}\mathrm{onv}$ are *reflective* subcategories, and Ctr and $WCtr$ are *reflectors* [7].

Remark. *We leave as open the problem of explicit construction of a left adjoint to the embedding $U_s^{ss} : \mathcal{SSC}\mathrm{onv} \to \mathcal{SC}\mathrm{onv}$, although its existence is known.*

Now we consider the "lower part" of the diagram.

Proposition 3.1. *Let X be a compactum, $\langle\langle \ldots \rangle\rangle : (0; +\infty) \to G$ be a continuous homomorphism into a compact Hausdorff Abelian topological group. Let $Z = P(X \times I \times G)$, $h_\lambda(x, t, g) = (x, \lambda t, \langle\langle \lambda \rangle\rangle g)$ for all $(x, t, g) \in Z$, $\lambda \in (0; 1]$. With an operation $c : Z \times Z \times I \to Z$ that is defined by the formula*

$$c(m_1, m_2, \lambda) = \lambda P h_\lambda(m_1) + (1 - \lambda) P h_{1-\lambda}(m_2),\ 0 < \lambda < 1,$$

and $c(m_1, m_2, 1) = m_1$, $c(m_1, m_2, 0) = m_2$ for all $m_1, m_2 \in Z$, Z is a semiconvex compactum.

The weak center of Z is equal to $P(X \times \{0\} \times G)$, and the mapping wbZ is equal to Pp_0, where $p_0 : X \times I \times G \to X \times \{0\} \times G$ takes each (x, t, g) to $(x, 0, g)$,

Proof. Properties (1)-(3) are obvious, only (4') has to be verified. Let the topologies on X and on G be defined by directed families of pseudometrics $(\rho_\beta)_{\beta \in \mathcal{B}}$ and $(\theta_\gamma)_{\gamma \in \Gamma}$ respectively, and all θ_γ be invariant, i.e. $\theta_\gamma(a, b) = \theta_\gamma(ac, bc)$ for all $a, b, c \in G$. Then the functions $d_{\beta,\gamma} : Z \times Z \times I \to Z$, which are defined by the formula, for $\beta \in \mathcal{B}$, $\gamma \in \Gamma$:

$$d_{\beta,\gamma}(m_1, m_2) = \sup\bigl\{|m_1(\varphi) - m_2(\varphi)| \bigm| |\varphi(x_1, t_1, g_1) - \varphi(x_2, t_2, g_2)| < \max\{\rho_\beta(x_1, x_2), |t_1 - t_2|, \theta_\gamma(g_1, g_2)\} \text{ for all } (x_1, t_1, g_1), (x_2, t_2, g_2) \in X \times I \times G\bigr\},\ m_1, m_2 \in Z,$$

form a family of pseudometrics, which is required by (4').

Observe that if $m \in Z$, $\operatorname{supp} m \ni (x,t,g)$, $t > 0$, then m is not equal to any $(\frac{1}{n},\dots,\frac{1}{n})(m_1,\dots,m_n)$, $m_1,\dots,m_n \in Z$, whenever $\frac{1}{n} < t$. Hence $WCtr(Z) \subset P(X \times \{0\} \times G)$. On the other hand, if $m \in P(X \times \{0\} \times G)$, $0 < \lambda < 1$, then

$$m = \lambda(m_1,m_2),\ m_1 = P\big(\mathbf{1}_X \times \mathbf{1}_I \times \langle\!\langle \frac{1}{\lambda} \rangle\!\rangle(-)\big)(m),\ m_2 = P\big(\mathbf{1}_X \times \mathbf{1}_I \times \langle\!\langle \frac{1}{1-\lambda} \rangle\!\rangle(-)\big)(m).$$

Since $m_1, m_2 \in P(X \times \{0\} \times G)$, we obtain $WCtr(Z) = P(X \times \{0\} \times G)$. For $m \in Z$, $m_0 = Pp_0(m)$, $\beta \in \mathcal{B}$, $\gamma \in \Gamma$, we have $m_0 \in P(X \times \{0\} \times G)$ and $d_{\beta,\gamma}(\langle\frac{1}{n}\rangle m, \langle\frac{1}{n}\rangle m_0) \to 0$ as $n \to \infty$, therefore $wbZ(m) = Pp_0(m)$. □

We are interested in the two particular cases: $G = \{e\}$ and $G = \mathit{bohr}\,\mathbb{R}^+$ with $\langle\!\langle \lambda \rangle\!\rangle = b_{\mathbb{R}^+}(\lambda)$.

Corollary 3.1. *The sets $P^aX \subset P(X \times I) \cong P(X \times I \times \{e\})$ and $P^bX \subset P(X \times I \times \mathit{bohr}\,\mathbb{R}^+)$ are semiconvex w.r.t. the just defined semiconvex combinations.*

Proof is straightforward. Hence P^aX and P^bX are semiconvex compacta, and it is easy to see that P^aX is strongly semiconvex. Moreover, for a continuous mapping of compacta $f : X \to Y$, the mappings $P^af : P^aX \to P^aY$ and $P^bf : P^bX \to P^bY$ are affine, thus we can regard P^a and P^b as functors $\mathcal{C}\mathrm{omp} \to \mathcal{SSC}\mathrm{onv}$ and $\mathcal{C}\mathrm{omp} \to \mathcal{SC}\mathrm{onv}$, resp.

There are embeddings $\eta^aX : X \hookrightarrow P^aX$ and $\eta^bX : X \hookrightarrow P^bX$, namely:

$$\eta^aX(x) = \delta_{(x,1)},\ \eta^bX(x) = \delta_{(x,1,b_{\mathbb{R}^+}(1))},\ x \in X.$$

Theorem 6. *For a compactum X, the pairs (P^aX, η^aX) and (P^bX, η^bX) are free objects over X in $\mathcal{SSC}\mathrm{onv}$ and $\mathcal{SC}\mathrm{onv}$, respectively, and the functors $P^a : \mathcal{C}\mathrm{omp} \to \mathcal{SSC}\mathrm{onv}$ and $P^b : \mathcal{C}\mathrm{omp} \to \mathcal{SC}\mathrm{onv}$ are left adjoints to $U^{ss} : \mathcal{SSC}\mathrm{onv} \to \mathcal{C}\mathrm{omp}$ and $U^s : \mathcal{SC}\mathrm{onv} \to \mathcal{C}\mathrm{omp}$, respectively.*

Proof. Let Y be a semiconvex compactum with a directed family $(d_\alpha)_{\alpha \in \mathcal{A}}$ that satisfies (4'), and $f : X \to Y$ a continuous mapping. Assume that $\bar{f} : P^bX \to Y$ is an affine continuous extension of f. Then, for all $x_1,\dots,x_n \in X$, $\lambda_1,\dots,\lambda_n \in (0;1]$ such that $\lambda_1 + \dots + \lambda_n = 1$, we have

$$\sum_{i=1}^{n} \lambda_i \delta(x_i, \lambda_1, b_{\mathbb{R}^+}(\lambda_i)) = (\lambda_1,\dots,\lambda_n)(\eta^b(x_1),\dots,\eta^b(x_n)),$$

therefore

$$\bar{f}\big(\sum_{i=1}^{n} \lambda_i \delta(x_i, \lambda_1, b_{\mathbb{R}^+}(\lambda_i))\big) = (\lambda_1,\dots,\lambda_n)(f(x_1),\dots,f(x_n)).$$

By continuity

$$\bar{f}\big(\sum_{i=1}^{\infty} \lambda_i \delta(x_i, \lambda_1, b_{\mathbb{R}^+}(\lambda_i))\big) = (\lambda_1,\lambda_2,\dots)(f(x_1),f(x_2),\dots)$$

also for all countable sequences $x_1, x_2, \dots \in X$, $\lambda_1, \lambda_2, \dots \in (0;1]$ such that $\sum_{i=1}^{\infty} \lambda_i = 1$.

Now let $m = \delta_{(x,0,g)} \in P(X \times \{0\} \times \mathit{bohr}\, \mathbb{R}^+)$. Take a net (n_β) in $\mathbb{N}$ which existence for $g^{-1} \in \mathit{bohr}\, \mathbb{R}^+$ is guaranteed by Lemma 1.1. Then $(n_\beta) \to +\infty$, $b_{\mathbb{R}^+}(\frac{1}{n_\beta}) \to g$ imply $(x, \frac{1}{n_\beta}, b_{\mathbb{R}^+}(\frac{1}{n_\beta})) \to (x, 0, g)$, hence

$$\bar{f}(n_\beta \frac{1}{n_\beta} \delta_{(x,\frac{1}{n_\beta},b_{\mathbb{R}^+}(\frac{1}{n_\beta}))}) = \langle \frac{1}{n_\beta} \rangle \bar{f}(\eta^b X(x)) = \langle \frac{1}{n_\beta} \rangle f(x) \to \bar{f}(\delta_{(x,0,g)}) = \bar{f}(m).$$

Therefore $\bar{f}(m) \in Y_0 = WCtr(Y)$, and the net $\langle \frac{1}{n_\beta} \rangle (wbY(f(x))$ in Y_0 by Theorem 3 converges to $\bar{f}(m)$ as well. Let $Iso(Y_0)$ be the group of all bijections on Y_0 that preserve all d_α, with the topology of uniform convergence w.r.t. each of d_α. Recall that by Theorem 2 there exist (and are unique) a continuous operation $\diamond : Y_0 \times Y_0 \times I \to Y_0$ and a continuous homomorphism $\langle \dots \rangle$ of $(0; +\infty)$ into $Iso(Y_0)$ such that $(Y_0, \diamond)$ is a convex compactum, all $\langle \lambda \rangle : (Y_0, \diamond) \to (Y_0, \diamond)$ are affine, and, for all $y_1, y_2 \in Y_0$, $\lambda \in (0; 1)$, the equality $\lambda(y_1, y_2) = \lambda \diamond (\langle \lambda \rangle y_1, \langle 1 - \lambda \rangle y_2)$ holds. The group $Iso(Y_0)$ is compact Hausdorff, hence there is a unique continuous homomorphism $\langle\langle \dots \rangle\rangle : \mathit{bohr}\, \mathbb{R}^+ \to Iso(Y_0)$ such that $\langle\langle b_{\mathbb{R}^+}(\lambda) \rangle\rangle = \langle \lambda \rangle$ for all $\lambda \in \mathbb{R}_+$. Then $\langle \frac{1}{n_\beta} \rangle (wbY(f(x)) = \langle\langle b_{\mathbb{R}^+}(\frac{1}{n_\beta}) \rangle\rangle (wbY(f(x)) \to \langle\langle g \rangle\rangle (wbY(f(x))$, thus obtain $\bar{f}(\delta_{(x,0,g)}) = \langle\langle g \rangle\rangle (wbY(f(x))$.

Now let $m = \sum_{i=1}^n \lambda_i \delta_{(x_i,0,g_i)} \in P(X \times \{0\} \times \mathit{bohr}\, \mathbb{R}^+)$, $\lambda_i \neq 0$ for $i = 1, \dots, n$, then

$$m = (\lambda_1, \dots, \lambda_n)\big(\delta_{(x_1,0,b_{\mathbb{R}^+}(\lambda_1^{-1})g_1)}, \dots, \delta_{(x_n,0,b_{\mathbb{R}^+}(\lambda_n^{-1})g_n)}\big),$$

hence

$$\begin{gathered} \bar{f}(m) = (\lambda_1, \dots, \lambda_n)\big(\bar{f}(\delta_{(x_1,0,b_{\mathbb{R}^+}(\lambda_1^{-1})g_1)}), \dots, \bar{f}(\delta_{(x_n,0,b_{\mathbb{R}^+}(\lambda_n^{-1})g_n)})\big) = \\ (\lambda_1, \dots, \lambda_n)\big(\langle\langle b_{\mathbb{R}^+}(\lambda_1^{-1})g_1 \rangle\rangle wbY(f(x_1)), \dots, \langle\langle b_{\mathbb{R}^+}(\lambda_n^{-1})g_n \rangle\rangle wbY(f(x_n))\big) = \\ \lambda_1 \langle \lambda_1 \rangle \langle \lambda_1^{-1} \rangle \langle\langle g_1 \rangle\rangle (wbY(f(x_1))) + \dots + \lambda_n \langle \lambda_n \rangle \langle \lambda_n^{-1} \rangle \langle\langle g_n \rangle\rangle (wbY(f(x_n))) = \\ \lambda_1 \langle\langle g_1 \rangle\rangle (wbY(f(x_1))) + \dots + \lambda_n \langle\langle g_n \rangle\rangle (wbY(f(x_n))). \end{gathered}$$

Since measures with finite supports are dense in $P(X \times \{0\} \times \mathit{bohr}\, \mathbb{R}^+)$, there can be at most one continuous mapping $P(X \times \{0\} \times \mathit{bohr}\, \mathbb{R}^+) \to Y_0$ that agrees with the above equality, and it is determined by the formula:

$$\bar{f}(m) = cY_0 \circ P(H_f)(m),\ m \in P(X \times \{0\} \times \mathit{bohr}\, \mathbb{R}^+),$$

here $cY_0 : PY_0 \to Y_0$ is a *barycenter map* which takes each probability measure on the convex compactum Y_0 to its barycenter, and $H_f : X \times \{0\} \times \mathit{bohr}\, \mathbb{R}^+ \to Y_0$ takes each $(x, 0, g)$ to $\langle\langle g \rangle\rangle (wbY(f(x))$.

Now we are ready to study a "mixed" case

$$m = \lambda_0 m_0 + \sum_{i=1}^N \lambda_i \delta(x_i, \lambda_i, b_{\mathbb{R}^+}(\lambda_i)) \in P^b(X),\ m_0 \in P(X \times \{0\} \times \mathit{bohr}\, \mathbb{R}^+),$$

N is either finite or ∞. Such m is a combination of measures of previously considered forms:

$$m = \lambda_0 \big(P\big(\mathbf{1}_{X \times \{0\}} \times b_{R^+}(\lambda_0^{-1})(-)\big)(m_0), \sum_{i=1}^N \frac{\lambda_i}{1 - \lambda_0} \delta(x_i, \frac{\lambda_i}{1 - \lambda_0}, b_{\mathbb{R}^+}(\frac{\lambda_i}{1 - \lambda_0}))\big),$$

hence

$$\bar{f}(m) = \lambda_0\big(\bar{f}\big(P\big(\mathbf{1}_{X\times\{0\}} \times b_{R^+}(\lambda_0^{-1})(-)\big)(m_0)\big), (\frac{\lambda_1}{1-\lambda_0}, \frac{\lambda_2}{1-\lambda_0}, \dots)(f(x_1), f(x_2), \dots)\big) =$$
$$(\lambda_0, \lambda_1, \lambda_2, \dots)(cY_0 \circ P(H_{f,\lambda_0})(m_0), f(x_1), f(x_2), \dots),$$

here $H_{f,\lambda_0} : X \times \{0\} \times \mathit{bohr}\,\mathbb{R}^+ \to Y_0$ takes each $(x, 0, g)$ to $\langle\lambda_0^{-1}\rangle\langle\langle g\rangle\rangle(wbY(f(x)))$.

We obtain formulae which determine $\bar{f}$ uniquely. It is easy to see that such $\bar{f}$ is affine. Due to size limitations we omit a routine but straightforward proof of its continuity. Its idea is that a "part" of a measure that is "close" to the weak center $P(X \times \{0\} \times \mathit{bohr}\,R^+) \subset P^bX$ can be retracted by a "small move" into the weak center, on which the continuity of $\bar{f}$ is known. Only a finite number of Dirac measures will be "left", and $\bar{f}$ also acts continuously at this "part".

Thus (P^bX, η^bX) is a free semiconvex compactum over a compactum X, and P^b is a required left adjoint to U_s.

Observe that, for the projection $\mathrm{pr}_{12} : X \times I \times \mathit{bohr}\,\mathbb{R}^+ \to X \times I$, the mapping $P\,\mathrm{pr}_{12} : P(X \times I \times \mathit{bohr}\,\mathbb{R}^+) \to P(X \times I)$ is affine and maps P^bX onto P^aX. If Y is strongly semiconvex, then the used action $\langle\langle\dots\rangle\rangle$ of $\mathit{bohr}\,\mathbb{R}^+$ on $WCtr(Y) = Ctr(Y)$ is trivial, i.e. $\langle\langle g\rangle\rangle = \mathbf{1}_{Ctr(Y)}$ for all $g \in \mathit{bohr}\,\mathbb{R}^+$, hence a rapid glance at the formulae shows that $\bar{f}(m_1) = \bar{f}(m_2)$ whenever $P\,\mathrm{pr}_{12}(m_1) = P\,\mathrm{pr}_{12}(m_2) \in P^aX$. Therefore there is a unique $\bar{\bar{f}} : P^aX \to Y$ such that $\bar{\bar{f}} \circ P\,\mathrm{pr}_{12}|_{P^bX} = \bar{f}$. Taking into account $P\,\mathrm{pr}_{12} \circ \eta^bX = \eta^aX$, we obtain that (P^aX, η^aX) is a free strongly semiconvex compactum over a compactum X, and P^a is a left adjoint functor to U^{ss}. □

Now we have left adjoints $WCtr$ to U^w_s and P^b to U^s and can combine them to obtain a left adjoint to $U^w : \mathcal{WSC}\mathrm{onv} \to \mathcal{C}\mathrm{omp}$. Recall that $WCtr(P^bX) = P(X \times \{0\} \times \mathit{bohr}\,\mathbb{R}^+) \cong P(X \times \mathit{bohr}\,\mathbb{R}^+)$, hence the latter space is an object of $\mathcal{WSC}\mathrm{onv}$.

Corollary 3.2. *The functor $P(- \times \mathit{bohr}\,\mathbb{R}^+) : \mathcal{C}\mathrm{omp} \to \mathcal{WSC}\mathrm{onv}$ is a left adjoint to $U^w : \mathcal{WSC}\mathrm{onv} \to \mathcal{C}\mathrm{omp}$, and a free object over a compactum X is of the form $(P(X \times \mathit{bohr}\,\mathbb{R}^+), \eta^wX)$, $\eta^wX : X \to P(X \times \mathit{bohr}\,\mathbb{R}^+)$ is defined by the equality $\eta^wX(x) = \delta_{(x, b_{\mathbb{R}^+}(1))}$, $x \in X$.*

Final remarks

Thus we have shown that spaces of normalized (weakly) atomized measures are free (strongly) semiconvex compacta. We can consider an atomized measure either as a result of concentration of some part of "mass" in atoms, or, conversely, as a limit of "totally atomized" distributions of mass. In the latter case, even if some atoms "have dissolved", there is a reason to consider "origins" of parts of the "liquid mass". This has been formalized in a more complicated definition of atomized measure, which takes into account the *compactness* of an underlying space.

Our constructions are functorial, but the respective functors are not as "good" as the probability measure functor, in particular, they do not preserve the class of singletons and therefore do not belong to the introduced by Schepin class of normal functors.

The obtained adjunctions lead to monads and therefore to respective categories of algebras. Nevertheless, these categories are not of much interest by themselves because by results of [9] the considered forgetful functors are *monadic*.

It is also interesting whether there is a simple explicit procedure of "making a semiconvex compactum stronger", more constructive than building equalizers.

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Запроваджено класс розпорошених мір на компактах, які є узагальненням регулярних дійснозначних мір. Показано також, що простір нормованих (слабко) розпорошених мір на компакті є вільним об'єктом над цим компактом у категорії (сильно) напівопуклих компактів.

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Введен класс распыленных мер на компактах, обобщающих регулярные действительнозначные меры. Также показано, что пространство нормированных (слабо) распыленных мер на компакте является свободным объектом над этим компактом в категории (сильно) полувыпуклых компактов.