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LEFT NOETHERIAN RINGS WITH DIFFERENTIALLY TRIVIAL PROPER QUOTIENT RINGS

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We characterize left Noetherian rings with differentially trivial proper quotient rings.

Introduction

Let R be an associative ring with an identity. As usually, a mapping $d : R \to R$ such that

$$d(a+b) = d(a) + d(b) \text{ and } d(ab) = d(a)b + ad(b)$$

for any $a, b \in R$ is called a derivation of R. A ring R having no non-zero derivations will be called differentially trivial. Differentially trivial left Noetherian rings were characterized by author in [1].

In this paper we prove the following

Theorem. *Let R be a left Noetherian ring with an identity. Then every proper quotient ring of R is differentially trivial if and only if it is of one of the following types:*

(i) *R is a differentially trivial Artinian ring;*

(ii) *R is a local ring with the differentially trivial residue field $R/J(R)$, where the Jacobson radical $J(R)$ is the heart of R;*

(iii) *$R = R_1 + R_2$ is a sum of ideals I_1 and I_2, where $I_1 \cap I_2$ is the heart of R and the quotient ring $R/(I_1 \cap I_2)$ is differentially trivial Artinian.*

As proved in [2], a left Artinian ring R is differentially trivial if and only if $R = R_1 \oplus \cdots \oplus R_m$ $(m \geq 1)$ is a ring direct sum of $R_1, \ldots, R_m$ and each R_i is either a differentially trivial field (i.e., R_i is algebraic over its prime subfield if the characteristic $\mathrm{char}(R_i) = 0$ and $R_i = \{a^{p_i} | a \in R_i\}$ if $\mathrm{char}(R_i) = p_i$) or isomorphic to some ring $\mathbb{Z}_{p^n}$ of integers modulo a

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prime power p^n. It is clear that every quotient ring of differentially trivial left Artinian ring is differentially trivial.

For convenience of the reader we recall some notation and terminology. If I is a non-zero ideal of R, then the quotient ring R/I is called proper. $N(R)$ will always denote the set of nilpotent elements of R, $J(R)$ the Jacobson radical of R, R^+ the additive group of R, We will also use other terminology from [3], [4] and [5].

1 Preliminaries

In the sequel we shall need the next

Lemma 1.1. *Let R be a ring with the identity 1. If all proper quotient rings of R are differentially trivial, then one of the following statements is satisfied:*

(1) *R is a differentially trivial ring;*

(2) *R has only trivial idempotents;*

(3) *$R = R_1 + R_2$ is a sum of ideals I_1 and I_2, where $I_1 \cap I_2$ is the heart of R and the quotient ring $R/(I_1 \cap I_2)$ is differentially trivial.*

Proof. Assume that e is a non-trivial idempotent of R. Put $f = 1 - e$. If R is commutative, then $R = eR \oplus fR$ is a ring direct sum and, consequently, R is differentially trivial. Therefore we suppose that R is not commutative. Then the ideal

$$I_0 = \bigcap \{I | I \text{ is a non-zero ideal of } R\ \}$$

is non-zero and $I_0^2 = (0)$. Clearly, R contains an idempotent e such that so $eI_0 = I_0$ and $fI_0 = (0)$.

1) Assume that $I_0 e = (0)$. Then $I_0 f = I_0$ and, as a consequence, $fRe = (0)$, $eRf = I_0$ and

$$R = eRe \oplus eRf \oplus fRf$$

is a group direct sum. We denote $eRe \oplus eRf$ by I_1 and $eRf \oplus fRf$ by I_2. Obviously, I_1, I_2 are ideals in R and $I_1 \cap I_2 = I_0$ is the heart of R.

2) Now suppose that $I_0 e \neq (0)$. Then $I_0 e = I_0$ and $I_0 f = (0)$. From this it follows that $eRf = fRe = (0)$ and

$$R = eRe \oplus fRf$$

is a group direct sum. Moreover, fRf is a two-sided ideal of R, $I_0 \leq eRe$ and we obtain that R is differentially trivial. $\square$

Example 1.1. *Let $R = \mathbb{Q}[X]/(X^2) = \mathbb{Q} + u\mathbb{Q}$ be a commutative $\mathbb{Q}$-algebra, where $u = X + (X^2)$. Then $u^2 = 0$, R is a local ring and the Jacobson radical $J(R) = u\mathbb{Q}$ is the heart of R. It is not difficult to prove that R has non-zero derivations and any proper quotient ring of R is differentially trivial (and so R is of type (ii) from Theorem).*

Example 1.2. *Let $R = \mathbb{Q}e_1 + \mathbb{Q}e_2 + \mathbb{Q}u$ be a central $\mathbb{Q}$-algebra with the basis $\{e_1, e_2, u\}$, where*

$$e_1^2 = e_1, e_2^2 = e_2, u^2 = 0, e_1e_2 = e_2e_1 = 0,$$
$$e_1u = u, ue_1 = 0, e_2u = 0, ue_2 = u.$$

Then R is an Artinian ring with the identity $1 = e_1 + e_2$, its Jacobson radical $J(R) = u\mathbb{Q}$ is the heart of R. Moreover, R has non-zero derivations and all proper quotient rings of R are differentially trivial (and thus R is of type (iii) from Theorem).

Lemma 1.2. *Let R be a left Noetherian ring with an identity. If R is a differentially trivial ring with differentially trivial proper quotient ring, then it is Artinian.*

Proof. 1) Assume that the additive group R^+ is torsion. If R is a domain, then, by Proposition 3 of [1], it is a field. Therefore we suppose that R is not domain. Let P be a prime ideal of R. Since R/P is a field, we conclude that R is Artinian by Akizuki Theorem (see [6, Chapter IV, §2, Theorem 2]).

2) Now let us R^+ be a torsion-free group. Then, by Theorem 8 of [1], $J(R) = N(R) = (0)$. If $pR \neq R$ for some prime p, then pR is not contained in some maximal ideal M of R and therefore $R = M \oplus pR$ is a ring direct sum, a contradiction. Hence $pR = R$ for any prime p and R^+ is a divisible group. If R is a domain, then R is a field or $aR = a^2R$ for any element $a \in R$. This gives that R is Artinian. Assume now that R is not domain. Then, by Proposition 3 of [1], every prime ideal is maximal in R and, by Akizuki Theorem, R is Artinian.

3) In view of Theorem 8 of [1], from 1) and 2) it follows that R is Artinian. □

2 Proof of Theorem

($\Leftarrow$) is obviously.

($\Rightarrow$) Let R be a left Noetherian rings with every proper quotient ring differentially trivial.

1) Assume that $N(R) = \{0\}$. Then $I_0 = (0)$ (where I_0 is as in Lemma 1.1) and R is commutative. And therefore there are prime ideals $P_1, \dots, P_n$ of R such that

$$\bigcap_{i=1}^{n} P_i = (0).$$

Then R is a subdirect product of differentially trivial domains $R/P_1, \dots, R/P_n$. By Theorem 2.3.6 of [4],

$$S = \{c \in R | c \text{ is a regular element of } R\}$$

is an Ore set and so the ring of quotients $S^{-1}R$ is an Artinian ring. Consequently

$$S^{-1}R = B_1 \oplus \cdots \oplus B_n$$

is a ring direct sum of fields $B_1, \dots, B_n$ such that the field of quotients $Q(R/P_i) \cong B_i$ $(i = 1, \dots, n)$. We have seen that $S^{-1}R$ is differentially trivial. Inasmuch as every derivation of R extended to some derivation of $S^{-1}R$, a ring R is differentially trivial.

2) If $N(R) \neq \{0\}$, then $N(R)$ is an ideal of R. By Lemma 1.2, $R/N(R)$ is Artinian. Then R is a local ring of type (ii) or, by Lemmas 1.1 and 1.2, a ring of one of types (i) or (iii).

$\square$

References

1. O. D. Artemovych, *Differentially trivial left Noetherian rings* // Comment. Math. Univ. Carolinae (2)**40** (1999), 201-208.
2. O. D. Artemovych, *Differentially trivial and rigid rings of finite rank* // Periodica Math. Hungarica (1-2) **36** (1998), 1–16.
3. I. Lambek, Lectures on Rings and Modules, Blaisdell Publ. C., Waltham, 1966.
4. J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, John Wiley and Sons, Chichester, 1987.
5. L. Fuchs, Infinite Abelian Groups, vol.I, Academic Press, New York London, 1970.
6. O. Zariski and P. Samuel, Commutative Algebra, vol.I, D. van Nostrand C., Addison-Wesley P. C., Reading, 1970.

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