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DETERMINING OF UNKNOWN FUNCTIONS OF DIFFERENT ARGUMENTS IN MINOR COEFFICIENT AND RIGHT-HAND SIDE OF SEMILINEAR ULTRAPARABOLIC EQUATION

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In this paper, we consider the inverse problem for semilinear ultraparabolic equation. The equation has two unknown functions of different arguments in its minor coefficient and in right-hand side function. The sufficient conditions of the existence and the uniqueness of solution on some interval $[0, T]$, where T depends on the coefficients of the equation, are obtained.

Key words and phrases: inverse problem, ultraparabolic equation, boundary-value problem, unique solvability.

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INTRODUCTION

Different problems for ultraparabolic equations appear in modeling of physical, biological and financial processes [4, 7, 10, 17]. The unique solvability of the direct problems for the ultraparabolic equations was investigated in the works [2, 4, 7, 10, 11, 17]. The conditions of the existence and the uniqueness of the solution for the inverse problems for semilinear ultraparabolic equations with single or several unknown parameters in its right-hand side function were found in [16,17], with two time dependent coefficients in minor term and in its right-hand side were found in [15], for the linear ultraparabolic equation with the unknown spatial type minor coefficient were found in [8].

In this paper, we find the conditions of the existence and the uniqueness of the solution for the inverse problem for the semilinear ultraparabolic equation on some time interval $[0, T]$, where T depends on the data of the problem. The unknown functions of different arguments are in the minor term and in the right-hand side of the equation. In order to obtain the main results, we set the initial, boundary and the integral type overdetermination conditions and we use the method of successive approximations.

The problems of determination of the minor coefficient or right-hand side function in other types of equations were studied in [1, 3, 6, 12–14, 18]. The authors used the methods of the integral equations, regularization and the Shauder principle [6, 8, 12, 14], the method of semigroups, of finite difference approximations, numerical and iterative methods [3, 13], the methods of successive approximations [15–17].

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1 STATEMENT OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^l$ be bounded domains with the boundaries $\partial\Omega \in C^2$ and $\partial D \in C^1$; $T \in (0,\infty)$, $x \in \Omega$, $y \in D$, $t \in (0,T)$, $G = \Omega \times D$, $\Pi_T = D \times (0,T)$, $Q_\tau = \Omega \times D \times (0,\tau)$, $\tau \in (0,T]$, $\Sigma_T = \partial\Omega \times D \times (0,T)$, $S_T = \Omega \times \partial D \times (0,T)$, $n, l \in \mathbb{N}$.

We shall use the spaces $L^\infty(\cdot)$, $L^2(\cdot)$, $W^{1,2}(\cdot)$, $C^k(\cdot)$, $C([0,T];L^2(G))$, $C^1(D;C^1(\overline{\Omega}))$ from [5, pp. 32, 37, 38, 44, 147] and introduce spaces

$$V_1(Q_T) := \{w : w, w_{x_i} \in L^2(Q_T), i = 1,\dots,n, w|_{\Sigma_T} = 0\};$$
$$V_2(Q_T) := \{w : w \in W^{1,2}(Q_T), w|_{S_T^1} = 0, w|_{\Sigma_T} = 0\};$$
$$V_3(Q_T) := \{w : w \in V_2(Q_T), w_{x_ix_j} \in L^2(Q_T), i,j = 1,\dots,n\}.$$

In this paper we shall study the following inverse problem: find the sufficient conditions of the existence and the uniqueness of a triple of functions $(u(x,y,t),\ c(t),\ q(x))$ that satisfies the equation

$$u_t + \sum_{i=1}^{l} \lambda_i(x,y,t)u_{y_i} - \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + (c(t) + b(x,y))u + g(x,y,t,u) = f_1(x,y,t)q(x) + f_2(x,y,t), \quad (x,y,t) \in Q_T, \tag{1}$$

and the conditions

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in G, \tag{2}$$
$$u|_{\Sigma_T} = 0, \qquad u|_{S_T^1} = 0, \tag{3}$$
$$\int_G K_1(x,y)u(x,y,t)\,dx\,dy = E_1(t), \quad t \in [0,T], \tag{4}$$
$$\int_{\Pi_T} K_2(y,t)u(x,y,t)\,dy\,dt = E_2(x), \quad x \in \Omega, \tag{5}$$

where $u(x,y,t)$, $c(t)$, $q(x)$ are unknown functions, ν is the outward unit normal vector to S_T, $S_T^1 := \Big\{(x,y,t) \in S_T : \sum_{i=1}^{l} \lambda_i(x,y,t)\cos(\nu,y_i) < 0\Big\}$.

Denote $S_T^2 := \Big\{(x,y,t) \in S_T : \sum_{i=1}^{l} \lambda_i(x,y,t)\cos(\nu,y_i) \geq 0\Big\}$, $\Gamma_2 = \partial D \setminus \Gamma_1$.

Suppose that the following assumptions hold:

(H1): there exists $\Gamma_1 \subset \partial D \subset \mathbb{R}^{l-1}$ such that $S_T^1 = \Omega \times \Gamma_1 \times (0,T)$;

(H2): $a_{ij} \in L^2(\Omega)$, $i,j = 1,\dots,n$, $\sum_{i=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2$ for almost all $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$, $a_0 > 0$;

(H3): $\lambda_i \in C(\overline{Q}_T)$, $\lambda_{iy_i} \in L^\infty(Q_T)$, $i = 1,\dots,l$;

(H4): $b \in L^\infty(G)$, $b(x,y) \geq b_0$ for almost all $(x,y) \in G$, where b_0 is a constant;

(H5): $g(x,y,t,\xi)$ is measurable with respect to (x,y,t) in Q_T for all $\xi \in \mathbb{R}^1$ and is continuous with respect to ξ for almost all $(x,y,t) \in Q_T$, moreover, there exists $g_0 > 0$ such that $|g(x,y,t,\xi) - g(x,y,t,\eta)| \leq g_0|\xi - \eta|$ for almost all $(x,y,t) \in Q_T$ and all $\xi, \eta \in \mathbb{R}^1$;

(H6): $f_1, f_2 \in C(Q_T)$;

(H7): $u_0 \in W^{1,2}(G)$, $u_0|_{\partial\Omega\times D} = 0$, $u_0|_{\Omega\times\Gamma_1} = 0$;

(H8): $K_1 \in C^1(D; C^1(\overline{\Omega}))$, $K_1|_{\partial\Omega\times D} = 0$, $K_1|_{\Omega\times\Gamma_2} = 0$, $K_2 \in C^1([0,T]; C^1(\overline{D}))$, $K_2(y,T) = 0$, $K_2(y,0) = 0$, $y \in D$, $K_2|_{\Gamma_2\times(0,T)} = 0$;

(H9): $E_1 \in W^{1,2}(0,T)$, $E_1(0) = \int_G K_1(x,y)u_0(x,y)\,dx\,dy$, $E_2 \in W_0^{1,2}(\Omega)$.

Definition 1. *A triple of functions $(u(x,y,t), c(t), q(x))$ is a solution of the problem* (1)–(5)*, if $u \in V_3(Q_T) \cap C([0,T]; L^2(G))$, $c \in C([0,T])$, $q \in L^2(\Omega)$, it satisfies* (1) *for almost all $(x,y,t) \in Q_T$ and the conditions* (2)*,* (4)*,* (5) *hold.*

Remark 1. *For the case $c(t) = c^*(t)$, $q(x) = q^*(x)$ from* (1)*, where $c^* \in C([0,T])$, $q^* \in L^2(\Omega)$ are known functions, the results of the unique solvability of the initial-boundary value problem* (1)–(3) *are proved with the use of [11, Theorem 1, 2; Lemma 1], [17, Theorem 3.4.3] and are represented in the following theorem.*

Theorem 1. *Suppose that the conditions (H1)–(H7) hold, and, besides:*

1) $a_{ijx_i} \in L^\infty(\Omega)$, $b_{y_k} \in L^\infty(G)$, $f_{sy_k} \in L^2(Q_T)$, $c^* \in C([0,T])$, $q^* \in L^2(\Omega)$, $i,j = 1,\dots,n$, $k = 1,\dots,l$, $s = 1,2$;

2) *there exists a constant g_1 such that for almost all $(x,y,t) \in Q_T$ and all $\xi \in \mathbb{R}^1$ the inequalities $|g_{y_i}(x,y,t,\xi)| \le g_1$, $i = 1,\dots,l$, hold true and $g(x,y,t,0)|_{S_T^1} = 0$;*

3) $f_s|_{S_T^1} = 0$, $s = 1,2$.

Then there exists a unique function $u^* \in V_2(Q_T) \cap C([0,T]; L^2(G))$ that satisfies the condition (2) *and the equality*

$$\int_{Q_T} \Big(u_t^* v + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i}^* v + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}^* v_{x_i} + (c^*(t) + b(x,y)) u^* v + g(x,y,t,u^*) v \Big)\,dx\,dy\,dt = \int_{Q_T} (f_1(x,y,t) q^*(x) + f_2(x,y,t)) v\,dx\,dy\,dt \tag{6}$$

for all functions $v \in V_1(Q_T)$. Moreover, $u^* \in V_3(Q_T) \cap C([0,T]; L^2(G))$, u^* satisfies the condition (2) *and the equation* (1) *for almost all $(x,y,t) \in Q_T$. The derivatives of u^* have the following estimates*

$$\int_{Q_T} \sum_{i=1}^{l} (u_{y_i}^*)^2\,dx\,dy\,dt \le M_0, \qquad \int_{Q_T} (u_t^*)^2\,dx\,dy\,dt \le M,$$

where the constants M_0, M depend on u_0, and on the coefficients and the right-hand side function of (1).

2 THE EQUIVALENT PROBLEM

In this section, we shall find the equivalent problem for the problem (1)–(5). Denote:

$$A_1(t) := E_i'(t) - \int_G K_1(x,y) f_2(x,y,t)\,dx\,dy,$$

$$A_2(x) := \int_{\Pi_T} K_2(y,t) f_2(x,y,t)\,dy\,dt + \sum_{i,j=1}^{n} (a_{ij}(x) E_{2x_i}(x))_{x_j},$$

$$B_1(x,y,t) := \sum_{i=1}^{l} (\lambda_i(x,y,t) K_1(x,y))_{y_i} + \sum_{i,j=1}^{n} (K_{1x_j}(x,y) a_{ij}(x))_{x_i} - K_1(x,y) b(x,y),$$

$$B_2(x,y,t) := -K_{2t}(y,t) - \sum_{i=1}^{l} (\lambda_i(x,y,t) K_2(x,y))_{y_i} + K_2(x,y) b(x,y),$$

$$F_1(x) := \int_{\Pi_T} K_2(y,t) f_1(x,y,t)\,dy\,dt.$$

Assume that

$$F_1(x) \neq 0 \text{ for all } x \in \Omega, \quad E_1(t) \neq 0 \text{ for all } t \in [0,T]. \tag{7}$$

Lemma 1. *The solution of the problem* (1)–(5) *satisfies the equalities*

$$\begin{aligned} c(t) &= \int_G \frac{K_1(x,y)\,(f_1(x,y,t)q(x) - g(x,y,t,u)) + B_1(x,y,t)u}{E_1(t)}\,dxdy - \frac{A_1(t)}{E_1(t)}, \quad t \in [0,T], \\ q(x) &= \int_{\Pi_T} \frac{K_2(y,t)c(t)u + B_2(x,y,t)u + K_2(y,t) g(x,y,t,u)}{F_1(x)}\,dy\,dt - \frac{A_2(x)}{F_1(x)}, \quad x \in \Omega. \end{aligned} \tag{8}$$

Proof. Let $(u(x,y,t), c(t), q(x))$ be a solution of the problem (1)–(5). After differentiation of (4) once with respect to t we derive the formula

$$\int_G K_1(x,y) u_t(x,y,t)\,dx\,dy = E_1'(t), \quad t \in [0,T]. \tag{9}$$

By using of the relations (1) and (9) we get

$$\begin{aligned} \int_G K_1(x,y) \Big(f_1(x,y,t) q(x) + f_2(x,y,t) - \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} - b(x,y) u \\ + \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} - c(t) u - g(x,y,t,u) \Big)\,dx\,dy = E_1'(t), \quad t \in [0,T]. \end{aligned} \tag{10}$$

Integrating by parts in (10), in view of the condition (H8), we obtain

$$\begin{aligned} -E_1(t)c(t) + \int_G \Big(K_1(x,y) f_1(x,y,t) q(x) + B_1(x,y,t) u \\ - K_1(x,y) g(x,y,t,u) \Big)\,dx\,dy = A_1(t), \quad t \in [0,T]. \end{aligned} \tag{11}$$

From (5) and (1) we get

$$\begin{aligned} \int_{\Pi_T} K_2(y,t) \Big(u_t - f_1(x,y,t) q(x) - f_2(x,y,t) + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} + b(x,y) u \\ - \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} + c(t) u + g(x,y,t,u) \Big)\,dx\,dy = E_2'(t), \quad t \in [0,T], \end{aligned}$$

and after integrating by parts in the first term of this formula, we derive

$$\int_{\Pi_T} \Big(K_2(y,t)(c(t)u + g(x,y,t,u)) + B_2(x,y,t)u\Big)\,dxdy - F_1(x)q(x) = A_2(x), \quad x \in \Omega. \quad (12)$$

Using the condition (7) after dividing the system of equations (11), (12) respectively on $E_1(t) \neq 0$ and $F_1(x) \neq 0$, we obtain (8). □

Lemma 2. *There exists a number T such that the solution of the system* (8) *exists for each fixed* $u^* \in V_3(Q_T) \cap C([0,T]; L^2(G))$.

Proof. Let us use the method of successive approximations. Let $c^1(t) = 0$, $q^1(x) = 0$, $t \in [0,T]$, $x \in \Omega$,

$$\begin{aligned} c^m(t) &= \int_G \frac{K_1(x,y)}{E_1(t)} f_1(x,y,t) q^{m-1}(x)\,dx\,dy \\ &\quad + \int_G \frac{B_1(x,y,t)u^* - K_1(x,y)g(x,y,t,u^*)}{E_1(t)}\,dx\,dy - \frac{A_1(t)}{E_1(t)}, \quad t \in [0,T],\ m \geq 2, \end{aligned} \quad (13)$$

$$\begin{aligned} q^m(x) &= \int_{\Pi_T} \frac{K_2(y,t)}{F_1(x)} c^{m-1}(t) u^*\,dy\,dt \\ &\quad + \int_{\Pi_T} \frac{B_2(x,y,t)u^* + K_2(y,t)g(x,y,t,u^*)}{F_1(x)}\,dy\,dt - \frac{A_2(x)}{F_1(x)}, \quad x \in \Omega,\ m \geq 2. \end{aligned} \quad (14)$$

The successive differences

$$c^m(t) - c^{m-1}(t) = \int_G \frac{K_1(x,y)f_1(x,y,t)}{E_1(t)} \big(q^{m-1}(x) - q^{m-2}(x)\big)\,dx\,dy, \quad t \in [0,T],\ m \geq 3, \quad (15)$$

$$q^m(x) - q^{m-1}(x) = \frac{1}{F_1(x)} \int_{\Pi_T} K_2(y,t)\big(c^{m-1}(t) - c^{m-2}(t)\big) u^*\,dy\,dt, \quad x \in \Omega,\ m \geq 3. \quad (16)$$

Squaring (15) and (16) and integrating (16) with respect to x for $m \geq 3$, give us inequalities

$$\begin{aligned} &(c^m(t) - c^{m-1}(t))^2 \\ &\quad \leq \frac{\operatorname{mes} D}{(E_1(t))^2} \int_G (K_1(x,y)f_1(x,y,t))^2\,dx\,dy \int_\Omega (q^{m-1}(x) - q^{m-2}(x))^2\,dx, \quad t \in [0,T], \end{aligned} \quad (17)$$

$$\begin{aligned} &\int_\Omega (q^m(x) - q^{m-1}(x))^2\,dx \\ &\quad \leq \operatorname{mes} D \int_{Q_T} \left(\frac{K_2(y,t)}{F_1(x)}\right)^2 (u^*)^2\,dx\,dy\,dt \int_0^T (c^{m-1}(t) - c^{m-2}(t))^2\,dt. \end{aligned} \quad (18)$$

From (17) and (18) it follows that

$$\int_\Omega (q^m(x) - q^{m-1}(x))^2\,dx \leq C_0 \int_\Omega (q^{m-2}(x) - q^{m-3}(x))^2\,dx, \quad m \geq 4,$$

where $C_0 := (\operatorname{mes} D)^2 \int_{Q_T} \frac{(K_1(x,y)f_1(x,y,t))^2}{(E_1(t))^2}\,dx\,dy\,dt \int_{Q_T} \left(\frac{K_2(y,t)}{F_1(x)}\right)^2 (u^*)^2\,dx\,dy\,dt$. Due to the fact that $u^* \in V_3(Q_T) \cap C([0,T]; L^2(G))$, there exists T, such that $C_0 < 1$. Then

$$\int_\Omega (q^m(x)-q^{m-1}(x))^2\,dx \le (C_0)^{\frac{m-3}{2}}K^2, \qquad m\ge 4,$$

where $K := \Big(\max\Big\{\int_\Omega (q^3(x)-q^2(x))^2\,dx;\ C_0^{\frac12}\int_\Omega (q^2(x)-q^1(x))^2\,dx\Big\}\Big)^{1/2}$. Then inequalities $\|q^{m+k}-q^m;\ L^2(\Omega)\| \le \sum_{i=m+1}^{m+k}\|q^i-q^{i-1};\ L^2(\Omega)\| \le \sum_{i=m+1}^{m+k}(C_0)^{\frac{i-3}{4}}K \le \frac{(C_0)^{\frac{m-2}{4}}K}{1-(C_1)^{\frac14}}$ hold for all $k\in\mathbb{N}$, $m\ge 4$. Besides,

$$\begin{aligned}\|c^{m+k}-c^m;C[0;T]\| &\le \sum_{i=m+1}^{m+k}\|c^i-c^{i-1};C[0;T]\|\\ &\le \sum_{i=m+1}^{m+k}\Big(\sup_{[0,T]}\frac{\operatorname{mes}D}{(E_1(t))^2}\int_G (K_1(x,y)f_1(x,y,t))^2\,dx\,dy\Big)^{\frac12}\|q^{i-1}-q^{i-2};L^2(\Omega)\|\\ &\le \sum_{i=m+1}^{m+k}\Big(\sup_{[0,T]}\frac{\operatorname{mes}D}{(E_1(t))^2}\int_G (K_1(x,y)f_1(x,y,t))^2\,dx\,dy\Big)^{\frac12}\frac{(C_0)^{\frac{m-3}{4}}K}{1-(C_0)^{\frac14}}, \quad m\ge 5.\end{aligned}$$

From here it follows that for any $\varepsilon>0$ there exists m_0 such that for all $k,\ m\in\mathbb{N}$, $m>m_0$, the inequalities $\|c^{m+k}-c^m;C([0,T])\|\le\varepsilon$ and $\|q^{m+k}-q^m;L^2(\Omega)\|\le\varepsilon$ are true. Hence, the sequence $\{c^m\}_{m=1}^\infty$ is fundamental in $C([0,T])$, and $\{q^m\}_{m=1}^\infty$ is fundamental in $L^2(\Omega)$. Passing to the limit in (13), (14) as $m\to\infty$, we obtain (8). □

Lemma 3. *Let the assumptions of Theorem 1 and* (7)*, (H8), (H9) hold. The triple of functions* $(u(x,y,t),c(t),q(x))$*, where* $u\in V_3(Q_T)\cap C([0,T];L^2(G))$*,* $c\in C([0,T])$*,* $q\in L^2(\Omega)$*, is a solution of the problem* (1)–(5) *if and only if it satisfies* (1) *for almost all* $(x,y,t)\in Q_T$*, and* (2)*,* (8) *hold.*

Proof. Necessity is proved in Lemma 1.

Sufficiency. Let $c^*\in C([0,T])$, $q^*\in L^2(\Omega)$, $u^*\in V_3(Q_T)\cap C([0,T];L^2(G))$ and let these functions satisfy (2), (8) and (1) for almost all $(x,y,t)\in Q_T$. Then u^* is a solution of the problem (1)–(3) with c^* and q^* instead of c and q in (1).

We set $E_1^*(t)=\int_G K_1(x,y)u^*(x,y,t)\,dx\,dy$, $t\in[0,T]$, $E_2^*(x)=\int_{\Pi_T}K_2(y,t)u^*(x,y,t)\,dy\,dt$, $x\in\Omega$. In exactly the same way as in the proof of necessity, we obtain

$$\begin{aligned}E_1^*(t)c^*(t) &= \int_G\Big(K_1(x,y)f_1(x,y,t)q^*(x)+B_1(x,y,t)u^*-K_1(x,y)g(x,y,t,u^*)\Big)\,dx\,dy\\ &\quad-(E_i^*(t))'+\int_G K_1(x,y)f_2(x,y,t)\,dx\,dy, \quad t\in[0,T],\\ F_1(x)q^*(x) &= \int_{\Pi_T}\Big(K_2(y,t)c^*(t)u^*+B_2(x,y,t)u^*+K_2(y,t)g(x,y,t,u^*)\Big)\,dy\,dt\\ &\quad-\int_{\Pi_T}K_2(y,t)f_2(x,y,t)\,dy\,dt-\sum_{i,j=1}^n(a_{ij}(x)E^*_{2x_i}(x))_{x_j}, \quad x\in\Omega.\end{aligned} \tag{19}$$

On the other hand $c^*(t)$, $q^*(t)$ and $u^*(x,y,t)$ satisfy (8), and therefore it is easy to get the following equalities

$$\begin{aligned} E_1(t)c^*(t) &= \int_G \Big(K_1(x,y)f_1(x,y,t)q^*(x)+B_1(x,y,t)u^*-K_1(x,y)g(x,y,t,u^*)\Big)\,dx\,dy \\ &\quad -(E_i(t))' + \int_G K_1(x,y)f_2(x,y,t)\,dx\,dy, \quad t\in[0,T], \\ F_1(x)q^*(x) &= \int_{\Pi_T}\Big(K_2(y,t)c^*(t)u + B_2(x,y,t)u^* + K_2(y,t)g(x,y,t,u^*)\Big)\,dy\,dt \\ &\quad -\int_{\Pi_T} K_2(y,t)f_2(x,y,t)\,dy\,dt - \sum_{i,j=1}^{n}(a_{ij}(x)E_{2x_i}(x))_{x_j}, \quad x\in\Omega. \end{aligned} \tag{20}$$

It follows from (19), (20) that

$$(E_1^*(t)-E_1(t))c^*(t) = -(E_1^*(t)-E_1(t))', \quad t\in[0,T], \tag{21}$$

$$\sum_{i,j=1}^{n}(a_{ij}(x)(E_2^*(x)-E_2(x))_{x_i})_{x_j} = 0, \quad x\in\Omega. \tag{22}$$

Integrating (21) with the use of the equality $E_1^*(0) = E_1(0) = \int_G K_1(x,y)u_0(x,y)\,dx\,dy$, we get $E_1^*(t) = E_1(t)$, $t\in[0,T]$. Besides from (22) and (H9) it follows that $E_2^*(x) = E_2(x)$, $x\in\Omega$. Hence, $u^*(x,y,t)$ satisfies the overdetermination conditions (4), (5). □

3 MAIN RESULTS

Denote

$$\begin{aligned} \lambda^1 &:= \max_i \operatorname*{ess\,sup}_{Q_T}|\lambda_{iy_i}(x,y,t)|, \; f_3 := \sup_\Omega \int_{\Pi_T}\big(f_1(x,y,t)\big)^2\,dy\,dt, \\ C_1 &:= \sup_{[0,T]}\frac{4\,\mathrm{mes}\,D}{(E_1(t))^2}\int_G\Big(K_1(x,y)f_1(x,y,t)\Big)^2\,dx\,dy, \\ C_2 &:= \sup_{[0;T]}\frac{4}{(E_1(t))^2}\int_G (B_1(x,y,t))^2+(g_0)^2(K_1(x,y))^2\,dx\,dy, \\ C_3 &:= \sup_{[0;T]}\frac{4}{(E_1(t))^2}\Big((A_1(t))^2+\int_G(K_1(x,y))^2\,dx\,dy\int_G(g(x,y,t,0))^2\,dx\,dy\Big), \\ C_4 &:= \sup_{\Omega\times[0,T]}\frac{4}{(F_1(x))^2}\int_D\big(K_2(y,t)\big)^2\,dy, \\ C_5 &:= \sup_\Omega\frac{8}{(F_1(x))^2}\int_{\Pi_T}(B_2(x,y,t))^2+2(g^0)^2(K_2(y,t))^2\,dy\,dt, \\ C_6 &:= \sup_\Omega\left(4\int_\Omega\frac{(A_1(x))^2}{(F_1(x))^2}\,dx+16\Big(\int_{\Pi_T}\frac{K_2(y,t)}{F_1(x)}g(x,y,t,0)\,dy\,dt\Big)^2\right), \end{aligned}$$

and $\gamma_0 := \gamma_0(\Omega)$ is the coefficient in Friedrichs' inequality

$$\int_\Omega |v(x)|^2\,dx \le \gamma_0\int_\Omega\sum_{i=1}^{n}|v_{x_i}(x)|^2\,dx, \quad v\in W_0^{1,2}(\Omega). \tag{23}$$

Assume that there exist numbers T and δ such that the following inequalities are true

$$
\begin{gathered}
T < \left(\frac{1}{C_4C_1M_1}\right)^{1/2}, \\
T^4C_4^2C_1M_1(f_3C_2M_1 + f_3C_3 - \delta C_1M_1^2) \\
+ T^3C_4C_1M_1f_3C_5 + T^2C_1C_4(C_6f_3 + \delta M_1^2 + M_1) < 1,
\end{gathered} \tag{24}
$$

$$
\frac{2a_0}{\gamma_0} + 2b_0 - \lambda^1 l - 2g_0 - 2M_3 - 3\delta > 0, \tag{25}
$$

where

$$
M_1 := \frac{1}{\delta}\int_{Q_T}(f_2(x,y,t))^2 + (g(x,y,t,0))^2\,dx\,dy\,dt + \int_G (u_0(x,y))^2\,dx\,dy;
$$

$$
M_2 := M_1 + \frac{f_3}{\delta(1 - C_4C_1M_1T^2)}\Big(C_4C_2M_1^2T^2 + C_4C_3M_1T^2 + C_5M_1T + C_6\Big);
$$

$$
M_3 := \left(\frac{C_1C_4C_2M_2^2T^2 + C_1(C_4C_3T + C_5)M_2T}{1 - C_4C_1M_2T^2} + C_2M_2 + C_1C_6 + C_3\right)^{1/2}.
$$

Denote

$$
M_4 := \frac{C_4C_2M_2^2T^2 + C_4C_3M_2T^2 + C_5M_2T + C_6}{1 - C_4C_1M_2T^2},
$$

$$
M_5 := 3\,\mathrm{mes}\,D\int_{Q_T}\left(\frac{K_1(x,y)f_1(x,y,t)}{E_1(t)}\right)^2 dx\,dy\,dt,
$$

$$
M_6 := 3\sup_{[0,T]}\int_G \frac{(B_1(x,y,t))^2 + (g_0)^2(K_1(x,y))^2}{(E_1(t))^2}\,dx\,dy,
$$

$$
M_7 := 4M_4T\sup_{\Omega\times[0,T]}\int_D\left(\frac{K_2(y,t)}{F_1(x)}\right)^2 dy,
$$

$$
M_8 := 4\sup_{\Omega}\int_\Pi \frac{(B_2(x,y,t))^2 + (K_2(y,t)g_0)^2 + (K_2(y,t)M_2)^2}{(F_1(x))^2}\,dy\,dt,
$$

$$
M_9 := \frac{M_6 + M_8}{1 - \max\{M_5; M_7\}}, \qquad M_{10} := \max\{M_4; f_3\},
$$

$$
M_{11} := \min\left\{T; \frac{1}{\frac{2a_0}{\gamma_0} + 2b_0 - \lambda^1 l - 2g_0 - 2M_2 - 2\delta}\right\}\frac{M_{10}}{\delta}, \qquad M_{12} := M_9M_{11},
$$

$$
M_{13} := 3\,\mathrm{mes}\,D\sup_{[0,T]}\int_G\left(\frac{K_1(x,y)f_1(x,y,t)}{E_1(t)}\right)^2 dx\,dy\,dt, \qquad M_{14} := \sqrt{M_{12}M_{13} + \frac{M_6M_{10}}{\delta}}.
$$

Theorem 2. *Let* $M_5 < 1$*,* $M_7 < 1$*,* $M_{12} < 1$*, the hypotheses* (7)*,* (25)*, (H1)–(H9) hold, and* $a_{ijx_i} \in L^\infty(\Omega)$*,* $b_{y_k} \in L^\infty(G)$*,* $f_{sy_k} \in L^2(Q_T)$*,* $f_s|_{S_T^1} = 0$*,* $i,j = 1,\dots,n$*,* $k = 1,\dots,l$*,* $s = 1,2$*. If* T *satisfies* (24)*, then a solution of the problem* (1)–(5) *exists in* Q_T*.*

Proof. We construct an approximation $(u^m(x,y,t), c^m(t), q^m(x))$ to the solution of the problem (1)–(5), where $c^1(t) := 0$, $q^1(x) := 0$, the functions $c^m(t)$ and $q^m(x)$, $m \geq 2$, satisfy the system of equalities

$$
c^m(t) = \int_G \frac{K_1(x,y)f_1(x,y,t)q^m(x) + B_1(x,y,t)u^{m-1} - K_1(x,y)g(x,y,t,u^{m-1})}{E_1(t)}\,dx\,dy - \frac{A_1(t)}{E_1(t)},
$$

$$
t \in [0;T],\ m \geq 2, \tag{26}
$$

$$q^m(x) = \int_{\Pi_T} \frac{K_2(y,t)c^m(t)u^{m-1} + B_2(x,y,t)u^{m-1} + K_2(y,t)g(x,y,t,u^{m-1})}{F_1(x)}\,dy\,dt - \frac{A_2(x)}{F_1(x)},$$
$$x \in \Omega,\ m \geq 2, \tag{27}$$

and u^m satisfies the equality

$$\int_{Q_\tau} \Big(u^m_t v + \sum_{i=1}^{l} \lambda_i(x,y,t) u^m_{y_i} v + \sum_{i,j=1}^{n} a_{ij}(x) u^m_{x_i} v_{x_j} + (c^m(t) + b(x,y))u^m v + g(x,y,t,u^m)v \Big)\,dx\,dy\,dt$$
$$= \int_{Q_\tau} (f_1(x,y,t)q^m(x) + f_2(x,y,t))v\,dx\,dy\,dt, \quad m \geq 1,\ \tau \in (0;T], \tag{28}$$

for all $v \in V_1(Q_T)$, as well as the condition

$$u^m(x,y,0) = u_0(x,y), \quad (x,y) \in G. \tag{29}$$

It follows from Theorem 1 that for each $m \in \mathbb{N}$ there exists a unique function $u^m \in V_2(Q_T) \cap C([0,T]; L^2(G))$ that satisfies (28), (29). Now we show that $c^m(t) \geq -M_3$ for all $m \in \mathbb{N}$, $t \in [0;T]$. Let $c^m(t) \geq c_{0m}$ for all $t \in [0,T]$, where $c_{0m} \in \mathbb{R}$. At first, we shall find the estimation for $\int_G |u^m(x,y,\tau)|^2\,dx\,dy$. Let us choose $v = u^m$ in (28):

$$\int_{Q_\tau} \Big(u^m_t u^m + \sum_{i=1}^{l} \lambda_i(x,y,t) u^m_{y_i} u^m + \sum_{i,j=1}^{n} a_{ij}(x) u^m_{x_i} u^m_{x_j} + (c^m(t) + b(x,y))(u^m)^2$$
$$+ g(x,y,t,u^m)u^m \Big)\,dx\,dy\,dt = \int_{Q_\tau} (f_1(x,y,t)q^m(x) + f_2(x,y,t))u^m\,dx\,dy\,dt, \tag{30}$$
$$\tau \in (0;T],\ m \geq 1.$$

Taking into account the hypotheses (H2)–(H7), from (30) we obtain the inequalities

$$\int_G (u^m(x,y,\tau))^2\,dx\,dy + \int_{S^2_\tau} \sum_{i=1}^{l} \lambda_i(x,y,t)(u^m)^2 \cos(\nu, y_i)\,d\sigma + 2a_0 \int_{Q_\tau} \sum_{i=1}^{n} (u^m_{x_i})^2\,dx\,dy\,dt$$
$$+ (2c_{0m} - \lambda^1 l + 2b_0 - 2g_0 - 3\delta) \int_{Q_\tau} (u^m)^2\,dx\,dy\,dt \leq \frac{1}{\delta} \int_{Q_\tau} \Big((f_1(x,y,t))^2 (q^m(x))^2 \tag{31}$$
$$+ (f_2(x,y,t))^2 + (g(x,y,t,0))^2 \Big)\,dx\,dy\,dt + \int_G (u_0(x,y))^2\,dx\,dy, \quad \tau \in (0;T],\ m \geq 1.$$

After using of the inequality (23) in the third term of (31), we get

$$\int_G (u^m(x,y,\tau))^2\,dx\,dy + \int_{S^2_\tau} \sum_{i=1}^{l} \lambda_i(x,y,t)(u^m)^2 \cos(\nu, y_i)\,d\sigma$$
$$+ \Big(\frac{2a_0}{\gamma_0} - \lambda^1 l + 2c_{0m} + 2b_0 - 2g_0 - 3\delta \Big) \int_{Q_\tau} (u^m)^2\,dx\,dy\,dt \leq \frac{1}{\delta} \int_{Q_\tau} \Big((f_1(x,y,t))^2 (q^m(x))^2 \tag{32}$$
$$+ (f_2(x,y,t))^2 + (g(x,y,t,0))^2 \Big)\,dx\,dy\,dt + \int_G (u_0(x,y))^2\,dx\,dy, \quad \tau \in (0;T],\ m \geq 1.$$

Using the assumption $\frac{2a_0}{\gamma_0} - \lambda^1 l + 2c_{0m} + 2b_0 - 2g_0 - 3\delta \geq 0$, from (32) we get the estimates

$$\int_G (u^m(x,y,\tau))^2\,dx\,dy \leq M_1 + \frac{1}{\delta} \int_{Q_\tau} |f_1(x,y,t)|^2 (q^m(x))^2\,dx\,dy\,dt, \quad \tau \in (0;T],\ m \geq 1. \tag{33}$$

Rising up the both sides of (26) to the square and using Hölder inequality, we get the estimate

$$(c^m(t))^2 \le C_1 \int_\Omega (q^m(x))^2\,dx + C_2 \int_G (u^{m-1})^2\,dx\,dy + C_3, \qquad t \in [0;T],\ m \ge 2. \tag{34}$$

Rising up the both sides of (27) to the square and using Hölder inequality, after integrating with respect to x, we get the estimate

$$\int_\Omega (q^m(x))^2\,dx \le C_4 \int_0^T (c^m(t))^2\,dt \int_{Q_T} (u^{m-1})^2\,dx\,dy\,dt + C_5 \int_{Q_T} (u^{m-1})^2\,dx\,dy\,dt + C_6,\ m \ge 2. \tag{35}$$

It is easy to prove the estimates for $m \ge 2$ after using (33), (34), (35)

$$\begin{aligned} &\int_G (u^m(x,y,t))^2\,dx\,dy \le M_1 + \frac{f_3}{\delta(1 - C_4C_1T\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt)} \\ &\times\left(C_4C_2\left(\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt\right)^2 + (C_4C_3T + C_5)\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt + C_6\right),\ t \in [0,T]; \\ &(c^m(t))^2 \le \frac{C_1C_4C_2\left(\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt\right)^2 + C_1(C_4C_3T + C_5)\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt}{1 - C_4C_1T\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt} \\ &+ C_2\int_G (u^{m-1})^2\,dx\,dy + C_1C_6 + C_3; \\ &\int_\Omega (q^m(x))^2\,dx \le \frac{C_4C_2\left(\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt\right)^2 + (C_4C_3T + C_5)\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt + C_6}{1 - C_4C_1T\int_{Q_T}(u^{m-1})^2\,dx\,dy\,dt}. \end{aligned} \tag{36}$$

Functions $\int_G (u^m(x,y,t))^2\,dx\,dy$ for all $m \in \mathbb{N}$ are bounded with the same constant M_2, when

$$M_1 + \frac{f_3}{\delta(1 - C_4C_1T^2M_2)}\left(C_4C_2M_2^2T^2 + (C_4C_3T + C_5)M_2T + C_6\right) \le M_2$$

or (under the assumption $1 - C_4C_1T^2M_2 > 0$)

$$T^2C_4M_2\big(\delta C_1(M_2 - M_1) + f_3(C_2M_2 + C_3)\big) + Tf_3C_5M_2 + (C_6f_3 - \delta(M_2 - M_1)) \le 0. \tag{37}$$

It is obvious that the inequality (37) has positive solutions under the conditions

$$\begin{cases} (f_3C_5M_2)^2 - 4C_4M_2\big(\delta C_1(M_2 - M_1) + f_3(C_2M_2 + C_3)\big)(C_6f_3 - \delta(M_2 - M_1)) > 0, \\ C_6f_3 - \delta(M_2 - M_1) < 0, \\ 1 - C_4C_1T^2M_2 > 0. \end{cases} \tag{38}$$

Taking into account the expression for M_2 in (38), we conclude that system (38) is fulfilled for all $T > 0$ from (24). Therefore from (36) it follows that for all $T > 0$ from (24)

$$\int_G (u^m(x,y,t))^2\,dx\,dy \le M_2, \quad t \in [0,T],\ m \ge 1,$$

$$|c^m(t)| \le M_3, \quad t \in [0,T],\ m \ge 1, \tag{39}$$

$$\int_\Omega (q^m(x))^2\,dx \le M_4, \quad m \ge 1. \tag{40}$$

Remark, that if we take $-M_3$ instead of c_{0m} and take into account (25), we get

$$\frac{2a_0}{\gamma_0} - \lambda^1 l + 2c_{0m} + 2b_0 - 2g_0 - 3\delta = \frac{2a_0}{\gamma_0} - \lambda^1 l - 2M_3 + 2b_0 - 2g_0 - 3\delta \geq 0.$$

Thus, for all $m \in \mathbb{N}$ the inequality $c^m(t) \geq -M_3$ holds, and we can choose $c_{0m} := -M_3$ for all $m \in \mathbb{N}$.

Now we show that $\{(u^m(x,y,t), c^m(t), q^m(x))\}_{m=1}^{\infty}$ converges to the solution of the problem (1)–(5). Denote $z^m := z^m(x,y,t) = u^m(x,y,t) - u^{m-1}(x,y,t)$, $r^m(t) := c^m(t) - c^{m-1}(t)$, $s^m(x) := q^m(x) - q^{m-1}(x)$, $K_m := \int_0^T (r^m(t))^2\,dt + \int_\Omega (s^m(x))^2\,dx$, $m \geq 2$.

Formulas (26), (27) for $t \in [0, T]$ and $m \geq 3$ imply the equalities

$$\begin{aligned} r^m(t) = \int_G \bigg(&\frac{K_1(x,y)}{E_1(t)} f_1(x,y,t) s^m(x)\,dx + \frac{B_1(x,y,t)}{E_1(t)} z^{m-1} \\ &- \frac{K_1(x,y)}{E_1(t)} \Big(g(x,y,t,u^{m-1}) - g(x,y,t,u^{m-2}) \Big) \bigg)\,dx\,dy, \quad t \in [0;T],\ m \geq 3, \\ s^m(x) = \int_{\Pi_T} \bigg(&\frac{K_2(y,t)}{F_1(x)} \Big(c^m(t) u^{m-1} - c^{m-1}(t) u^{m-2} \Big) + \frac{B_2(x,y,t)}{F_1(x)} z^{m-1} \\ &+ \frac{K_2(y,t)}{F_1(x)} \Big(g(x,y,t,u^{m-1}) - g(x,y,t,u^{m-2}) \Big) \bigg)\,dy\,dt, \quad x \in \Omega,\ m \geq 3. \end{aligned} \tag{41}$$

We square both sides of equalities (41) and integrate the result with respect to t, take into account that under the hypotheses (H5) we have

$$\int_{Q_\tau} (g(x,y,t,u^m) - g(x,y,t,u^{m-1})) z^m\,dx\,dy\,dt \leq g_0 \int_{Q_\tau} (z^m)^2\,dx\,dy\,dt, \quad \tau \in (0;T],\ m \geq 2,$$

and $|c^m(t)u^{m-1} - c^{m-1}(t)u^{m-2}| = |c^m(t)z^{m-1} + r^m(t)u^{m-2}| \leq |c^m(t)z^{m-1}| + |r^m(t)u^{m-2}|$, then we obtain

$$\int_0^T (r^m(t))^2\,dt \leq M_5 \int_\Omega (s^m(x))^2\,dx + M_6 \int_{Q_T} (z^{m-1})^2\,dx\,dy\,dt, \quad m \geq 3, \tag{42}$$

$$\int_\Omega (s^m(x))^2\,dx \leq M_7 \int_0^T (r^m(t))^2\,dt + M_8 \int_{Q_T} (z^{m-1})^2\,dx\,dy\,dt, \quad m \geq 3. \tag{43}$$

After adding (42) and (43), we get

$$K_m \leq M_9 \int_{Q_T} (z^{m-1})^2\,dx\,dy\,dt, \quad m \geq 3. \tag{44}$$

It follows from (29) that $z^m(x,y,0) = 0,\ (x,y) \in G,\ m \geq 2$. Hence, from (28) we get

$$\begin{aligned} \frac{1}{2} \int_G (z^m(x,y,\tau))^2\,dx\,dy + \int_{Q_\tau} \bigg(&\sum_{i=1}^{l} \lambda_i(x,y,t) z^m_{y_i} z^m + \sum_{i,j=1}^{n} a_{ij}(x) z^m_{x_i} z^m_{x_j} + b(x,y)(z^m)^2 \\ &+ \big(g(x,y,t,u^m) - g(x,y,t,u^{m-1})\big) z^m + \big(c^m(t)u^m - c^{m-1}(t)u^{m-1}\big) z^m \bigg)\,dx\,dy\,dt \\ &= \int_{Q_\tau} f_1(x,y,t) s^m(x) z^m\,dx\,dy\,dt, \quad \tau \in (0;T],\ m \geq 2. \end{aligned} \tag{45}$$

We note that $(c^m(t)u^m - c^{m-1}(t)u^{m-1})z^m = c^m(t)(z^m)^2 + r^m(t)u^{m-1}z^m$, and therefore

$$\begin{aligned}\int_{Q_\tau} &(c^m(t)u^m - c^{m-1}(t)u^{m-1})z^m \,dx\,dy\,dt \\ &\geq \Big(-M_2 - \frac{\delta}{2}\Big)\int_{Q_\tau}(z^m)^2\,dx\,dy\,dt - \frac{1}{2\delta}\int_0^\tau (r^m(t))^2\Big(\int_G (u^{m-1}(x,y,t))^2\,dx\,dy\Big)\,dt \\ &\geq \Big(-M_2 - \frac{\delta}{2}\Big)\int_{Q_\tau}(z^m)^2\,dx\,dy\,dt - \frac{M_4}{2\delta}\int_0^\tau (r^m(t))^2\,dt, \qquad \tau\in(0,T],\ m\geq 2.\end{aligned} \tag{46}$$

For the last term in (45) we have

$$\int_{Q_\tau} f_1(x,y,t)s^m(x)z^m\,dx\,dy\,dt \leq \frac{\delta}{2}\int_{Q_\tau}(z^m)^2\,dx\,dy\,dt + \frac{f_3}{2\delta}\int_\Omega (s^m(x))^2\,dx.$$

Then, taking into account (H2)–(H7) and (46), from (45) we get inequalities

$$\begin{aligned}&\int_G (z^m(x,y,\tau))^2\,dx\,dy + \int_{S^2_\tau}\sum_{i=1}^{l}\lambda_i(x,y,t)(z^m)^2\cos(\nu,y_i)\,d\sigma + 2a_0\int_{Q_\tau}\sum_{i,j=1}^{n}(z^m_{x_i})^2\,dx\,dy\,dt \\ &+ \big(2b_0 - \lambda^1 l - 2g_0 - 2M_2 - 2\delta\big)\int_{Q_\tau}(z^m)^2\,dx\,dy\,dt \leq \frac{M_4}{\delta}\int_0^T (r^m(t))^2\,dt + \frac{f_3}{\delta}\int_\Omega (s^m(x))^2\,dx, \\ &\qquad \tau\in(0;T],\ m\geq 2.\end{aligned} \tag{47}$$

After applying (23) to the third term of (47), we get the estimate

$$\begin{aligned}&\int_G (z^m(x,y,\tau))^2\,dx\,dy + \int_{S^2_\tau}\sum_{i=1}^{l}\lambda_i(x,y,t)(z^m)^2\cos(\nu,y_i)\,d\sigma \\ &\qquad + \Big(2b_0 - l\lambda^1 - 2g_0 + \frac{2a_0}{\gamma_0} - 2M_2 - 2\delta\Big)\int_{Q_\tau}(z^m)^2\,dx\,dy\,dt \\ &\qquad \leq \frac{M_4}{\delta}\int_0^T (r^m(t))^2\,dt + \frac{f_3}{\delta}\int_\Omega (s^m(x))^2\,dx, \quad \tau\in(0;T],\ m\geq 2.\end{aligned} \tag{48}$$

In view of the conditions (24), (25), from (48) we find the estimates

$$\int_G (z^m(x,y,\tau))^2\,dx\,dy \leq \frac{M_{10}}{\delta}K_m,\ \tau\in(0;T],\ m\geq 2, \tag{49}$$

and

$$\int_{Q_T}(z^m)^2\,dx\,dy\,dt \leq M_{11}K_m, \qquad m\geq 2. \tag{50}$$

It follows from (44) and (50) that

$$K_m \leq M_{12}K_{m-1} \leq (M_{12})^{m-2}K_2, \qquad m\geq 3. \tag{51}$$

From (41), it is easy to find the estimate

$$(r^m(t))^2 \leq M_{13}\int_\Omega (s^m(x))^2\,dx + M_6\int_G (z^{m-1}(x,y,t))^2\,dx\,dy, \quad t\in[0,T],\ m\geq 3. \tag{52}$$

Further, with the use of (49) and (51), from (52) we get

$$|r^m(t)| \leq M_{14}K^{\frac{1}{2}}_{m-1}, \quad t\in[0,T],\ m\geq 3. \tag{53}$$

By using (51), (53) and the assumption $M_{12} < 1$ we can show the estimate

$$\begin{aligned}\|c^{m+k} - c^m; C([0,T])\| &\le \sum_{i=m+1}^{m+k} \sup_{[0,T]} |r^i(t)| \le M_{14} \sum_{i=m+1}^{m+k} K_{i-1}^{\frac{1}{2}} \\ &\le \sum_{i=m+1}^{m+k} M_{14}(M_{12})^{\frac{i-3}{2}} K_2^{\frac{1}{2}} \le \frac{M_{14}(M_{12})^{\frac{m-2}{2}} K_2^{\frac{1}{2}}}{1-(M_{12})^{\frac{1}{2}}}, \quad k \in \mathbb{N},\ m \ge 3.\end{aligned} \tag{54}$$

Besides,

$$\|q^{m+k} - q^m; L^2(\Omega)\| \le \sum_{i=m+1}^{m+k} K_{i-1}^{\frac{1}{2}} \le \frac{(M_{12})^{\frac{m-2}{2}} K_2^{\frac{1}{2}}}{1-(M_{12})^{\frac{1}{2}}}, \quad k \in \mathbb{N},\ m \ge 3. \tag{55}$$

It follows from (54), (55) that for any $\varepsilon > 0$ there exists m_0 such that for all k, $m \in \mathbb{N}$, $m > m_0$, the inequalities $\|c^{m+k} - c^m; C([0,T])\| \le \varepsilon$ and $\|q^{m+k} - q^m; L^2(\Omega)\| \le \varepsilon$ are true. Hence, the sequence $\{c^m\}_{m=1}^{\infty}$ is fundamental in $C([0,T])$, and $\{q^m\}_{m=1}^{\infty}$ is fundamental in $L^2(\Omega)$. Thus, it follows from (49) and (47) that $\{u^m\}_{m=1}^{\infty}$ is fundamental in $L^2(Q_T) \cap C([0,T]; L^2(G))$ and $\{u^m_{x_i}\}_{m=1}^{\infty}$ is fundamental in $L^2(Q_T)$ and, hence,

$$\begin{aligned} u^m \to u \text{ in } L^2(Q_T) \cap C([0,T]; L^2(G)), \qquad & u^m_{x_i} \to u_{x_i} \text{ in } L^2(Q_T),\ i = 1,\dots,n, \\ c^m \to c \text{ in } C([0,T]), \qquad & q^m \to q \text{ in } L^2(\Omega), \qquad \text{as } m \to \infty. \end{aligned} \tag{56}$$

Theorem 1 implies the following estimates $\int_{Q_T} \sum_{i=1}^{l} (u^m_{y_i})^2\, dx\, dy\, dt \le M_0$, $\int_{Q_T} (u^m_t)^2\, dx\, dy\, dt \le M$, and, by virtue of the inequalities (39), (40), the constants $M_0,\ M$ are independent of m and these estimations are true for all $m \in \mathbb{N}$. Thus, we can select a subsequence of the sequence $\{u^m\}_{m=1}^{\infty}$ (we preserve the same notation for this subsequence), such that

$$u^m_{y_i} \to u_{y_i} \text{ weakly in } L^2(Q_T),\ i = 1,\dots,l, \qquad u^m_t \to u_t \text{ weakly in } L^2(Q_T) \tag{57}$$

as $m \to \infty$. Taking into account (56), (57), from (26) and (27) we get that the triple of functions $(u(x,y,t), c(t), q(x))$ satisfies the system of equations (6) and

$$\begin{aligned}\int_{Q_\tau} \Big(u_t v + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} v + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_j} + (c(t) + b(x,y)) uv + g(x,y,t,u) v \Big)\, dx\, dy\, dt \\ = \int_{Q_\tau} \big(f_1(x,y,t) q(x) + f_2(x,y,t) \big) v\, dx\, dy\, dt\end{aligned} \tag{58}$$

for all $v \in V_1(Q_T)$, $\tau \in (0;T]$. It follows from (58) that

$$\begin{aligned}\int_{\Omega} \Big(u_t w + \sum_{i=1}^{l} \lambda_i(x,y,t) u_{y_i} w + \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} w_{x_j} + (c(t) + b(x,y)) uw + g(x,y,t,u) w \Big)\, dx \\ = \int_{\Omega} \big(f_1(x,y,t) q(x) + f_2(x,y,t) \big) w\, dx\end{aligned} \tag{59}$$

for almost all $(y,t) \in D \times (0;T)$ and for all $w \in W_0^{1,2}(\Omega)$. From (59) we derive that u for almost all $(y,t) \in D \times (0;T)$ is a weak solution of the Dirichlet problem for the elliptic equation

$$\sum_{i,j=1}^{n} \left(a_{ij}(x) u_{x_i} \right)_{x_j} = F(x,y,t), \quad x \in \Omega, \quad u|_{\partial\Omega} = 0, \tag{60}$$

where

$$F(x,y,t) = f_1(x,y,t)q(x) + f_2(x,y,t) - u_t - \sum_{i=1}^{l} \lambda_i(x,y,t)u_{y_i} - (c(t)+b(x,y))u - g(x,y,t,u).$$

Since condition (3) is satisfied and $F(\cdot,y,t) \in L^2(\Omega)$ for almost all $(y,t) \in D \times (0;T)$, it follows from Theorem 7.3 in [9, p. 130], that there exists the unique weak solution u of the problem (60) and $u_{x_ix_j}(\cdot,y,t) \in L^2(\Omega)$, hence, $u(\cdot,y,t) \in W_0^{2,2}(\Omega)$ for almost all $(y,t) \in D \times (0;T)$. Hence, $u \in V_3(Q_T) \cap C([0,T];L^2(G))$, the triple $(u(x,y,t), c(t), q(x))$ satisfies (1) for almost all $(x,y,t) \in Q_T$, and by virtue of Lemma 3 $(u(x,y,t), c(t), q(x))$ is a solution of the problem (1)–(5) in Q_T. □

Theorem 3. *Assume that the hypotheses of Theorem 2 are satisfied. Then the problem* (1)–(5) *has at most one solution.*

Proof. Assume that $(u_{(1)}(x,y,t), c_{(1)}(t), q_{(1)}(x))$ and $(u_{(2)}(x,y,t), c_{(2)}(t), q_{(2)}(x))$ are two solutions of the problem (1)–(5). Then the triple of functions $(\tilde{u}(x,y,t), \tilde{c}(t), \tilde{q}(x))$, where $\tilde{u}(x,y,t) = u_{(1)}(x,y,t) - u_{(2)}(x,y,t)$, $\tilde{c}(t) = c_{(1)}(t) - c_{(2)}(t)$, $\tilde{q}(x) = q_{(1)}(x) - q_{(2)}(x)$, satisfies the condition $\tilde{u}(x,y,0) \equiv 0$, as well as the equality

$$\int_{Q_\tau} \Big(\tilde{u}_t v + \sum_{i=1}^{l} \lambda_i(x,y,t)\tilde{u}_{y_i} v + \sum_{i,j=1}^{n} a_{ij}(x)\tilde{u}_{x_i} v_{x_j} + b(x,y)\tilde{u}v + (c_{(1)}(t)u_{(1)} - c_{(2)}(t)u_{(2)})v$$
$$+(g(x,y,t,u_{(1)}) - g(x,y,t,u_{(2)}))v \Big)\, dx\, dy\, dt = \int_{Q_\tau} f_1(x,y,t)\tilde{q}(x) v\, dx\, dy\, dt, \quad \tau \in [0,T], \tag{61}$$

for all $v \in V_1(Q_T)$ and the system of equalities

$$\begin{aligned} \tilde{c}(t) &= \int_G \frac{K_1(x,y)f_1(x,y,t)\tilde{q}(x) + B_1(x,y,t)\tilde{u}}{E_1(t)}\, dx\, dy \\ &\quad - \int_G \frac{K_1(x,y)(g(x,y,t,u_{(1)}) - g(x,y,t,u_{(2)}))}{E_1(t)} dxdy, \quad t \in [0,T]; \\ \tilde{q}(x) &= \int_{\Pi_T} \Big(\frac{K_2(y,t)}{F_1(x)}(c_{(1)}(t)u_{(1)} - c_{(2)}(t)u_{(2)}) + \frac{B_2(x,y,t)}{F_1(x)}\tilde{u} \\ &\quad + \frac{K_2(y,t)}{F_1(x)}(g(x,y,t,u_{(1)}) - g(x,y,t,u_{(1)})) \Big)\, dy\, dt, \quad x \in \Omega, \end{aligned} \tag{62}$$

holds. After choosing $v = \tilde{u}$ in (61) we get

$$\int_{Q_\tau} \Big(\tilde{u}_t \tilde{u} + \sum_{i=1}^{l} \lambda_i(x,y,t)\tilde{u}_{y_i}\tilde{u} + \sum_{i,j=1}^{n} a_{ij}(x)\tilde{u}_{x_i}\tilde{u}_{x_j} + (c_{(1)}(t)u_{(1)} - c_{(2)}(t)u_{(2)})\tilde{u} + b(x,y)(\tilde{u})^2$$
$$+(g(x,y,t,u_{(1)}) - g(x,y,t,u_{(2)}))\tilde{u} \Big)\, dx\, dy\, dt = \int_{Q_\tau} f_1(x,y,t)\tilde{q}(x)\tilde{u}\, dx\, dy\, dt, \quad \tau \in (0;T]. \tag{63}$$

It is easy to get from (62) and (H5) the inequalities

$$\int_0^T (\tilde{c}(t))^2\, dt + \int_\Omega (\tilde{q}(x))^2\, dx \le M_9 \int_{Q_T} (\tilde{u})^2\, dx\, dy\, dt. \tag{64}$$

From (63) by the same way as from (45) we got (50), we find the following estimate

$$\int_{Q_T} (\tilde{u})^2 \, dx \, dy \, dt \leq M_{11} \left(\int_0^T (\tilde{c}(t))^2 \, dt + \int_\Omega (\tilde{q}(x))^2 \, dx \right) \tag{65}$$

and taking into account (64) from (65) we obtain

$$(1 - M_{12}) \int_{Q_T} (\tilde{u})^2 \, dx \, dy \, dt \leq 0.$$

Since $M_{12} < 1$, we conclude that $\int_{Q_T} (\tilde{u})^2 \, dx \, dy \, dt = 0$, hence, $u_{(1)} = u_{(2)}$ in Q_T. Then (64) implies $\tilde{c}(t) \equiv 0$, $\tilde{q}(x) \equiv 0$, and, therefore, $c_{(1)}(t) \equiv c_{(2)}(t)$, $q_{(1)}(x) \equiv q_{(2)}(x)$ in Q_T. □

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Процах Н.П., Флюд В.М. *Визначення невідомих функцій різних аргументів в молодшому члені та правій частині слабко нелінійного ультрапараболічного рівняння* // Карпатські матем. публ. — 2020. — Т.12, №2. — C. 317–332.

У статті розглянуто обернену задачу для слабко нелінійного ультрапараболічного рівняння. Рівняння містить дві невідомі функції різних аргументів в молодшому члені та в правій частині. Отримано достатні умови існування та єдиності розв'язку на інтервалі $[0, T]$, де T залежить від коефіцієнтів рівняння.

Ключові слова і фрази: обернена задача, ультрапараболічне рівняння, мішана задача, однозначна розв'язність.