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# THE MULTIDIMENSIONAL $g$-FRACTION WITH NONEQUIVALENT VARIABLES CORRESPONDING TO THE FORMAL MULTIPLE POWER SERIES

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In this paper we consider the multidimensional $g$-fraction with nonequivalent variables which is the generalization of the continued $g$-fraction. The correspondence between the formal multiple power series and the above mentioned fraction is studied.

## Introduction

One of the methods of expanding the functions of multiple variables, given by the formal multiple power series (FMPS), into the branched continued fractions (BCF) is the construction of corresponding BCF [1, 3, 6, 9–12, 14].

Let

$$f_n(\mathbf{z}) = \frac{P_{m_n}(\mathbf{z})}{Q_{l_n}(\mathbf{z})}, \quad n \geq 1,$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, $N \in \mathbb{N}$, $P_{m_n}(\mathbf{z})$, $Q_{l_n}(\mathbf{z})$ are polynomials of degrees $m_n$ and $l_n$ respectively, be a sequence of the rational functions of multiple variables. The function $f_n(\mathbf{z})$ expands into FMPS in neighborhood of zero, if the denominator $Q_{l_n}(\mathbf{z})$ is not equal to zero in the point $\mathbf{z} = (0, 0, \ldots, 0)$.

The rational function $f_n(\mathbf{z})$ is called corresponding to FMPS

$$f(\mathbf{z}) = \sum_{|m(N)| \geq 0} a_{m(N)} \mathbf{z}^{m(N)},$$

where $m(N) = m_1, m_2, \ldots, m_N$ is multiindex, $m_i \in \mathbb{Z}_+$, $1 \leq i \leq N$, $|m(N)| = m_1 + m_2 + \cdots + m_N$, $\mathbf{z}^{m(N)} = z_1^{m_1} z_2^{m_2} \cdot \ldots \cdot z_N^{m_N}$, $\mathbf{z} \in \mathbb{C}^N$, $a_{m(N)} \in \mathbb{C}$, with order of correspondence $\nu_n$, if

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the expansion $f_n(\mathbf{z})$ into FMPS coincides with $f(\mathbf{z})$ for all homogeneous polynomials to the degree $\nu_n - 1$ inclusively. The sequence $\{f_n(\mathbf{z})\}$ corresponds to FMPS, if

$$\lim_{n\to+\infty} \nu_n = +\infty.$$

The correspondence BCF to FMPS $f(\mathbf{z})$ means that the sequence of approximants of BCF corresponds to $f(\mathbf{z})$.

We consider the multidimensional $g$-fraction with nonequivalent variables

$$\frac{s_0}{1+\displaystyle\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{g_{i(k)}(1-g_{i(k-1)})z_{i_k}}{1}} = \cfrac{s_0}{1+\displaystyle\sum_{i_1=1}^{N}\cfrac{g_{i(1)}z_{i_1}}{1+\displaystyle\sum_{i_2=1}^{i_1}\cfrac{g_{i(2)}(1-g_{i(1)})z_{i_2}}{1+\displaystyle\sum_{i_3=1}^{i_2}\cfrac{g_{i(3)}(1-g_{i(2)})z_{i_3}}{1+\ddots}}}}, \tag{1}$$

where $s_0 > 0$, $i(k) = i_1, i_2, \ldots, i_k$ is multiindex, $0 < g_{i(k)} < 1$, $k \geq 1$, $1 \leq i_n \leq i_{n-1}$, $1 \leq n \leq k$, $i_0 = N$, $g_{i(0)} = 0$, $\mathbf{z} \in \mathbb{C}^N$, which is generalization of the continued $g$-fraction

$$\frac{s_0}{1+\displaystyle\mathop{\mathrm{D}}_{n=1}^{\infty}\frac{g_n(1-g_{n-1})z}{1}} = \cfrac{s_0}{1+\cfrac{g_1 z}{1+\cfrac{g_2(1-g_1)z}{1+\cfrac{g_3(1-g_2)z}{1+\ddots}}}},$$

where $s_0 > 0$, $0 < g_n < 1$, $n \geq 1$, $g_0 = 0$, $z \in \mathbb{C}$.

Continued $g$-fractions are used, in particular, for analytic extension of functions, for the finding of zeros, poles and domains of univalent for some analytic and meromorphic functions, for the solution of the power moment problem [13, 16, 17].

The first multidimensional generalization of continued $g$-fraction was considered in [2, 7], where the circular domain of convergence for suggested generalization was investigated. The convergence of multidimensional $g$-fractions is investigated in [4, 8, 9]. The algorithm for the expansion of the formal multiple power series into corresponding multidimensional $g$-fraction is constructed in [6, 9].

In the present paper we study the correspondence between the FMPS

$$\sum_{|m(N)|\geq 0} (-1)^{|m(N)|} s_{m(N)} \mathbf{z}^{m(N)}, \tag{2}$$

where $s_{m(N)} \in \mathbb{R}$, $\mathbf{z} \in \mathbb{C}^N$, and the multidimensional $g$-fraction with nonequivalent variables (1). We prove that the fraction (1) converges in the domain

$$D = \{\mathbf{z} \in \mathbb{C}^N : \ |z_k| < 1/N, \ 1 \leq k \leq N\} \tag{3}$$

to function $g(\mathbf{z})$, which is holomorphic in this domain. We also establish that the sum of FMPS (2), which corresponds to the multidimensional $g$-fraction with nonequivalent variables (1), has the same value as this fraction in domain (3).

## 1 The correspondence between the FMPS and BCF

The correspondence between the multidimensional $g$-fraction with nonequivalent variables (1) and the FMPS (2) means that the expansion of each $n$th approximant, $n \geq 1$, into the FMPS coincides with the given series for all homogeneous polynomials to the degree $\nu_n - 1$ inclusively. The $\nu_n$ is called the order of correspondence.

We introduce the notation for the remainders of fraction (1):

$$Q_{i(s)}^{(s)}(\mathbf{z}) = 1, \quad Q_{i(p)}^{(s)}(\mathbf{z}) = 1 + \mathop{\mathrm{D}}_{r=p+1}^{s} \sum_{i_r=1}^{i_{r-1}} \frac{g_{i(r)}(1 - g_{i(r-1)})z_{i(r)}}{1},$$

where $s \geq 0,\ 0 \leq p \leq s-1,\ 1 \leq i_k \leq i_{k-1},\ 1 \leq k \leq p,\ i_0 = N$. Under this notation the following recurrent relations hold

$$Q_{i(p)}^{(s)}(\mathbf{z}) = 1 + \sum_{i_{p+1}=1}^{i_p} \frac{g_{i(p+1)}(1 - g_{i(p)})z_{i(p+1)}}{Q_{i(p+1)}^{(s)}(\mathbf{z})}, \tag{4}$$

where $s \geq 0,\ 0 \leq p \leq s-1,\ 1 \leq i_k \leq i_{k-1},\ 1 \leq k \leq p,\ i_0 = N$.

Let

$$g_n(\mathbf{z}) = \frac{s_0}{Q_{i(0)}^{(n-1)}(\mathbf{z})}$$

be $n$th approximant of BCF (1), $n \geq 1$.

**Theorem 1.** *For the multidimensional $g$-fraction with nonequivalent variables* (1) *there exists the unique formal multiple power series of form* (2) *to which this fraction will correspond. The order of correspondence is $\nu_n = n$.*

*Proof.* Since $Q_{i(r)}^{(s)}(\mathbf{0}) = 1$, where $\mathbf{0} = (0, 0, \ldots, 0)$, for any multiindex $i(r),\ s \geq 0,\ 0 \leq r \leq s-1,\ 1 \leq i_k \leq i_{k-1},\ 1 \leq k \leq r,\ i_0 = N$, then the fraction $1/Q_{i(r)}^{(s)}(\mathbf{z})$ has a formal expansion into FMPS of form (2).

Let for each index $n,\ n \geq 1$, the series

$$\sum_{|m(N)| \geq 0} (-1)^{|m(N)|} s_{m(N)}^{(n)} \mathbf{z}^{m(N)},$$

be a formal expansion of approximant $g_n(\mathbf{z})$ of BCF (1). Let $Q_{i(r)}^{(s)}(\mathbf{z}) \neq 0$ for all indices. Applying the method suggested in [1, p. 28] and recurrent relations (4), we find a formula for the difference of two approximants of BCF (1) for $m > n \geq 2$, namely

$$g_m(\mathbf{z}) - g_n(\mathbf{z}) = (-1)^n s_0 \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \cdots \sum_{i_n=1}^{i_{n-1}} \frac{\prod_{k=1}^{n} g_{i(k)}(1 - g_{i(k-1)})z_{i_k}}{\prod_{k=0}^{n} Q_{i(k)}^{(m-1)}(\mathbf{z}) \prod_{k=0}^{n-1} Q_{i(k)}^{(n-1)}(\mathbf{z})}$$

From this formula we have

$$g_m(\mathbf{z}) - g_n(\mathbf{z}) = \sum_{|m(N)|\geq 0} (-1)^{|m(N)|}\big(s_{m(N)}^{(m)} - s_{m(N)}^{(n)}\big)\mathbf{z}^{m(N)}, \quad m > n \geq 2,$$

in neighborhood of zero.

Hence for each $m,\ m > n \geq 2$, the relations $s_{m(N)}^{(m)} = s_{m(N)}^{(n)}$ hold for any multiindex $m(N),\ |m(N)| \leq n-1$.

BCF (1) corresponds to FMPS

$$L(\mathbf{z}) = \sum_{|m(N)|\geq 0} (-1)^{|m(N)|} s_{m(N)}^{(|m(N)|+1)}\mathbf{z}^{m(N)},$$

since for $n \geq 2$

$$L(\mathbf{z}) - g_n(\mathbf{z}) = \sum_{|m(N)|\geq n} (-1)^{|m(N)|}\big(s_{m(N)}^{(|m(N)|+1)} - s_{m(N)}^{(n)}\big)\mathbf{z}^{m(N)}.$$

The unique implies that for arbitrary $n,\ n \geq 2$, the all coefficients $s_{m(N)}^{(n)}$ of the $n$th approximant $g_n(\mathbf{z})$ expansion of BCF (1) into FMPS by form (2) are uniquely determined. □

The following theorem deals with the convergence of corresponding multidimensional $g$-fraction with nonequivalent variables to FMPS.

**Theorem 2.** *The multidimensional $g$-fraction with nonequivalent variables* (1) *converges in the domain* (3) *to function $g(\mathbf{z})$ which is holomorphic in this domain. The sum of the formal multiple power series* (2)*, which corresponds to the multidimensional $g$-fraction with nonequivalent variables* (1)*, has the same value as this fraction in the domain* (3).

*Proof.* Using relations (4), by mathematical induction we show that the following inequalities are valid

$$\big|Q_{i(r)}^{(s)}(\mathbf{z})\big| > g_{i(r)}, \tag{5}$$

where $s \geq 1,\ 1 \leq r \leq s,\ 1 \leq i_k \leq i_{k-1},\ 1 \leq k \leq r,\ i_0 = N$.

For $r = s$ relations (5) are obvious. By induction hypothesis that (5) hold for $r = p+1$, where $p+1 \leq s$, we prove (5) for $r = p$. Indeed, the implement of the relations (4) lead to

$$\big|Q_{i(p)}^{(s)}(\mathbf{z})\big| \geq 1 - \sum_{i_{p+1}=1}^{i_p} \frac{g_{i(p+1)}(1-g_{i(p)})|z_{i_{p+1}}|}{\big|Q_{i(p+1)}^{(s)}(\mathbf{z})\big|} > g_{i(p)}.$$

By virtue of estimates (5), $Q_{i(p+1)}^{(s)}(\mathbf{z}) \neq 0$. Therefore, replacing $g_{i(p+1)}$ by $\big|Q_{i(p+1)}^{(s)}(\mathbf{z})\big|$, inequalities (5) are obtained for $r = p$.

From relations (5) it follows, that $Q_{i(r)}^{(s)}(\mathbf{z}) \neq 0$ for all indices. Thus, the approximants $g_n(\mathbf{z}),\ n \geq 1$, of BCF (1) form the sequence of functions holomorphic in domain (3).

Let

$$D_c = \{\mathbf{z} \in \mathbb{C}^N :\ |z_k| < c/N,\ 1 \leq k \leq N\}, \quad 0 < c < 1. \tag{6}$$

be a domain contained in $D$. Applying relations (5), for the arbitrary $\mathbf{z} \in D_c$, $D_c \subset D$, we obtain for $n \geq 2$

$$|g_n(\mathbf{z})| = \frac{s_0}{\left|Q_{i(0)}^{(n-1)}(\mathbf{z})\right|} \leq \frac{s_0}{1 - \sum\limits_{i_1=1}^{N} \frac{g_{i(1)}|z_{i_1}|}{\left|Q_{i(1)}^{(n-1)}(\mathbf{z})\right|}} < \frac{s_0}{1-c} = M(D_c),$$

where the constant $M(D_c)$ depends only on the domain $D_c$, i.e. the sequence $\{g_n(\mathbf{z})\}$ is uniformly bounded in the domain of form (6).

Let $K$ be an arbitrary compact subset of domain (3). Let us cover $K$ with the domain of form (6). From this cover we choose the finite subcover $\{D_{c_j}\}_{j=1}^{s}$. Let

$$M(K) = \max\{M(D_{c_j}) : \ 1 \leq j \leq s\}.$$

Then, taking into account $g_1(\mathbf{z}) = s_0$, for arbitrary $\mathbf{z} \in K$ we obtain

$$|g_n(\mathbf{z})| \leq M(K)$$

for $n \geq 1$, i.e. the sequence $\{g_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (3).

According to theorem 2 [5] BCF (1) converges in the domain

$$\Delta_r = \{\mathbf{z} \in \mathbb{C}^N : \ |z_k| \leq r < 1/(8N), \ 1 \leq k \leq N\}.$$

Evidently $\Delta_r \subset D$ for each $r$, $0 < r < 1/(8N)$, in particular, say $\Delta_{1/(9N)} \subset D$. Applying theorem 2.17 [1] we come to the conclusion that the multidimensional $g$-fraction with nonequivalent variables (1) converges uniformly on each compact subset of the domain (3) to function $g(\mathbf{z})$, which is to be holomorphic in this domain.

Now, we prove the second statement of this theorem. Since the sequence $\{g_n(\mathbf{z})\}$ converges uniformly on each compact subset of the domain $D$ to function $g(\mathbf{z})$, which is holomorphic in $D$, then according to Weierstrass's theorem [15, p. 288] for arbitrary $|m(N)|$ we have

$$\frac{\partial^{|m(N)|} g_n(\mathbf{z})}{\partial \mathbf{z}^{m(N)}} \to \frac{\partial^{|m(N)|} g(\mathbf{z})}{\partial \mathbf{z}^{m(N)}},$$

where $\partial \mathbf{z}^{m(N)} = \partial z_1^{m_1} \partial z_2^{m_2} \cdot \ldots \cdot \partial z_N^{m_N}$, on each compact subset of the domain $D$. And now, according to theorem 1 the expansion of each approximant $g_n(\mathbf{z})$, $n \geq 1$, into FMPS coincides with series (2) for all homogeneous polynomials to the degree $n-1$ inclusively. Then for $n \to \infty$

$$\lim_{n\to\infty} \left. \frac{\partial^{|m(N)|} g_n(\mathbf{z})}{\partial \mathbf{z}^{m(N)}} \right|_{\mathbf{z}=\mathbf{0}} = m(N)!(-1)^{|m(N)|} s_{m(N)},$$

where $m(N)! = m_1! m_2! \cdot \ldots \cdot m_N!$.

Hence,

$$g(\mathbf{z}) = \sum_{|m(N)|\geq 0} \frac{\partial^{|m(N)|} g(\mathbf{z})}{m(N)!\partial \mathbf{z}^{m(N)}}\bigg|_{\mathbf{z}=\mathbf{0}} \mathbf{z}^{m(N)} = \sum_{|m(N)|\geq 0} (-1)^{|m(N)|} s_{m(N)} \mathbf{z}^{m(N)}$$

for all $\mathbf{z} \in D$. $\square$

## CONCLUSION

The question of the class of functions of multiple variables which are presented by the multidimensional $g$-fraction with nonequivalent variables remains open.

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У статті розглянуто багатовимірний $g$-дріб з нерівнозначними змінними, який є узагальненням неперервного $g$-дробу. Досліджено відповідність між формальним кратним степеневим рядом і багатовимірним $g$-дробом з нерівнозначними змінними.

Дмитришин Р.И. *Многомерная g-дробь с неравноправными переменными, соответствующая формальному кратному степенному ряду // Карпатские математические публикации.* — 2009. — Т.1, №2. — С. 145–151.

В статье рассмотрена многомерная $g$-дробь с неравноправными переменными, которая является обобщением непрерывной $g$-дроби. Исследовано соответствие между формальным кратным степенным рядом и многомерной $g$-дробью с неравноправными переменными.