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ANALYTICITY AND UNIFORM STABILITY IN THE INVERSE SPECTRAL PROBLEM FOR IMPEDANCE STURM–LIOUVILLE OPERATORS

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We prove that the inverse spectral mapping reconstructing the impedance function of the Sturm–Liouville operators on $[0, 1]$ in impedance form from their spectral data (two spectra or one spectrum and the corresponding norming constants) is analytic and uniformly stable in a certain sense.

1 INTRODUCTION

The main goal of this paper is to establish analyticity and uniform continuity of solutions to the inverse spectral problems for a certain class of Sturm–Liouville operators on $[0, 1]$ in the so-called *impedance* form. Namely, the spectral problems of interest are

$$-(a^2(x)y'(x))' = \lambda a^2(x)y(x), \quad x \in [0, 1], \quad (1)$$

subject to suitable boundary conditions, e.g., the Neumann ones

$$y'(0) = y'(1) = 0 \quad (2)$$

or Neumann–Dirichlet ones

$$y'(0) = y(1) = 0. \quad (3)$$

Here $a > 0$ is an *impedance function*, which will be supposed to belong to the Sobolev space $W_2^1(0, 1)$, so that the logarithmic derivative $\tau := (\log a)'$ (called the *logarithmic impedance* below) is in $L_2(0, 1)$. Without loss of generality we may assume that $a(0) = 1$, so that $a(x) = \exp(\int_0^x \tau(s) ds)$. Such spectral problems arise in many applications, e.g., in modelling propagation of sound waves in a duct [44], torsional vibrations of the earth [17] or longitudinal vibrations in a thin straight rod [13].

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The corresponding differential operators S_N and S_D given by the differential expression $\ell(y) := a^{-2}(a^2y)'$ and boundary conditions (2) and (3) respectively are self-adjoint in the weighted Hilbert space $L_2((0, 1); a^2 dx)$ and have simple discrete spectra accumulating at $+\infty$. We denote by $0 = \lambda_0 < \lambda_1 < \dots$ the eigenvalues of S_N and by $0 < \mu_0 < \mu_1 < \dots$ those of S_D . The inverse spectral problem is to reconstruct the impedance function a or its logarithm τ from the spectra of S_N and/or S_D .

For the standard Sturm–Liouville operators, i.e., those generated by the differential expression

$$-\frac{d^2}{dx^2} + q,$$

with q a real-valued locally integrable *potential*, it was proved by Borg [7] in 1946 that, generically, knowledge of the spectrum corresponding to one set of boundary conditions (e.g. Neumann ones or Neumann–Dirichlet ones) does not allow to unambiguously determine q . (An exceptional situation where this is possible was pointed out by Ambartzumyan [5] in 1929.) However, two such spectra do uniquely determine q .

The same holds true for the inverse spectral problem of reconstructing the impedance function a of the operators S_N or S_D . In fact, these operators are unitarily equivalent to self-adjoint operators T_N and T_D acting in $L_2(0, 1)$ and generated by the differential expression

$$\ell(\tau) := -\frac{1}{a} \frac{d}{dx} a^2 \frac{d}{dx} \frac{1}{a} = -\left(\frac{d}{dx} + \tau\right) \left(\frac{d}{dx} - \tau\right) \quad (4)$$

and the boundary conditions

$$y^{[1]}(0) = y^{[1]}(1) = 0 \quad (5)$$

and

$$y^{[1]}(0) = y(1) = 0 \quad (6)$$

respectively. Here and hereafter $f^{[1]}(x) := f'(x) - \tau(x)f(x)$ shall denote the *quasi-derivative* of a function f . Moreover, for $a \in W_2^2(0, 1)$ the differential expression $\ell(\tau)$ can be recast in the potential form

$$\ell(\tau) = -\frac{d^2}{dx^2} + \tau' + \tau^2$$

with potential $q = \tau' + \tau^2$. For $a \in W_2^1(0, 1)$ the reduction to the potential form is still possible, but the potential q becomes a distribution from $W_2^{-1}(0, 1)$ [39]. Sturm–Liouville and Schrödinger operators with singular potentials (that are, e.g., point interactions, measures, or distributions) have been widely studied; we refer the reader, e.g., to the books [1, 3] and to review paper [40] where additional references can be found. Inverse problems for distributional potentials in the space $W_2^{-1}(0, 1)$ have also been successfully treated; see, e.g., [24, 41].

This suggests the following method of solving the inverse spectral problem for impedance Sturm–Liouville operators under consideration: first, one recasts the problem (1) in the potential form, then uses one of the algorithms reconstructing the potential q from the spectral data $((\lambda_n), (\mu_n))$ of T_N and T_D , and, finally, finds τ by solving the Riccati differential equation $\tau' + \tau^2 = q$. However, this equation may not possess global solutions on

$[0, 1]$, whence it is desirable to find a way to reconstruct the impedance a or its logarithmic derivative τ directly from the spectral data for the operators T_N and T_D .

In the papers [2, 6, 8, 32, 35] several approaches to reconstruction of the impedance $a \in W_2^1(0, 1)$ were suggested and the corresponding spectral data were completely described. These necessary and sufficient conditions require that the spectra (λ_n) and (μ_n) must

- (i) interlace, i.e., that $\lambda_n < \mu_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}_+$, and
- (ii) satisfy the asymptotic relations

$$\sqrt{\lambda_n} = \pi n + \rho_{2n}, \quad \sqrt{\mu_n} = \pi(n + \frac{1}{2}) + \rho_{2n+1},$$

where the sequence (ρ_n) belongs to ℓ_2 .

Moreover, the induced mapping from the spectral data $((\lambda_n), (\mu_n))$ into the impedance function a providing a solution to the inverse spectral problem was shown in [6] and [32] to be locally continuous in a certain sense. In particular, this yields local stability of the inverse spectral problem; see also similar stability results for the related problem of reconstructing the potential q in [4, 7, 16, 19–21, 31, 33, 34, 36–38, 46]. Here we introduce a metric on the set of the spectral data $((\lambda_n), (\mu_n))$ by e.g. identifying such data with the sequence (ρ_n) in the representation of item (ii) above. Typically, this local stability states that, for a fixed $M > 0$, there are positive ε and L with the following property: if potentials q_1 and q_2 (resp., logarithmic impedances τ_1 and τ_2) are such that $\|q_1\|_* \leq M$ and $\|q_2\|_* \leq M$ (resp., $\|\tau_1\|_* \leq M$ and $\|\tau_2\|_* \leq M$) and the corresponding spectral data $\nu_1 := ((\lambda_{1,n}), (\mu_{1,n}))$ and $\nu_2 := ((\lambda_{2,n}), (\mu_{2,n}))$ satisfy $\|\nu_1 - \nu_2\| \leq \varepsilon$, then

$$\|q_1 - q_2\|_* \leq L\|\nu_1 - \nu_2\| \tag{7}$$

(resp., then

$$\|\tau_1 - \tau_2\|_* \leq L\|\nu_1 - \nu_2\|) \tag{8}$$

for a suitable norm $\|\cdot\|_*$. For instance, local stability results with respect to the $L_2(0, 1)$ -norm were established in [32, 38] in the regular case $q \in L_2(0, 1)$, and in [6, 8, 32] for impedance Sturm–Liouville operators. In [16, 33] the case $L_\infty(0, 1)$ was treated; earlier Hochstadt in [20, 21] proved stability if only finitely many eigenvalues in one spectrum are changed. The papers [19, 36] studied to what extent only finitely many eigenvalues in one or both spectra determine the potential, and the latter problem in the non-self-adjoint setting was recently discussed in [31]. Also, stability of the inverse spectral problems on semi-axis was proved in [30, 37], and the inverse scattering problem on the line was studied in [10, 18].

However, the above results cannot be considered satisfactory, as they refer to the norm of the potential q (resp. of the logarithmic impedance τ) to be recovered and thus specify neither the allowed noise level ε nor the Lipschitz constant L . Therefore we need a global stability result that asserts (7) whenever the spectral data ν_1 and ν_2 run through bounded sets \mathcal{N} and with L only depending on \mathcal{N} .

Recently, such a uniform stability in the inverse spectral problem for Sturm–Liouville operators on $[0, 1]$ was established by Shkalikov and Savchuk [43]. They considered operators

with real-valued potentials from the Sobolev spaces $W_2^s(0, 1)$ with $s > -1$. (For negative s , such potentials are distributions; see [40] for the review on Sturm–Liouville operators with distributional potentials.) Their approach for solving the inverse spectral problem was based on the so called Prüfer angle and used extensively the implicit function theorem. In our work [22] analyticity and global stability of the inverse spectral mapping for $s \in [-1, 0]$ was established using a different approach that generalizes the classical method due to Gelfand and Levitan [12] and Marchenko [29] and has been successfully applied to reconstruction of Sturm–Liouville operators with singular potentials in [24, 25].

The main aim of this paper is to prove analyticity and Lipschitz continuity on bounded subsets of the inverse spectral mapping $((\lambda_n), (\mu_n)) \mapsto \tau$ for the class of the Sturm–Liouville operators in impedance form with logarithmic impedance $\tau \in L_2(0, 1)$. To this end we use the approach of [2] to the inverse spectral problem for impedance Sturm–Liouville operators based on the Krein equation [27] and further develop the methods of [22]. Also, we discuss the analogous properties in the inverse spectral problem of reconstruction of τ from the Neumann spectrum (λ_n) and the corresponding norming constants α_n defined in Subsection 2.1.

We mention that the methods of [2] could be used to treat logarithmic impedances τ belonging to $L_p(0, 1)$ with $p \in [1, \infty)$. However, apart from some technicalities caused by more complicated properties of the Fourier transform in $L_p(0, 1)$ for $p \neq 2$, the approach would remain the same and we decided to sacrifice the generality to simplicity of presentation. See Section 5 for discussion of possible generalizations.

The paper is organised as follows. In the next section, we state the main results of the paper and recall the method of reconstructing the impedance Sturm–Liouville operators from their spectral data using the Gelfand–Levitan–Marchenko and Krein equations. In Section 3, we show analyticity and uniform continuity in the inverse problem of reconstructing the logarithmic impedance τ from the spectrum of the operator $T_N(\tau)$ and the sequence of the corresponding norming constants. Reconstruction from two spectra (those of $T_N(\tau)$ and $T_D(\tau)$) is discussed in Section 4; there the problem is reduced to the one studied in Section 3 by showing that the norming constants depend analytically and Lipschitz continuously on these spectra. The last Section 5 discusses some ways of extending the results to a wider class of operators. Finally, three appendices contain auxiliary results on some related nonlinear mappings in $L_2(0, 1)$, on relation between some analytic functions of sine type and their zeros, and on the special Banach algebra that were used in the proofs.

2 PRELIMINARIES AND MAIN RESULTS

In this section we state the main results of the paper and recall the method of solution of the inverse spectral problem based on the Gelfand–Levitan–Marchenko [28] and Krein [27] equations. All the missing details can be found in [2].

2.1 Spectral data

Throughout this subsection, τ designates a fixed real-valued function in $L_2(0, 1)$. We denote by λ_n and μ_n , $n \in \mathbb{Z}_+$, the eigenvalues of the operators $T_N(\tau)$ and $T_D(\tau)$ respectively defined

via (4)–(6) and recall that these eigenvalues interlace, i.e., $\lambda_n < \mu_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}_+$, and satisfy the relations

$$\sqrt{\lambda_n} = \pi n + \rho_{2n}, \quad \sqrt{\mu_n} = \pi(n + \frac{1}{2}) + \rho_{2n+1} \quad (9)$$

with some $\ell_2(\mathbb{Z}_+)$ -sequence $\boldsymbol{\rho} = (\rho_n)$.

For $\lambda = \omega^2 \in \mathbb{C}$, the equation $\ell(\tau)u = \omega^2 u$ subject to the initial conditions $u(0) = 1$ and $u^{[1]}(0) = 0$ has the solution

$$c(x, \omega) = \cos \omega x + \int_0^x k(x, t) \cos \omega t dt, \quad (10)$$

where k is the kernel of the so called *transformation operator*. Clearly, $\cos \omega x$ is a solution of the “unperturbed” equation $\ell(0)u = \omega^2 u$ with $\tau = 0$; it is mapped into the solution $c(\cdot, \omega)$ for a generic τ by means of the transformation operator via (10). The function k vanishes for a.e. $(x, t) \in [0, 1]^2$ with $x < t$ and, for every $x \in [0, 1]$, $k(x, \cdot)$ belongs to $L_2(0, 1)$ and the mapping $x \mapsto k(x, \cdot)$ is continuous from $[0, 1]$ into $L_2(0, 1)$. Also, there exists a kernel k_1 with similar properties such that

$$c^{[1]}(x, \omega) = -\omega \sin \omega x - \omega \int_0^x k_1(x, t) \sin \omega t dt; \quad (11)$$

we recall that $f^{[1]} := f' - \tau f$ is the quasi-derivative of a function f .

Set $\omega_{2n} := \sqrt{\lambda_n}$ and $\omega_{2n+1} := \sqrt{\mu_n}$, $n \in \mathbb{Z}_+$. Then $c(\cdot, \omega_{2n})$ is an eigenfunction of the operator $T_N(\tau)$ corresponding to the eigenvalue $\lambda_n = \omega_{2n}^2$, and we call the number¹ $\alpha_n := 1/(2\|c(\cdot, \omega_{2n})\|^2)$ the *norming constant* for this eigenvalue. It is known [2] that

$$\alpha_n = 1 + \beta_n, \quad (12)$$

where the sequence $\boldsymbol{\beta} := (\beta_n)_{n \in \mathbb{Z}_+}$ belongs to ℓ_2 . Moreover, the norming constants α_n can be determined from the spectra of the operators $T_N(\tau)$ and $T_D(\tau)$ as follows. We set $C(\omega) := c(1, \omega)$ and $S(\omega) := c^{[1]}(1, \omega)$; due to (10) and (11) these are entire functions of exponential type 1 with zeros $\pm\sqrt{\mu_n}$ and $\pm\sqrt{\lambda_n}$ respectively. The Hadamard canonical products for S and C are

$$S(\omega) = \omega^2 \prod_{n=1}^{\infty} \frac{\omega_{2n}^2 - \omega^2}{\pi^2 n^2}, \quad C(\omega) = \prod_{n=0}^{\infty} \frac{\omega_{2n+1} - \omega^2}{\pi^2 (n + \frac{1}{2})^2}, \quad (13)$$

so that S and C are uniquely determined by their zeros. Then we have (cf. [2])

$$\alpha_n = -\frac{\omega_{2n}}{\dot{S}(\omega_{2n})C(\omega_{2n})}, \quad (14)$$

where the dot denotes the derivative in ω .

Here and hereafter, $\|f\|$ shall stand for the $L_2(0, 1)$ -norm of a function f .

2.2 The main results

We introduce the set \mathcal{N} of pairs $((\lambda_n)_{n \in \mathbb{Z}_+}, (\mu_n)_{n \in \mathbb{Z}_+})$ with the following properties:

- the sequences (λ_n) and (μ_n) strictly interlace, i.e., $\lambda_n < \mu_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}_+$;
- the sequence $\boldsymbol{\rho} := (\rho_k)_{k \in \mathbb{Z}_+}$, with $\rho_{2n} := \sqrt{\lambda_n} - \pi n$ and $\rho_{2n+1} := \sqrt{\mu_n} - \pi(n + \frac{1}{2})$, belongs to ℓ_2 .

In this way every element $\boldsymbol{\nu} := ((\lambda_n), (\mu_n))$ of \mathcal{N} is identified with a sequence (ρ_n) in ℓ_2 thus inducing a metric on \mathcal{N} . Namely, if $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ are elements of \mathcal{N} and $\boldsymbol{\rho}_1 := (\rho_{1,n})$ and $\boldsymbol{\rho}_2 := (\rho_{2,n})$ are the corresponding ℓ_2 -sequences of remainders, then

$$\text{dist}_{\mathcal{N}}(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) := \|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\|_{\ell_2}.$$

In what follows, $\boldsymbol{\nu}_0$ shall stand for the element of \mathcal{N} corresponding to $\boldsymbol{\rho} = 0$; then we get $\text{dist}_{\mathcal{N}}(\boldsymbol{\nu}, \boldsymbol{\nu}_0) = \|(\rho_n)\|_{\ell_2}$.

According to [2], every element of \mathcal{N} gives the eigenvalue sequences of the operators $T_{\text{N}}(\tau)$ and $T_{\text{D}}(\tau)$ for a unique real-valued function $\tau \in L_2(0, 1)$ and, conversely, for every real-valued $\tau \in L_2(0, 1)$ the spectra of the corresponding Sturm–Liouville operators $T_{\text{N}}(\tau)$ and $T_{\text{D}}(\tau)$ form an element of \mathcal{N} . When the logarithmic impedance τ varies over a bounded subset of $L_2(0, 1)$, then the corresponding spectral data $((\lambda_n), (\mu_n))$ remain in a bounded subset of \mathcal{N} . Moreover, the Prüfer angle technique (cf. [41, 42]) yields then a positive d such that all the corresponding spectral data $((\lambda_n), (\mu_n))$ are d -separated, i.e., that $\mu_n - \lambda_n \geq d$ and $\lambda_{n+1} - \mu_n \geq d$ for every $n \in \mathbb{Z}_+$. Summarizing, we conclude that the uniform stability of the inverse spectral problem we would like to establish is only possible on bounded sets of spectral data in \mathcal{N} that are d -separated for some $d > 0$.

This motivates the following definition.

Definition 2.1. For $d \in (0, \pi/2)$ and $r > 0$, we denote by $\mathcal{N}(d, r)$ the set of all $\boldsymbol{\nu} \in \mathcal{N}$ that are d -separated and satisfy $\text{dist}_{\mathcal{N}}(\boldsymbol{\nu}, \boldsymbol{\nu}_0) \leq r$.

In these notations, the first main result of the paper reads as follows.

Theorem 1. For every $d \in (0, \pi/2)$ and $r > 0$, the inverse spectral mapping

$$\mathcal{N}(d, r) \ni \boldsymbol{\nu} \mapsto \tau \in L_2(0, 1) \tag{15}$$

is analytic and Lipschitz continuous.

See [9] for analyticity of mapping between Banach spaces. In fact, as in [22], we prove first the analyticity and Lipschitz continuity of the inverse spectral problem of reconstructing τ from the Neumann spectrum (λ_n) and the norming constants (α_n) (see Theorem 2 below), and then derive Theorem 1 by showing that the norming constants depend analytically and Lipschitz continuously on the two spectra.

More exactly, we denote by \mathcal{L} the family of strictly increasing sequences $\boldsymbol{\lambda} := (\lambda_n)_{n \in \mathbb{Z}_+}$ such that $\rho_{2n} := \sqrt{\lambda_n} - \pi n$ form an element of ℓ_2 and pull back the topology on \mathcal{L} from that

of ℓ_2 by identifying such $\boldsymbol{\lambda}$ with $(\rho_{2n}) \in \ell_2$. For $d \in (0, \pi)$ and $r > 0$, we denote by $\mathcal{L}(d, r)$ the closed convex subset of \mathcal{L} consisting of sequences $(\lambda_n)_{n \in \mathbb{Z}_+}$ such that $\|(\rho_{2n})\|_{\ell_2} \leq r$ and $\lambda_{n+1} - \lambda_n \geq d$ for all $n \in \mathbb{Z}_+$. Next, we write \mathcal{A} for the set of sequences $\boldsymbol{\alpha} := (\alpha_n)_{n \in \mathbb{Z}}$ of positive numbers such that the sequence (β_n) with $\beta_n := \alpha_n - 1$ belongs to ℓ_2 . This induces the topology of ℓ_2 on \mathcal{A} ; we further consider closed subsets $\mathcal{A}(d, r)$ of \mathcal{A} consisting of all (α_n) satisfying the inequalities $\alpha_n \geq d$ for all $n \in \mathbb{Z}$ and the relation $\|(\beta_n)\|_{\ell_2} \leq r$.

It is known [2] that, given an element $(\boldsymbol{\lambda}, \boldsymbol{\alpha}) \in \mathcal{L} \times \mathcal{A}$, there is a unique real-valued $\tau \in L_2(0, 1)$ such that $\boldsymbol{\lambda}$ is the sequence of eigenvalues and $\boldsymbol{\alpha}$ the sequence of norming constants for the Sturm–Liouville operator $T_N(\tau)$. Some further properties of the induced mapping are described in the following theorem.

Theorem 2. *For every $d \in (0, \pi)$ and $d' \in (0, 1)$ and every positive r and r' , the inverse spectral mapping*

$$\mathcal{L}(d, r) \times \mathcal{A}(d', r') \ni (\boldsymbol{\lambda}, \boldsymbol{\alpha}) \mapsto \tau \in L_2(0, 1)$$

is analytic and Lipschitz continuous.

2.3 Solution of the inverse spectral problem using the Krein equation

The classical algorithm of reconstructing the potential $q = \tau' + \tau^2$ of a Sturm–Liouville operator uses the so called Gelfand–Levitan–Marchenko (GLM) equation relating the spectral data $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ and the transformation operator K , see e.g. the monographs [28, 29] for details. The derivation of the GLM equation sketched below follows the reasoning of [24], to which we refer the reader for further details.

First we notice that due to the asymptotics of λ_n and α_n the series in

$$h(s) := 1 + 2 \sum_{n=0}^{\infty} [\alpha_n \cos(2\omega_{2n}s) - \cos(2\pi ns)] \quad (16)$$

converges in $L_2(0, 1)$ (in fact, h is an even function on $(-1, 1)$). Next, denote by F an integral operator in $L_2(0, 1)$ with kernel

$$f(x, t) := \frac{1}{2} \left[h\left(\frac{x+t}{2}\right) + h\left(\frac{x-t}{2}\right) \right]. \quad (17)$$

Starting with the resolution of identity for the operator $T_N(\tau)$,

$$I = 2 \sum_{n=0}^{\infty} \alpha_n(\cdot, c_n)c_n,$$

with $c_n = c(\cdot, \omega_{2n})$ being the eigenfunction corresponding to the eigenvalue $\lambda_n = \omega_{2n}^2$, and using the relations (10) and the definition of F , after straightforward transformations one arrives at the equality

$$I = (I + K)(I + F)(I + K^*).$$

Actually, the above equality rewritten in terms of the kernels k and f of the operators K and F produces the GLM equation,

$$k(x, t) + f(x, t) + \int_0^x k(x, s)f(s, t) ds = 0, \quad x > t. \quad (18)$$

Given the spectral data and thus the kernel f , one solves the GLM equation for the kernel k and then determines the potential q from the relation

$$q(x) = 2 \frac{d}{dx} k(x, x). \quad (19)$$

However, this approach does not work for impedance Sturm–Liouville operators under consideration since formula (19) is then meaningless: indeed, the kernel k is not regular enough to have a well-defined restriction $k(x, x)$ to the diagonal and the potential $q = \tau' + \tau^2$ is a distribution rather than a regular function. Instead, one can use the method of Krein that reconstructs the function $\tau \in L_2(0, 1)$ directly. The original method was suggested by Krein [27] for smooth functions τ and was further developed for the class of impedance Sturm–Liouville operators with $\tau \in L_p(0, 1)$, $p \in [1, \infty)$ in [2].

Namely, with the function h of (16), one considers a different GLM-type integral equation (called the *Krein equation*)

$$r(x, t) + h(x - t) + \int_0^x r(x, s)h(s - t) ds = 0, \quad 0 < t < x < 1, \quad (20)$$

of which the GLM equation (18) is the even part (in the sense that if r is a solution to (20), then the function

$$k(x, t) := \frac{1}{2} \left[r(x, \frac{x-t}{2}) + r(x, \frac{x+t}{2}) \right]$$

solves (18)). It can be proved (see the next section) that equation (20) possesses a unique solution r and, moreover, the function τ satisfies the equality

$$\tau = r(\cdot, 0). \quad (21)$$

This formula will be the basis of the reconstruction algorithm and stability analysis.

3 STABILITY OF THE INVERSE SPECTRAL PROBLEM: NORMING CONSTANTS

In this section, we prove Theorem 2 on analytic and Lipschitz continuous dependence of the logarithmic potential τ determining the impedance Sturm–Liouville operator $T_N(\tau)$ on its eigenvalues λ_n and norming constants α_n .

We shall study the correspondence between the data $(\boldsymbol{\lambda}, \boldsymbol{\alpha}) \in \mathcal{L}(d, r) \times \mathcal{A}(d', r')$ and the functions τ through the chain of mappings

$$(\boldsymbol{\lambda}, \boldsymbol{\alpha}) \mapsto h \mapsto r \mapsto \tau,$$

in which h is the function of (16), r is the kernel solving the Krein equation (20), and, finally, τ is given by (21).

Lemma 3.1. *The mapping*

$$\mathcal{L}(d, r) \times \mathcal{A}(d', r') \ni (\boldsymbol{\lambda}, \boldsymbol{\alpha}) \mapsto h \in L_2(0, 1)$$

is analytic and Lipschitz continuous.

Proof. We have $h = 1 + h_{\boldsymbol{\lambda}} + h_{\boldsymbol{\lambda}, \boldsymbol{\alpha}}$, where

$$h_{\boldsymbol{\lambda}}(s) := 2 \sum_{n=0}^{\infty} [\cos(2\omega_{2n}s) - \cos(2\pi ns)], \quad h_{\boldsymbol{\lambda}, \boldsymbol{\alpha}}(s) := 2 \sum_{n=0}^{\infty} \beta_n \cos(2\omega_{2n}s);$$

recall that the numbers $\rho_{2n} = \omega_{2n} - \pi n$ and $\beta_n := \alpha_n - 1$ form sequences in ℓ_2 that induce the topology of \mathcal{L} and \mathcal{A} .

Introduce the function $f_{\boldsymbol{\lambda}} \in L_2(0, 1)$ whose Fourier coefficients are $\hat{f}_{\boldsymbol{\lambda}}(0) = 0$ and

$$\hat{f}_{\boldsymbol{\lambda}}(n) = -\hat{f}_{\boldsymbol{\lambda}}(-n) := \rho_{2n}$$

for $n \in \mathbb{N}$; then we have $h_{\boldsymbol{\lambda}} = \Phi_1(f_{\boldsymbol{\lambda}})$ with the mapping Φ_1 of Lemma A.1. Therefore the function $h_{\boldsymbol{\lambda}}$ depends analytically and Lipschitz continuously on $f_{\boldsymbol{\lambda}}$ in bounded sets. Since the mapping sending $(\rho_{2n}) \in \ell_2$ into $f_{\boldsymbol{\lambda}} \in L_2(0, 1)$ is linear and *quasi-isometric* in the sense that $\|f_{\boldsymbol{\lambda}}\| = \sqrt{2}\|(\rho_{2n})\|$, we conclude that the mapping $\boldsymbol{\lambda} \mapsto h_{\boldsymbol{\lambda}}$ is analytic and Lipschitz continuous on bounded sets.

Next, let $g_{\boldsymbol{\alpha}}$ be the function in $L_2(0, 1)$ whose Fourier coefficients are

$$\hat{g}_{\boldsymbol{\alpha}}(n) = \hat{g}_{\boldsymbol{\alpha}}(-n) := \beta_n$$

for $n \in \mathbb{Z}_+$. Then $h_{\boldsymbol{\lambda}, \boldsymbol{\alpha}} = \Phi_2(f_{\boldsymbol{\lambda}}, g_{\boldsymbol{\alpha}})$ with Φ_2 being the mapping of Lemma A.2. The properties of Φ_2 and of the mapping $(\beta_n) \mapsto g_{\boldsymbol{\alpha}}$ then establish the required dependence of $h_{\boldsymbol{\lambda}, \boldsymbol{\alpha}}$ on $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$. The lemma is proved. \square

Solubility of the Krein equation crucially relies on the following property of the convolution operator $H = H(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ defined via

$$(Hf)(x) := \int_0^1 h(x-t)f(t) dt,$$

with the function h of (16).

Lemma 3.2. *For every $d \in (0, \pi)$, $d' \in (0, 1)$, and positive r and r' , there exists $\varepsilon > 0$ with the following property: if $(\boldsymbol{\lambda}, \boldsymbol{\alpha})$ is an arbitrary element of $\mathcal{L}(d, r) \times \mathcal{A}(d', r')$ and h is the function of (16), then for the corresponding convolution operator H we have $I + H \geq \varepsilon I$.*

Proof. Observing that

$$\int_0^1 \cos 2\pi n(x-t)f(t) dt = \cos 2\pi nx \int_0^1 \cos 2\pi nt f(t) dt + \sin 2\pi nx \int_0^1 \sin 2\pi nt f(t) dt$$

and that the functions $1, \sqrt{2} \sin 2\pi nx, \sqrt{2} \cos 2\pi nx$, $n \in \mathbb{N}$, form an orthonormal basis of $L_2(0, 1)$, we find that

$$\begin{aligned} ((I + H)f, f) &= (f, f) + 2 \lim_{k \rightarrow \infty} \sum_{n=0}^k \alpha_n [|(f, \cos 2\omega_{2n}s)|^2 + |(f, \sin 2\omega_{2n}s)|^2] \\ &\quad - |(f, 1)|^2 - 2 \lim_{k \rightarrow \infty} \sum_{n=1}^k [|(f, \cos 2\pi ns)|^2 + |(f, \sin 2\pi ns)|^2] \\ &= 2 \sum_{n=0}^{\infty} \alpha_n [|(f, \cos 2\omega_{2n}s)|^2 + |(f, \sin 2\omega_{2n}s)|^2] \\ &= 2\alpha_0 |(f, 1)|^2 + \sum_{n=1}^{\infty} \alpha_n [|(f, e^{-2\omega_{2n}is})|^2 + |(f, e^{2\omega_{2n}is})|^2]. \end{aligned}$$

It follows from the results of [14, Ch. VI], [45, Ch. 4] that the system

$$\mathcal{E}_{\lambda} := \{e^{-2\omega_{2n}is}\}_{n \in \mathbb{N}} \cup \{1\} \cup \{e^{2\omega_{2n}is}\}_{n \in \mathbb{N}}$$

is a Riesz basis of $L_2(0, 1)$. Moreover, it was shown in [23] that there exists $m = m(d, r) > 0$ that gives a lower bound of \mathcal{E}_{λ} for every $\lambda \in \mathcal{L}(d, r)$. Since the inclusion $\alpha \in \mathcal{A}(d', r')$ implies that $\alpha_n \geq d'$ for all $n \in \mathbb{Z}$, we get

$$((I + H)f, f) = 2\alpha_0 |(f, 1)|^2 + \sum_{n=1}^{\infty} \alpha_n [|(f, e^{-2\omega_{2n}is})|^2 + |(f, e^{2\omega_{2n}is})|^2] \geq d'm \|f\|^2,$$

and the proof is complete. \square

To study solubility of the Krein equation (20), we shall regard it as a relation between the corresponding integral operators. To this end we recall several notions that will be used. The ideal \mathfrak{S}_2 of Hilbert–Schmidt operators in $L_2(0, 1)$ consists of integral operators whose kernels are square integrable on $\Omega := [0, 1] \times [0, 1]$. The linear set \mathfrak{S}_2 becomes a Hilbert space under the scalar product

$$\langle A, B \rangle_2 := \text{tr}(AB^*) := \int_0^1 \int_0^1 a(x, y) \overline{b(x, y)} dx dy,$$

where a and b are the kernels of A and B respectively; in particular, $\|A\|_{\mathfrak{S}_2} := \langle A, A \rangle_2^{1/2}$ is the corresponding norm.

As an example, the inequality

$$\int_0^1 \int_0^1 |h(x - y)|^2 dx dy \leq 2\|h\|^2$$

implies that the convolution operator H belongs to \mathfrak{S}_2 and, moreover, $\|H\|_{\mathfrak{S}_2}^2 \leq 2\|h\|^2$.

Denote by \mathfrak{S}_2^+ the subspace of \mathfrak{S}_2 consisting of all Hilbert–Schmidt operators with lower-triangular kernels. In other words, $A \in \mathfrak{S}_2$ belongs to \mathfrak{S}_2^+ if the kernel a of A satisfies

$a(x, y) = 0$ for a.e. $0 \leq x < y \leq 1$. For an arbitrary $A \in \mathfrak{S}_2$ with kernel a the cut-off a^+ of a given by

$$a^+(x, y) = \begin{cases} a(x, y) & \text{for } x \geq y, \\ 0 & \text{for } x < y \end{cases}$$

generates an operator $A^+ \in \mathfrak{S}_2^+$, and the corresponding mapping $\mathcal{P}^+ : A \mapsto A^+$ turns out to be an orthoprojector in \mathfrak{S}_2 onto \mathfrak{S}_2^+ , i.e. $(\mathcal{P}^+)^2 = \mathcal{P}^+$ and $\langle \mathcal{P}^+A, B \rangle_2 = \langle A, \mathcal{P}^+B \rangle_2$ for all $A, B \in \mathfrak{S}_2$; see details in [15, Ch. I.10].

With these notations, the Krein equation (20) can be recast as

$$R + \mathcal{P}^+H + \mathcal{P}^+(RH) = 0 \quad (22)$$

or

$$(\mathcal{I} + \mathcal{P}_H^+)R = -\mathcal{P}^+H,$$

where \mathcal{P}_X^+ is the linear operator in \mathfrak{S}_2 defined by $\mathcal{P}_X^+Y = \mathcal{P}^+(YX)$ and \mathcal{I} is the identity operator in \mathfrak{S}_2 . Therefore solubility of the Krein equation and continuity of its solutions on H is strongly connected with the properties of the operator \mathcal{P}_H^+ .

Lemma 3.3. *For every $X \in \mathcal{B}$, the operator \mathcal{P}_X^+ is bounded in \mathfrak{S}_2 . Moreover, for every convolution operator H from the set*

$$\mathfrak{H} := \{H = H(\lambda, \alpha) \mid (\lambda, \alpha) \in \mathcal{L}(d, r) \times \mathcal{A}(d', r')\} \subset \mathfrak{S}_2$$

the operator $\mathcal{I} + \mathcal{P}_H^+$ is invertible in $\mathcal{B}(\mathfrak{S}_2^+)$ and the inverse $(\mathcal{I} + \mathcal{P}_H^+)^{-1}$ depends analytically and Lipschitz continuously on $H \in \mathfrak{H}$ in the topology of \mathfrak{S}_2 .

Proof. Boundedness of \mathcal{P}_X^+ is a straightforward consequence of the inequality

$$\|\mathcal{P}_X^+Y\|_{\mathfrak{S}_2} \leq \|YX\|_{\mathfrak{S}_2} \leq \|X\|_{\mathcal{B}}\|Y\|_{\mathfrak{S}_2},$$

cf. [14, Ch. 3]. Assume next that $I + X \geq \varepsilon I$ in $L_2(0, 1)$; then for $Y \in \mathfrak{S}_2^+$ we find that

$$\langle (\mathcal{I} + \mathcal{P}_X^+)Y, Y \rangle_2 = \langle Y, Y \rangle_2 + \langle YX, Y \rangle_2 = \text{tr}(Y(I + X)Y^*).$$

Since $Y(I + X)Y^* \geq \varepsilon YY^*$ and the trace is a monotone functional, we get

$$\langle (\mathcal{I} + \mathcal{P}_X^+)Y, Y \rangle_2 \geq \varepsilon \langle Y, Y \rangle_2,$$

i.e., $\mathcal{I} + \mathcal{P}_X^+ \geq \varepsilon \mathcal{I}$ in \mathfrak{S}_2^+ .

Applying now Lemma 3.2, we conclude that for every $H \in \mathfrak{H}$ it holds $\mathcal{I} + \mathcal{P}_H^+ \geq \varepsilon \mathcal{I}$ with ε of that lemma depending only on d, d', r , and r' ; therefore, $\mathcal{I} + \mathcal{P}_H^+$ is boundedly invertible in $\mathcal{B}(\mathfrak{S}_2^+)$ and

$$\|(\mathcal{I} + \mathcal{P}_H^+)^{-1}\|_{\mathcal{B}(\mathfrak{S}_2^+)} \leq \varepsilon^{-1}.$$

Since \mathcal{P}_H^+ depends linearly on H , it follows that the mapping $H \mapsto (\mathcal{I} + \mathcal{P}_H^+)^{-1}$ from \mathfrak{S}_2 into $\mathcal{B}(\mathfrak{S}_2^+)$ is analytic and Lipschitz continuous on the set \mathfrak{H} . The proof is complete. \square

Corollary 3.1. *For every $H \in \mathfrak{H}$, the Krein equation (22) has a unique solution*

$$R := -(\mathcal{I} + \mathcal{P}_H^+)^{-1} \mathcal{P}^+ H \in \mathfrak{S}_2^+;$$

moreover, R depends analytically and Lipschitz continuously in \mathfrak{S}_2^+ on $H \in \mathfrak{H} \subset \mathfrak{S}_2$.

It follows that the kernel $r(x, t)$ of R is square integrable in the domain Ω and depends analytically and Lipschitz continuously in $L_2(\Omega)$ on H . However, we need to know that $r(\cdot, 0)$ is well defined and belongs to $L_2(0, 1)$.

To this end we use the Krein equation to find that

$$r(x, t) = -h(x - t) - \int_0^1 r(x, s)h(s - t) ds$$

as a function of x depends continuously in $L_2(0, 1)$ on $t \in [0, 1]$. Indeed, since the shift $f(\cdot) \mapsto f(\cdot - t)$ is a continuous operation in $L_2(\mathbb{R})$, $h(\cdot - t)$ enjoys the required property. Next, since the kernels r and h belong to $L_2(\Omega)$, we find that

$$\begin{aligned} & \int_0^1 \left| \int_0^1 r(x, s)h(s - t) ds \right|^2 dx \\ & \leq \int_0^1 dx \int_0^1 |r(x, s)|^2 ds \int_0^1 |h(s - t)|^2 ds \\ & \leq 2 \int_0^1 |h(s)|^2 ds \int_0^1 \int_0^1 |r(x, s)|^2 ds dx < \infty. \end{aligned} \quad (23)$$

Thus the function

$$\int_0^1 r(x, s)h(s - t) ds \quad (24)$$

of the variable $x \in [0, 1]$ belongs to $L_2(0, 1)$; moreover, continuity of the shifts $h(\cdot - t)$ and estimate (23) show that function (24) depends continuously in $L_2(0, 1)$ on $t \in [0, 1]$. We thus conclude that indeed $r(\cdot, t)$ depends continuously in $L_2(0, 1)$ on $t \in [0, 1]$. In particular, $r(x, 0)$ is a well-defined function in $L_2(0, 1)$.

Finally, we again use the Krein equation and (21) to get the relation

$$\tau(x) = r(x, 0) = -h(x) - \int_0^1 r(x, s)h(s) ds.$$

The integral on the right-hand side is a bilinear expression in h and r . In view of the analytic dependence of r on h stated in Corollary 3.1 and estimates (23), this yields analyticity and Lipschitz continuity of $r(x, 0)$ on $h \in L_2(0, 1)$. On account of Lemma 3.2, the proof of Theorem 2 is complete.

4 RECONSTRUCTION FROM TWO SPECTRA

We recall that the norming constants α_n for the Sturm–Liouville operator $T_N(\tau)$ can be determined from the spectra (λ_n) and (μ_n) of $T_N(\tau)$ and $T_D(\tau)$ by the formula (14),

$$\alpha_n = -\frac{\omega_{2n}}{\dot{S}(\omega_{2n})C(\omega_{2n})},$$

where the entire functions S and C are given by the canonical products (13) over $\lambda_n = \omega_{2n}^2$ and $\mu_n = \omega_{2n+1}^2$ respectively. This induces a mapping $\nu \mapsto \alpha$ from the spectral data $\nu := ((\lambda_n), (\mu_n)) \in \mathcal{N}$ into the norming constants $\alpha := (\alpha_n) \in \mathcal{A}$. In this section, we shall establish Theorem 1 by proving the following result.

Theorem 3. *For every $d \in (0, \pi/2)$ and $r > 0$, the mapping*

$$\mathcal{N}(d, r) \ni \nu \mapsto \alpha \in \mathcal{A} \quad (25)$$

is analytic and Lipschitz continuous; moreover, there exist positive constants d' and r' such that the range of this mapping belongs to $\mathcal{A}(d', r')$.

By definition, \mathcal{A} consists of elements of the commutative unital Banach algebra A introduced in Appendix C. We observe that the metrics on \mathcal{A} agrees with the norm of A , and thus the results of Appendix C yield the following statement.

Proposition 4.1. *For every positive d and r , the set $\mathcal{A}(d, r)$ consists of invertible elements of A . Moreover, the mapping $\alpha \mapsto \alpha^{-1}$ is analytic and Lipschitz continuous in A on $\mathcal{A}(d, r)$, and its range lies in $\mathcal{A}((1+r)^{-1}, rd^{-1})$.*

In view of Proposition 4.1, it suffices to prove Theorem 3 with α replaced by α^{-1} . The elements of the sequence α^{-1} are $\alpha_n^{-1} = -\dot{S}(\omega_{2n})C(\omega_{2n})/\omega_{2n}$. We shall show that the sequences

$$\gamma := ((-1)^{n+1}\dot{S}(\omega_{2n})/\omega_{2n})_{n \in \mathbb{Z}_+}, \quad \delta := ((-1)^n C(\omega_{2n}))_{n \in \mathbb{Z}_+}$$

form elements of \mathcal{A} . Thus Theorem 3 will be proved if we show that the mappings

$$\mathcal{N}(d, r) \ni \nu \mapsto \gamma \in \mathcal{A}, \quad \mathcal{N}(d, r) \ni \nu \mapsto \delta \in \mathcal{A} \quad (26)$$

enjoy the properties required therein for the mapping (25).

To begin with, integral representations (10) and (11) of the solution $c(\cdot, \omega)$ and its quasi-derivative $c^{[1]}(\cdot, \omega)$ yield the formulae

$$S(\omega) = -\omega \sin \omega - \omega \int_0^1 k_1(1, t) \sin \omega t \, dt, \quad (27)$$

$$C(\omega) = \cos \omega + \int_0^1 k(1, t) \cos \omega t \, dt \quad (28)$$

for the functions S and C . Therefore both expressions $-\dot{S}(\omega_{2n})/\omega_{2n}$ and $C(\omega_{2n})$ can be recast in the form

$$\cos \omega_{2n} + \int_0^1 g(t) \cos \omega_{2n} t \, dt$$

with $g(t) = tk_1(1, t)$ for the former expression and $g(t) = k(1, t)$ for the latter. The sequences γ and δ have therefore similar structures; namely, their n -th element equals

$$\cos \rho_{2n} + (-1)^n \int_0^1 g(t) \cos \omega_{2n} t \, dt \quad (29)$$

for respective g ; here, as usual, $\rho_{2n} := \omega_{2n} - \pi n$.

Clearly, the mapping $(\rho_{2n}) \mapsto (\cos \rho_{2n} - 1)$ is analytic in ℓ_2 . Its Lipschitz continuity follows from the inequality $|\cos x - \cos y| \leq |x - y|$; also, the inequality $1 - \cos x \leq x^2/2$ yields the estimate

$$\|(\cos \rho_{2n} - 1)\|_{\ell_2} \leq \frac{1}{2} \|(\rho_{2n})\|_{\ell_2}^2. \quad (30)$$

Set

$$\tilde{g}(s) := \begin{cases} g(1 - 2s), & s \in [0, \frac{1}{2}), \\ g(2s - 1), & s \in [\frac{1}{2}, 1]; \end{cases}$$

then straightforward transformations give

$$\begin{aligned} v_n &:= (-1)^n \int_0^1 g(t) \cos \omega_{2n} t \, dt = (-1)^n \int_0^1 \tilde{g}(s) e^{i\omega_{2n}(1-2s)} \, ds \\ &= \int_0^1 \tilde{g}(s) e^{i\rho_{2n}(1-2s)} e^{-2\pi i n s} \, ds. \end{aligned} \quad (31)$$

Therefore the above number v_n gives the n -th Fourier coefficient of the function $u := \Psi(f_\lambda, \tilde{g})$, where Ψ is the mapping of Lemma A.3 and f_λ is the function introduced in the proof of Lemma 3.1. It follows from Lemma A.3 that the sequence $(\hat{u}(n))_{n \in \mathbb{Z}}$ of Fourier coefficients of u depends analytically and boundedly Lipschitz continuously in ℓ_2 on f_λ and \tilde{g} . We prove in the lemma below that the functions $k(1, \cdot)$ and $k_1(1, \cdot)$ (and thus the corresponding transformates \tilde{g}) depend in the same manner on $\nu = (\lambda, \mu) \in \mathcal{N}(d, r)$.

Lemma 4.1. *The mappings*

$$\begin{aligned} \mathcal{N}(d, r) \ni (\lambda, \mu) &\mapsto k(1, \cdot) \in L_2(0, 1), \\ \mathcal{N}(d, r) \ni (\lambda, \mu) &\mapsto k_1(1, \cdot) \in L_2(0, 1) \end{aligned}$$

are analytic and Lipschitz continuous.

Proof. Since both mappings can be treated similarly, we only consider the second one. By definition, we have $S(\omega_{2n})/\omega_{2n} = 0$, and thus the numbers $\omega_{2n} = \pi n + \rho_{2n}$, $n \in \mathbb{Z}$, are zeros of the odd entire function $S(\omega)/\omega$ of (27). The required properties of the mapping $\lambda \mapsto k_1(1, \cdot)$ follow now from the results of [26]; see Appendix B. \square

The above reasoning justifies the inclusion $\alpha^{-1} \in \mathcal{A}$ as well as analyticity and Lipschitz continuity of the mappings of (26). It remains to prove that there exist positive d' and r' such that, for every $\nu \in \mathcal{N}(h, r)$, the corresponding elements γ and δ belong to $\mathcal{A}(d', r')$.

Existence of such an r' follows from the uniform estimates of the ℓ_2 -norms of the sequences $(\cos \rho_{2n} - 1)$ of (30) and the fact that

$$\sum_{n \in \mathbb{Z}_+} |v_n|^2 \leq \|\Psi(f_\lambda, \tilde{g})\|^2,$$

see (31) and the discussion following it. Indeed, in view of Lemma A.3 the function $u = \Psi(f_\lambda, \tilde{g})$ remains in the bounded subset of $L_2(0, 1)$ when f_λ and \tilde{g} vary over bounded subsets

of $L_2(0, 1)$, and the latter is the case when ν runs over $\mathcal{N}(d, r)$ by the definition of the functions f_λ and \tilde{g} and Lemma 4.1.

Next, in view of formula (13) and the interlacing property of λ_n and μ_n , the numbers $\gamma_n = (-1)^n \dot{S}(\omega_{2n})/\omega_{2n}$ and $\delta_n = (-1)^n C(\omega_{2n})$ are all of the same sign and thus are all positive in view of the asymptotic relation (29). The uniform positivity of γ_n and δ_n (and thus existence of a positive d' such that $1/\alpha_n = \gamma_n \delta_n \geq d'$) follows immediately from the lemma below.

Lemma 4.2. *For every $d \in (0, \pi/2)$ and $r > 0$ we have*

$$\sup_{(\lambda, \mu)} \sup_{n \in \mathbb{Z}_+} \log |\dot{S}(\omega_{2n})/\omega_{2n}| < \infty, \quad \sup_{(\lambda, \mu)} \sup_{n \in \mathbb{Z}_+} \log |C(\omega_{2n})| < \infty,$$

where S and C are constructed via (13) from the sequences λ and μ , and the suprema are taken over $(\lambda, \mu) \in \mathcal{N}(d, r)$.

Proof. We assume first that $n \neq 0$. By (13), we have

$$\dot{S}(\omega_{2n})/\omega_{2n} = -\frac{2\omega_{2n}^2}{\pi^2 n^2} \prod_{k \in \mathbb{N}, k \neq n} \frac{\omega_{2k}^2 - \omega_{2n}^2}{\pi^2 k^2}.$$

Dividing both sides by

$$\cos \pi n = \left. \frac{d \sin z}{dz} \right|_{z=\pi n} = -2 \prod_{k \in \mathbb{N}, k \neq n} \frac{k^2 - n^2}{k^2},$$

we conclude that

$$|\dot{S}(\omega_{2n})/\omega_{2n}| = \frac{\omega_{2n}^2}{\pi^2 n^2} \prod_{k \in \mathbb{N}, k \neq n} \frac{\omega_{2k}^2 - \omega_{2n}^2}{\pi^2 (k^2 - n^2)},$$

for $n = 0$ the direct calculations give

$$\lim_{\omega \rightarrow 0} |\dot{S}(\omega)/\omega| = 2 \prod_{k \in \mathbb{N}} \frac{\omega_{2k}^2}{\pi^2 k^2}.$$

Recall that $\rho_{2k} := \omega_{2k} - \pi k$ and set

$$a_{n, \pm k} := \frac{\rho_{2n} \mp \rho_{2k}}{\pi(n \mp k)},$$

with $a_{0,0} = 1$ and $a_{n,n} = 0$ if $n \in \mathbb{N}$; then

$$\frac{\omega_{2k}^2 - \omega_{2n}^2}{\pi^2 (k^2 - n^2)} = (1 + a_{n,k})(1 + a_{n,-k})$$

and²

$$|\dot{S}(\omega_{2n})/\omega_{2n}| = \prod_{k \in \mathbb{Z}} (1 + a_{n,k}).$$

In what follows, all summations and multiplications over the index set \mathbb{Z} will be taken in the principal value sense and the symbol V.p. will be omitted.

Since the sequence (λ_n) is $2d$ -separated for every $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathcal{N}(d, r)$, we have $1 + a_{k,n} \geq 2d/\pi$ for all $n \in \mathbb{Z}_+$ and all $k \in \mathbb{Z}$. Therefore, with

$$K := \max_{x \geq -1+2d/\pi} \left| \frac{\log(1+x) - x}{x^2} \right| < \infty,$$

we get the estimate

$$\left| \log \prod_{k \in \mathbb{Z}} (1 + a_{n,k}) \right| \leq \left| \sum_{k \in \mathbb{Z}} a_{n,k} \right| + K \sum_{k \in \mathbb{Z}} a_{n,k}^2, \quad (32)$$

provided the two series converge.

Clearly,

$$\sum_{k \neq n} \frac{1}{n-k} = 0,$$

and thus

$$\left| \sum_{k \in \mathbb{Z}} a_{n,k} \right| = \left| \frac{1}{\pi} \sum_{k \neq n} \frac{\rho_{2k}}{k-n} \right| \leq \frac{r}{\sqrt{3}}$$

by the Cauchy–Bunyakovski–Schwarz inequality (recall that $\sum_{k \in \mathbb{Z}} \rho_{2k}^2 \leq r^2$ by the definition of the set $\mathcal{N}(d, r)$ and $\sum_{k \neq n} (k-n)^{-2} = \pi^2/3$). Next, the inequality

$$a_{n,k}^2 \leq \frac{2\rho_{2k}^2}{\pi^2(k-n)^2} + \frac{2\rho_{2n}^2}{\pi^2(k-n)^2}$$

for $k \neq n$ yields

$$\sum_{k \in \mathbb{Z}} a_{n,k}^2 \leq 4r^2 \sum_{k \neq n} \frac{1}{\pi^2(k-n)^2} = \frac{4r^2}{3}.$$

It follows from (32) that

$$\left| \log \prod_{k \in \mathbb{Z}} (1 + a_{n,k}) \right| \leq (\sqrt{3}r + 4Kr^2)/3,$$

where the constant K only depends on d .

Similarly, we find that

$$|C(\omega_{2n})| = \left| \prod_{k \in \mathbb{Z}_+} \frac{\omega_{2k+1}^2 - \omega_{2n}^2}{\pi^2(k + \frac{1}{2})^2} \right| = \prod_{k \in \mathbb{Z}_+} \frac{\omega_{2k+1}^2 - \omega_{2n}^2}{\pi^2(k + \frac{1}{2})^2 - \pi^2 n^2}$$

and then mimic the above reasoning to establish the other uniform bound. The lemma is proved. \square

Proof of Theorem 3. Combining the results of Lemmata 4.1 and 4.2, we conclude that the mappings (26) enjoy all the properties stated in Theorem 3, and thus so does the mapping $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \mapsto \boldsymbol{\alpha}^{-1}$. In virtue of Proposition 4.1 this completes the proof of the theorem. \square

Proof of Theorem 1. Analyticity and Lipschitz continuity on bounded sets of the inverse spectral mapping

$$\mathcal{N} \ni \boldsymbol{\nu} \mapsto \tau \in L_2(0, 1)$$

is the direct consequence of those for the mappings (25) and (15) established in Theorems 3 and 2 respectively. \square

5 SOME EXTENSIONS

The results proved above for the class of impedance Sturm–Liouville operators with real-valued impedance functions $a \in W_2^1(0, 1)$, i.e., for Sturm–Liouville operators $T_N(\tau)$ and $T_D(\tau)$ with $\tau = a' + a^2 \in L_2(0, 1)$ allow quite a straightforward generalization to wider classes of operators.

Firstly, it is not important that the boundary conditions considered are of Dirichlet or Dirichlet–Neumann type. In fact, the analysis proceeds in much the same way for generic Robin-type boundary conditions at one or both endpoints.

Secondly, as in [2] one can treat the case $\tau \in L_p(0, 1)$, with $p \in [1, \infty)$. The asymptotic representation of the eigenvalues and norming constants become then as in (9) and (12), but the sequences of remainders (ρ_n) and (β_n) form now sequences of sine or cosine Fourier coefficients of functions in the respective $L_p(0, 1)$ space, see details in [2, 26].

Finally, also the τ in the Sobolev space scale $W_2^s(0, 1)$ can be treated; see similar results for the potential Sturm–Liouville inverse problem in [25, 41]. Again the sequences of remainders (ρ_n) and (β_n) are then sine or cosine Fourier coefficients of functions in the same space, and they form Banach algebra under multiplication with properties similar to those of the algebra A discussed in Appendix C.

For such more general settings the above-described approach is applicable and, save for some more involved technicalities, proceeds in much the same way and establishes analytic and Lipschitz continuous dependence of the impedance function a on the spectral data for the impedance Sturm–Liouville operators considered.

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A SOME AUXILIARY RESULTS

We recall that the convolution $f * g$ of two functions in $L_2(0, 1)$ is a function in $L_2(0, 1)$ given by

$$(f * g)(x) := \int_0^1 f(x - t)g(t) dt,$$

where f is extended to $(-1, 0)$ as a periodic function with period 1. The (discrete) Fourier transform \hat{f} of $f \in L_2(0, 1)$ is a function over \mathbb{Z} given by

$$\hat{f}(n) := \int_0^1 f(t)e^{-2\pi nit} dt.$$

It is well known that the Fourier transform is a unitary mapping from $L_2(0, 1)$ to $\ell_2(\mathbb{Z})$ and that $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$; as a result, we have the inequality

$$\|f * g\| \leq \|f\|\|g\|$$

for all $f, g \in L_2(0, 1)$.

Lemma A.1. For a function $f \in L_2(0, 1)$, set

$$\Phi_1(f)(x) := \text{V.p.} \sum_{n=-\infty}^{\infty} [e^{2\hat{f}(n)ix} - 1]e^{2\pi nix}.$$

Then the series determines a function in $L_2(0, 1)$, and the mapping

$$L_2(0, 1) \ni f \mapsto \Phi_1(f) \in L_2(0, 1)$$

is analytic and Lipschitz continuous on bounded subsets.

Proof. We start with observing that the series $\sum_{n \in \mathbb{Z}} \hat{f}^k(n)e^{2\pi nis}$ is the Fourier series for the function $f^{(k)}$, the k -fold convolution of f with itself, and that $\|f^{(k)}\| \leq \|f\|^k$. Developing $e^{\hat{f}(n)is}$ into the Taylor series, we find that

$$\begin{aligned} \Phi_1(f) &= \text{V.p.} \sum_{n=-\infty}^{\infty} \left[\sum_{k=1}^{\infty} \frac{\hat{f}^k(n)(2is)^k}{k!} \right] e^{2\pi nis} \\ &= \sum_{k=1}^{\infty} \frac{(2is)^k}{k!} \text{V.p.} \sum_{n=-\infty}^{\infty} \hat{f}^k(n)e^{2\pi nis} \\ &= \sum_{k=1}^{\infty} \frac{(2is)^k}{k!} f^{(k)}. \end{aligned}$$

The change of the summation order in the second equality above is justified by the fact that, for $k > 1$, the summands in the double series are dominated by $C^k \hat{f}^2(n)/k!$ with $C := 2 \max_{n \in \mathbb{Z}} \{|\hat{f}(n)|\} + 1$. Therefore the double series over the index set $\{(n, k) \mid n \in \mathbb{Z}, k > 1\}$ converges absolutely and the Fubini theorem applies. This formula represents $\Phi_1(f)$ as an absolutely convergent series (which is a Taylor series expansion of $\Phi_1(f)$ in the variable f) and thus proves the analyticity in $L_2(0, 1)$ of the mapping $f \mapsto \Phi_1(f)$.

Lipschitz continuity of that mapping on bounded sets follows from the estimate

$$\begin{aligned} \|\Phi_1(f_1) - \Phi_1(f_2)\| &= \left\| \sum_{k=1}^{\infty} \frac{(2is)^k}{k!} [f_1^{(k)} - f_2^{(k)}] \right\| \\ &\leq \sum_{k=1}^{\infty} \frac{2^k}{(k-1)!} \|f_1 - f_2\| (\|f_1\| + \|f_2\|)^{k-1} \leq \exp\{4r\} \|f_1 - f_2\|, \end{aligned}$$

which is valid as soon as the L_2 -norms of f_1 and f_2 are not greater than r . The proof is complete. \square

Lemma A.2. For f and g in $L_2(0, 1)$, set

$$\Phi_2(f, g) := \text{V.p.} \sum_{n=-\infty}^{\infty} \hat{g}(n) \exp\{2[\pi n + \hat{f}(n)]is\}.$$

Then the function $\Phi_2(f, g)$ belongs to $L_2(0, 1)$ and the mapping

$$\Phi_2 : L_2(0, 1) \times L_2(0, 1) \rightarrow L_2(0, 1)$$

is analytic and Lipschitz continuous on bounded subsets.

Proof. Transformations similar to those used in the proof of the above lemma show that

$$\Phi_2(f, g) = \sum_{k=1}^{\infty} \frac{(2is)^k}{k!} [f^{(k)} * g].$$

The mapping Φ_2 is linear (and thus analytic) in g , and its analyticity in f as well as Lipschitz continuity on bounded subsets is established in the same manner as for the mapping Φ_1 of Lemma A.1. \square

Lemma A.3. For f and g in $L_2(0, 1)$, set

$$\Psi(f, g) := \text{V.p.} \sum_{n \in \mathbb{Z}} (-1)^n \int_0^1 g(t) \exp\{[\pi n + \hat{f}(n)]i(1 - 2t)\} dt e^{2\pi i n x}.$$

Then the function $\Psi(f, g)$ belongs to $L_2(0, 1)$ and the mapping

$$\Psi : L_2(0, 1) \times L_2(0, 1) \rightarrow L_2(0, 1)$$

is analytic and Lipschitz continuous on bounded subsets.

Proof. The coefficient of $e^{2\pi i n x}$ in the above series for Ψ can be written as

$$\int_0^1 g(t) \exp\{i(1 - 2t)\hat{f}(n)\} e^{-2\pi i n t} dt \quad (33)$$

and gives the n -th Fourier coefficient of the function $h := \sum_{k=0}^{\infty} h_k/k!$, with $h_0 := g$, $h_k := f^{(k)} * M^k g$ for $k \geq 1$, and M being the operator of multiplication by the function $i(1 - 2x)$. In other words, we have $\Psi(f, g) = h$. Since $\|f * g\| \leq \|f\| \|g\|$ for every f and g in $L_2(0, 1)$, the functions h_k belong to $L_2(0, 1)$ and their norms there obey the estimate

$$\|h_k\| \leq \|f\|^k \|M^k g\| \leq \|f\|^k \|g\|.$$

Thus the series for h converges absolutely and, since every h_k is a multi-linear function of f and g , the mapping Ψ is analytic. Its Lipschitz continuity on bounded subsets is established in the usual manner, and the proof is complete. \square

B ANALYTICITY OF SOME RELATED MAPPINGS

Here we give a brief account on the results of [26] and also establish some of their extensions needed to prove Lemma 4.1. It was shown in [26] that for every $f \in L_2(0, 1)$ there exists a unique function $g \in L_2(0, 1)$ such that all zeros (counting multiplicities) of the entire function

$$G_g(z) := \sin z + \int_0^1 g(t) e^{iz(1-2t)} dt \quad (34)$$

are given by the numbers $\pi n + \hat{f}(n)$, $n \in \mathbb{Z}$. Such pairs of f and g in fact satisfy the relation

$$H(f, g) := s(f) + g + \sum_{k=1}^{\infty} \frac{(M^k g) * f^{(k)}}{k!} = 0; \quad (35)$$

here

$$s(f) := \sum_{k=0}^{\infty} \frac{(-1)^k f^{(2k+1)}}{(2k+1)!},$$

$f^{(k)}$ is the k -fold convolution of f with itself, and M is the operator of multiplication by $i(1-2x)$. The function H is analytic from $L_2(0,1) \times L_2(0,1)$ into $L_2(0,1)$, and its partial derivatives $\partial_f H(f,g)$ and $\partial_g H(f,g)$ are given by

$$\partial_f H(f,g)(h_1) = \left(c(f) + \sum_{k=1}^{\infty} \frac{(M^k g) * f^{(k-1)}}{(k-1)!} \right) * h_1, \quad (36)$$

$$\partial_g H(f,g)(h_2) = h_2 + \sum_{k=1}^{\infty} \frac{(M^k h_2) * f^{(k)}}{k!} \quad (37)$$

with

$$c(f) := \sum_{k=0}^{\infty} \frac{(-1)^k f^{(2k)}}{(2k)!}.$$

Using the implicit function theorem, it was shown that the induced mapping $\varphi : f \mapsto g$ is analytic. In order to establish its Lipschitz continuity, we shall study the above partial derivatives in more detail.

Namely, we assume that $f \in L_2(0,1)$ is such that the corresponding sequence $\boldsymbol{\omega} = (\omega_n)_{n \in \mathbb{Z}}$ with $\omega_n := \pi n + \hat{f}(n)$ belongs to $\mathcal{L}(d,r)$ and that $g = \varphi(f)$. Set $S_{\boldsymbol{\omega}}$ to be the canonical product of (13); then $S_{\boldsymbol{\omega}}(z)/z$ can also be represented as (34). Direct calculations show that the n -th Fourier coefficient of the function of (36) is equal to

$$(-1)^n \hat{h}_1(n) \left[\cos \omega_n + \int_0^1 i(1-2t)g(t)e^{i\omega_n(1-2t)} dt \right] = (-1)^n \hat{h}_1(n) \dot{S}_{\boldsymbol{\omega}}(\omega_n).$$

By Lemma 4.2 there are positive numbers k_1 and k_2 such that

$$k_1 \leq |\dot{S}_{\boldsymbol{\omega}}(\omega_n)/\omega_{2n}| \leq k_2$$

for all $\boldsymbol{\omega} \in \mathcal{L}(d,r)$ and all $n \in \mathbb{Z}$. Therefore the partial derivative $\partial_f H(f,g)$ is a bounded and boundedly invertible operator in $L_2(0,1)$; moreover, for every fixed $d > 0$ and $r > 0$, the norms of $\partial_f H(f,g)$ and their inverses are uniformly bounded for all $f \in L_2(0,1)$ generating the sequences $\boldsymbol{\omega}$ in the set $\mathcal{L}(d,r)$.

Similarly, the n -th Fourier coefficient of the function of (37) is equal to

$$(-1)^n \int_0^1 h_2(t)e^{i\omega_n(1-2t)} dt.$$

By the results of [23], there exist positive m_1 and m_2 such that, for all $\boldsymbol{\omega} \in \mathcal{L}(d,r)$, the sequences $(e^{i\omega_n(1-2x)})_{n \in \mathbb{Z}}$ form Riesz bases of $L_2(0,1)$ of lower bound m_1 and upper bound m_2 . Therefore the operator $H_g := \partial_g H(f,g)$,

$$H_g : h_2 \mapsto \sum_{n \in \mathbb{Z}} (-1)^n (h_2, e^{i\omega_n(1-2x)}) e^{2\pi n i x},$$

is bounded and boundedly invertible in $L_2(0, 1)$, with $\|H_g\| \leq m_2^{1/2}$ and $\|H_g^{-1}\| \leq m_1^{-1/2}$.

We now use the implicit mapping theorem to conclude that the mapping $\varphi : f \mapsto g$ is analytic in $L_2(0, 1)$. The uniform bounds on the inverses of the partial derivatives $\partial_f H(f, g)$ and $\partial_g H(f, g)$ established above imply that, for every $d \in (0, \pi)$ and $r > 0$, this mapping is Lipschitz continuous on the set of functions $f \in L_2(0, 1)$ generating the sequences $\omega \in \mathcal{L}(d, r)$.

C THE BANACH ALGEBRA A

The space $\ell_2 = \ell_2(\mathbb{Z}_+)$ is a commutative Banach algebra under the pointwise multiplication $(x_n) \cdot (y_n) = (x_n \cdot y_n)$. Its unital extension A consists of elements of ℓ_∞ of the form $a\mathbf{1} + \mathbf{x}$ with $a \in \mathbb{C}$, the unity $\mathbf{1} \in \ell_\infty$ having all its elements equal to 1, and $\mathbf{x} = (x_n) \in \ell_2$. The norm in A is given by

$$\|a\mathbf{1} + \mathbf{x}\|_A = |a| + \|\mathbf{x}\|.$$

An element $a\mathbf{1} + \mathbf{x}$ is invertible in A if and only if $a \neq 0$ and $a + x_n \neq 0$ for all $n \in \mathbb{Z}$; in this case the inverse is equal to $a^{-1}\mathbf{1} + \mathbf{y}$, where $\mathbf{y} = (y_n)$ with $y_n := -x_n/a(a + x_n)$. Since under the above assumptions we have $\inf_n |a + x_n| > 0$, we see that \mathbf{y} indeed belongs to ℓ_2 ; moreover,

$$\|(a\mathbf{1} + \mathbf{x})^{-1}\|_A \leq |a|^{-1} (1 + \|\mathbf{x}\| / \inf_n |a + x_n|).$$

The mapping $\hat{\mathbf{x}} \mapsto \hat{\mathbf{x}}^{-1}$ is analytic on the open set of all invertible elements of A ; in addition, it is Lipschitz continuous on the sets

$$\mathcal{S}_\varepsilon := \{a\mathbf{1} + \mathbf{x} \mid |a| \geq \varepsilon, \inf_n |a + x_n| \geq \varepsilon\}.$$

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Доведено, що обернене спектральне відображення, що відновлює імпедансну функцію операторів Штурма–Ліувілля на $[0, 1]$ в імпедансній формі за спектральними даними (двома спектрами або одним спектром та нормівними множниками) є аналітичним та рівномірно стійким в певному сенсі.

Гринив Р.О. *Аналитичность и равномерная устойчивость в обратной задаче для импедансных операторов Штурма–Лиувилля* // Карпатские математические публикации. — 2010. — Т.2, №1. — С. 35–58.

Доказано, что обратное спектральное отображение, восстанавливающее импедансную функцию операторов Штурма–Лиувилля на $[0, 1]$ в импедансной форме по спектральным данным (двум спектрам или одному спектру и нормирующим множителям) является аналитическим и равномерно устойчивым в некотором смысле.