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GROWTH ESTIMATES FOR THE MAXIMAL TERM AND CENTRAL EXPONENT OF THE DERIVATIVE OF A DIRICHLET SERIES

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Let $A \in (-\infty, +\infty]$, $\Phi : [a, A) \to \mathbb{R}$ be a continuous function such that $x\sigma - \Phi(\sigma) \to -\infty$ as $\sigma \uparrow A$ for every $x \in \mathbb{R}$, $\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma) : \sigma \in [a, A)\}$ be the Young-conjugate function of Φ, $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ and $\Gamma(x) = (\widetilde{\Phi}(x) - \ln x)/x$ for all sufficiently large x, (λ_n) be a nonnegative sequence increasing to $+\infty$, and $F(s) = \sum\limits_{n=0}^{\infty} a_n e^{s\lambda_n}$ be a Dirichlet series such that its maximal term $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geq 0\}$ and central index $\nu(\sigma, F) = \max\{n \geq 0 : |a_n|e^{\sigma\lambda_n} = \mu(\sigma, F)\}$ are defined for all $\sigma < A$. It is proved that if $\ln \mu(\sigma, F) \leq (1 + o(1))\Phi(\sigma)$ as $\sigma \uparrow A$, then the inequalities

$$\varlimsup_{\sigma\uparrow A} \frac{\mu(\sigma, F')}{\mu(\sigma, F)\overline{\Phi}^{-1}(\sigma)} \leq 1, \qquad \varlimsup_{\sigma\uparrow A} \frac{\lambda_{\nu(\sigma,F')}}{\Gamma^{-1}(\sigma)} \leq 1,$$

hold, and these inequalities are sharp.

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INTRODUCTION

We fix a nonnegative sequence (λ_n) increasing to $+\infty$, and consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}. \tag{1}$$

For this series, by $\sigma_a(F)$ we denote its abscissa of absolute convergence. Put

$$\beta(F) = \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}, \tag{2}$$

and let

$$E_1(F) = \left\{\sigma \in \mathbb{R} : \ |a_n|e^{\sigma\lambda_n} = o(1), n \to \infty\right\},$$
$$E_2(F) = \left\{\sigma \in \mathbb{R} : \ |a_n|e^{\sigma\lambda_n} = O(1), n \to \infty\right\}.$$

It is easy to see that for $j = 1, 2$ we have

$$\beta(F) = \begin{cases} -\infty, & \text{if} \quad E_j(F) = \varnothing, \\ \sup E_j(F), & \text{if} \quad E_j(F) \neq \varnothing, \end{cases}$$

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i.e., the interval $(-\infty, \beta(F))$ is the domain of existence for the maximal term

$$\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geq 0\}$$

of series (1). Since $\beta(F') = \beta(F)$, this interval is also the domain of existence for the maximal term of the derivative of series (1).

It is well known (for instance, see [8, pp. 114–115]) that for every Dirichlet series of the form (1) we have

$$\sigma_a(F) \leq \beta(F) \leq \sigma_a(F) + \tau, \quad \tau := \overline{\lim_{n\to\infty}} \frac{\ln n}{\lambda_n}, \tag{3}$$

and these inequalities are sharp. Moreover, it was shown in [4] that for any $A, B \in [-\infty, +\infty]$ such that $A \leq B \leq A + \tau$ there exists a Dirichlet series of the form (1) for which $\sigma_a(F) = A$ and $\beta(F) = B$.

We assume that every Dirichlet series of the form (1) considered below is not reduced to a constant, that is, for this series we have $a_n\lambda_n \neq 0$ for at least one integer $n \geq 0$. By this assumption, the central index

$$\nu(\sigma, F) = \max\{n \geq 0 : |a_n|e^{\sigma\lambda_n} = \mu(\sigma, F)\}$$

of series (1) and the central index of the derivative of this series are defined for all $\sigma < \beta(F)$.

Let $A \in (-\infty, +\infty]$, and $\Phi : D_\Phi \to \mathbb{R}$ be a real function. We say that $\Phi \in \Omega_A$ if the domain D_Φ of Φ is an interval of the form $[a, A)$, Φ is continuous on D_Φ, and the following condition

$$\forall x \in \mathbb{R} : \quad \lim_{\sigma\uparrow A}(x\sigma - \Phi(\sigma)) = -\infty \tag{4}$$

holds. It is easy to see that in the case $A < +\infty$ condition (4) is equivalent to the condition $\Phi(\sigma) \to +\infty$, $\sigma \to A - 0$, and in the case $A = +\infty$ this condition is equivalent to the condition $\Phi(\sigma)/\sigma \to +\infty$, $\sigma \to +\infty$. For $\Phi \in \Omega_A$ by $\widetilde{\Phi}$ we denote the Young-conjugate function of Φ, i.e.,

$$\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma) : \sigma \in D_\Phi\}, \quad x \in \mathbb{R}.$$

Note (see Lemma 1 below), that the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ is continuous and increasing to A on some interval of the form $(x_0, +\infty)$. Hence the inverse function $\overline{\Phi}^{\,-1}$ is defined on some interval of the form (A_0, A) and $\overline{\Phi}^{\,-1}$ is continuous and increasing to $+\infty$ on (A_0, A).

We say that $\Phi \in \Omega'_A$, if $\Phi \in \Omega_A$, Φ is continuously differentiable on D_Φ, and Φ' is positive and increasing on D_Φ.

Let $\Phi \in \Omega'_A$. It is clear that $\Phi'(\sigma) \uparrow +\infty$ as $\sigma \uparrow A$. In addition, Φ' has an inverse function $\varphi : [x_0, +\infty) \to D_\Phi$. Set

$$\widehat{\Phi}(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}, \quad \sigma \in D_\Phi.$$

It is easy to prove that $\overline{\Phi}(x) = \widehat{\Phi}(\varphi(x))$ for every $x \in (x_0, +\infty)$. This implies that $\Phi'(\widehat{\Phi}^{-1}(\sigma)) = \overline{\Phi}^{\,-1}(\sigma)$ for all $\sigma \in (A_0, A)$.

M.M. Sheremeta [9] proved the following two theorems.

Theorem A. *Suppose that $A \in (-\infty, +\infty]$, $\Phi \in \Omega'_A$, and the condition*

$$\ln \Phi'(\sigma) = o(\Phi(\sigma)), \quad \sigma \uparrow A, \tag{5}$$

holds. Then for every Dirichlet series of the form (1) *such that* $\sigma_a(F) = A$ *and*

$$\varlimsup_{\sigma\uparrow A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)} = 1 \tag{6}$$

we have

$$\varlimsup_{\sigma\uparrow A} \frac{\mu(\sigma, F')}{\mu(\sigma, F)\overline{\Phi}^{-1}(\sigma)} \le 1. \tag{7}$$

Theorem B. *Suppose that* $A \in (-\infty, +\infty]$, $\Phi \in \Omega'_A$, *there exists a number* $\alpha \in (0, 1]$ *such that the function* $h(\sigma) = (\Phi'(\sigma))^{\alpha}/\Phi(\sigma)$ *is nonincreasing on* $[\sigma_0, A)$, *and* $\lambda_n = o(\lambda_{n+1})$ *as* $n \to +\infty$. *If*

$$F(s) = \sum_{n=0}^{\infty} e^{-\widetilde{\Phi}(\lambda_n)} e^{s\lambda_n}, \tag{8}$$

then

$$\varlimsup_{\sigma\uparrow A} \frac{\mu(\sigma, F')}{\mu(\sigma, F)\overline{\Phi}^{-1}(\sigma)} = 1. \tag{9}$$

Remark 1. *Clearly, if for a Dirichlet series of the form* (1) *with* $\sigma_a(F) = A$ *equality* (6) *holds, then for this series we have* $\beta(F) = A$.

Remark 2. *It can be proved that for series* (8) *by the conditions of Theorem B relation* (6) *holds (this is also clear from considerations given in [9]).*

Remark 3. *In the proofs of Theorems A and B suggested in [9], the obvious inequalities*

$$\lambda_{\nu(\sigma,F)} \le \frac{\mu(\sigma, F')}{\mu(\sigma, F)} \le \lambda_{\nu(\sigma,F')}, \quad \sigma < \beta(F), \tag{10}$$

were used, and, in fact, the following more exactly results were proved: by the conditions of Theorem A for every Dirichlet series of the form (1) *the inequality*

$$\varlimsup_{\sigma\uparrow A} \frac{\lambda_{\nu(\sigma,F')}}{\overline{\Phi}^{-1}(\sigma)} \le 1 \tag{11}$$

holds, and by the conditions of Theorem B for series (8) *we have*

$$\varlimsup_{\sigma\uparrow A} \frac{\lambda_{\nu(\sigma,F)}}{\overline{\Phi}^{-1}(\sigma)} = 1. \tag{12}$$

Therefore, for every Dirichlet series of the form (1) with $\sigma_a(F) = A$, by some conditions on a function $\Phi \in \Omega'_A$, equality (6) implies estimates (7) and (11), and these estimates are sharp.

In [9], M.M. Sheremeta conjectured that in Theorem A condition (5) may be unnecessary, that is, Theorem A is true without any additional condition on a function $\Phi \in \Omega'_A$. Below we confirm this conjecture. Moreover, we prove that inequality (7) is sharp in the case of an arbitrary function $\Phi \in \Omega'_A$. In addition, in the case of an arbitrary $\Phi \in \Omega'_A$ we obtain a sharp growth estimate for the central exponent $\lambda_{\nu(\sigma,F')}$ of the derivative of a Dirichlet series, which, generally, does not coincide with estimate (11).

1 MAIN RESULTS

Let $A \in (-\infty, +\infty]$. For a Dirichlet series of the form (1) with $\beta(F) = A$ and a function $\Phi \in \Omega_A$ we put

$$t_\Phi(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)}.$$

Setting $\overline{\Phi}^{\,-1}(\sigma) = +\infty$ for all $\sigma \in [A, +\infty]$, we have

$$t_\Phi(F) = \varlimsup_{n\to\infty} \frac{\lambda_n}{\overline{\Phi}^{\,-1}\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right)} \tag{13}$$

(see [7] and also [5]).

The following theorem confirms the above conjecture of M.M. Sheremeta.

Theorem 1. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$.*

(i) For every Dirichlet series of the form (1) *with $\beta(F) = A$ and $t_\Phi(F) \le 1$ we have* (7).

(ii) There exists a Dirichlet series of the form (1) *with $\beta(F) = A$ and $t_\Phi(F) = 1$ such that equality* (9) *holds.*

Let $\Phi \in \Omega_A$. Since $\overline{\Phi}$ is continuous and increasing to A on some interval of the form $(x_0, +\infty)$, there exists $\alpha > e$ such that the function

$$\Gamma(x) = \overline{\Phi}(x) - \frac{\ln x}{x}, \quad x \in [\alpha, +\infty), \tag{14}$$

is continuous and increasing to A. Hence the inverse function Γ^{-1} is defined on some interval of the form $[A_1, A)$ and Γ^{-1} is continuous and increasing to $+\infty$ on $[A_1, A)$.

Theorem 2. *Suppose that $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and Γ is defined by* (14).

(i) For every Dirichlet series of the form (1) *with $\beta(F) = A$ and $t_\Phi(F) \le 1$ we have*

$$\varlimsup_{\sigma\uparrow A} \frac{\lambda_{\nu(\sigma,F')}}{\Gamma^{-1}(\sigma)} \le 1.$$

(ii) There exists a Dirichlet series of the form (1) *with $\beta(F) = A$ and $t_\Phi(F) = 1$ such that*

$$\varlimsup_{\sigma\uparrow A} \frac{\lambda_{\nu(\sigma,F')}}{\Gamma^{-1}(\sigma)} = 1. \tag{15}$$

Using Theorem 2, we show that without additional conditions on a function $\Phi \in \Omega'_A$ estimate (11) may not be satisfied for some Dirichlet series of the form (1) with $\sigma_a(F) = A$ such that (6) holds. Indeed, let $\Phi(\sigma) = -\ln|\sigma|$ for all $\sigma \in [-1, 0)$. It is easy to make sure that

$$\overline{\Phi}^{\,-1}(\sigma) \sim \frac{1}{|\sigma|}\ln\frac{1}{|\sigma|}, \qquad \Gamma^{-1}(\sigma) \sim \frac{2}{|\sigma|}\ln\frac{1}{|\sigma|} \qquad \text{as } \sigma\uparrow A.$$

By Theorem 2 there exists a Dirichlet series of the form (1) with $\beta(F) = 0$ and $t_\Phi(F) = 1$ such that equality (15) holds, that is

$$\varlimsup_{\sigma\uparrow 0} \frac{\lambda_{\nu(\sigma,F')}}{\frac{1}{|\sigma|}\ln\frac{1}{|\sigma|}} = 2.$$

Suppose that $\ln n = o(\lambda_n)$ as $n \to \infty$. Then by (3) we have $\sigma_a(F) = 0$. Estimate (11) takes the form

$$\varlimsup_{\sigma\uparrow 0} \frac{\lambda_{\nu(\sigma,F')}}{\frac{1}{|\sigma|}\ln\frac{1}{|\sigma|}} \le 1$$

and, obviously, this estimate is false.

Theorems 1 and 2 are consequences of the following two theorems.

Theorem 3. *Let $A \in (-\infty, +\infty]$ and $\Phi \in \Omega_A$. For every Dirichlet series of the form* (1) *such that $\beta(F) = A$ and*

$$\ln \mu(\sigma, F) \le \Phi(\sigma), \quad \sigma \in [\sigma_1, A), \tag{16}$$

we have

$$\frac{\mu(\sigma, F')}{\mu(\sigma, F)} \le \overline{\Phi}^{-1}(\sigma), \quad \sigma \in [\sigma_2, A).$$

Theorem 4. *Let $A \in (-\infty, +\infty]$ and $\Phi \in \Omega_A$. There exists a Dirichlet series of the form* (1) *such that for an infinite set E of positive integers we have*

$$a_n = \begin{cases} e^{-\widetilde{\Phi}(\lambda_n)}, & \text{if} \quad n \in E, \\ 0, & \text{if} \quad n \notin E, \end{cases}$$

and this series satisfies (12).

Remark 4. *Since for each $\Phi \in \Omega_A$ we have $\widetilde{\Phi}(x)/x = \overline{\Phi}(x) \to A$ as $x \to +\infty$, for a Dirichlet series of the form* (1) *whose existence follows from Theorem 4 we obtain $\beta(F) = A$ by* (2).

Remark 5. *If $\Phi \in \Omega_A$ and a Dirichlet series of the form* (1) *with $\beta(F) = A$ satisfies* (16)*, then, by Theorem 3 and the left of inequalities* (10)*, for all $\sigma \in [\sigma_2, A)$ we obtain $\lambda_{\nu(\sigma,F)} \le \overline{\Phi}^{-1}(\sigma)$. Since $\lambda_{\nu(\sigma,F)} = (\ln \mu(\sigma, F))'_+$ for every $\sigma < \beta(F)$, this fact is easy to prove without using Theorem 3 (see [2, Lemma 5] or [3, Lemma 4]).*

In order to prove Theorems 1, 2, 3 and 4, we will need some auxiliary results, which are given in the next section.

2 AUXILIARY RESULTS

The following lemma is well known (see, for example, [1, § 3.2], [7]).

Lemma 1. *Suppose that $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and, for all $x \in \mathbb{R}$, $\varphi(x) = \max\{\sigma \in D_\Phi : x\sigma - \Phi(\sigma) = \widetilde{\Phi}(x)\}$. Then the following statements are true:*

(i) the function φ is nondecreasing on $\mathbb{R}$;

(ii) the function φ is continuous from the right on $\mathbb{R}$;

(iii) $\varphi(x) \to A$, $x \to +\infty$;

(iv) the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$ at every point $x \in \mathbb{R}$;

(v) if $x_0 = \inf\{x > 0 : \Phi(\varphi(x)) > 0\}$, then the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ increases to A on $(x_0, +\infty)$;

(vi) the function $\alpha(x) = \Phi(\varphi(x))$ *is nondecreasing on* $[0, +\infty)$.

In the following two lemmas, which are proved in [2], φ and x_0 are defined by Φ in the same way as in Lemma 1.

Lemma 2. *Let* $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0 + 0)$, *and* $\sigma \in (\sigma_0, A)$ *be a fixed number. Then the minimum value of the function*

$$h(y) = \frac{\Phi(y)}{y - \sigma}, \quad y \in (\sigma, A),$$

is $\overline{\Phi}^{-1}(\sigma)$ *and this value is attained at the point* $y = \varphi(\overline{\Phi}^{-1}(\sigma))$.

Lemma 3. *Let* $\delta \in (0, 1)$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0 + 0)$, *and* $y(\sigma) = \varphi(\overline{\Phi}^{-1}(\sigma))$ *for all* $\sigma \in (\sigma_0, A)$. *Then*

$$\overline{\Phi}^{-1}\left(\sigma + \frac{\delta\Phi(y(\sigma))}{\overline{\Phi}^{-1}(\sigma)}\right) \leq \frac{\overline{\Phi}^{-1}(\sigma)}{1 - \delta}, \quad \sigma \in (\sigma_0, A).$$

The following lemma is proved in [6].

Lemma 4. *Let* $A \in (-\infty, +\infty]$. *If for a Dirichlet series of the form* (1) *there exists an increasing sequence* $(n_k)_{k=0}^{\infty}$ *of nonnegative integers such that* $a_n = 0$ *for all* $n < n_0$, $a_{n_k} \neq 0$ *for every* $k \geq 0$, *and*

$$\varkappa_k := \frac{\ln|a_{n_k}| - \ln|a_{n_{k+1}}|}{\lambda_{n_{k+1}} - \lambda_{n_k}} \uparrow A, \quad k \uparrow \infty, \qquad |a_n| \leq |a_{n_k}| e^{\varkappa_k(\lambda_{n_k} - \lambda_n)}, \quad n \in (n_k, n_{k+1}),\ k \geq 0,$$

then $\beta(F) = A$ *and, in addition,* $\nu(\sigma, F) = n_0$ *for every* $\sigma < \varkappa_0$ *and* $\nu(\sigma, F) = n_{k+1}$ *for all* $\sigma \in [\varkappa_k, \varkappa_{k+1})$ *and* $k \geq 0$.

Lemma 5. *Suppose that* h *is a function increasing on* $[\alpha, \beta)$, $h(\alpha) = a$, $\lim_{x\uparrow\beta} h(x) = b$, *and*

$$h^{-1}(\sigma) := \inf\{x \in [\alpha, \beta) : h(x) > \sigma\}, \quad \sigma \in [a, b).$$

Then the following statements are true:

(i) h^{-1} *is nondecreasing continuous on* $[a, b)$;

(ii) $h^{-1}(a) = \alpha$, $\lim_{\sigma\uparrow b} h^{-1}(\sigma) = \beta$;

(iii) $h(x + 0) = \max\{\sigma \in [a, b) : h^{-1}(\sigma) \leq x\}$ *for each* $x \in [\alpha, \beta)$.

Proof. Let

$$E(\sigma) = \{x \in [\alpha, \beta) : h(x) > \sigma\}, \quad \sigma \in [a, b).$$

If $\sigma_1, \sigma_2 \in [a, b)$ and $\sigma_1 < \sigma_2$, then $E(\sigma_2) \subset E(\sigma_1)$, and hence

$$h^{-1}(\sigma_1) = \inf E(\sigma_1) \leq \inf E(\sigma_2) = h^{-1}(\sigma_2).$$

Therefore, h^{-1} is nondecreasing on $[a, b)$.

If $x \in [\alpha, \beta)$ and $h(x) = \sigma$, then $h^{-1}(\sigma) = x$, i.e., the interval $[\alpha, \beta)$ is the range of h^{-1}. This and the monotonicity of the function h^{-1} imply its continuity, as well as both equalities in (ii).

Let us prove (iii). Let $x_0 \in [\alpha, \beta)$ and $\sigma_0 = \max\{\sigma \in [a, b) : h^{-1}(\sigma) \leq x_0\}$. Then $h^{-1}(\sigma_0) = x_0$. Therefore, if $x \in (x_0, \beta)$, then $h(x) > \sigma_0$, and hence $h(x_0 + 0) \geq \sigma_0$. Suppose that $h(x_0 + 0) = \sigma_3 > \sigma_0$. Then $h(x) > \sigma_3$ for all $x \in (x_0, \beta)$, that is, $(x_0, \beta) \subset E(\sigma_3)$. Thus

$$h^{-1}(\sigma_3) = \inf E(\sigma_3) \leq x_0.$$

This and the definition of σ_0 imply that $\sigma_3 \leq \sigma_0$, which contradicts the assumption that $h(x_0 + 0) > \sigma_0$. Hence, $h(x_0 + 0) = \sigma_0$. □

3 PROOF OF THEOREMS

Proof of Theorem 3. Suppose that $A \in (-\infty, +\infty]$ and $\Phi \in \Omega_A$. Consider a Dirichlet series of the form (1) with $\beta(F) = A$ which satisfies (16).

Let σ_0 be defined as in Lemma 2, and $\varphi(x) = \widetilde{\Phi}'_+(x)$ for all $x \in \mathbb{R}$. Condition (16) implies the existence of a number $\sigma_2 \in (\sigma_0, A)$ such that

$$\max\{1, \ln \mu(y, F) - \ln \mu(\sigma, F)\} \le \Phi(y), \quad y, \sigma \in [\sigma_2, A).$$

By taking here $y = y(\sigma)$, where $y(\sigma) = \varphi(\overline{\Phi}^{-1}(\sigma))$, and using Lemma 2, we get

$$\frac{\ln \mu(y(\sigma), F) - \ln \mu(\sigma, F)}{y(\sigma) - \sigma} \le \overline{\Phi}^{-1}(\sigma), \quad \sigma \in [\sigma_2, A). \tag{17}$$

Fix an arbitrary $\sigma \in [\sigma_2, A)$. If $\lambda_{\nu(\sigma,F')} \le \overline{\Phi}^{-1}(\sigma)$, then

$$\frac{\mu(\sigma, F')}{\mu(\sigma, F)} \le \overline{\Phi}^{-1}(\sigma)$$

by the right of inequalities (10). Therefore, we can further assume that $\lambda_{\nu(\sigma,F')} > \overline{\Phi}^{-1}(\sigma)$.

For every integer $n \ge 0$ we have

$$|a_n| e^{\sigma\lambda_n} = |a_n| e^{y(\sigma)\lambda_n} e^{(\sigma - y(\sigma))\lambda_n} \le \mu(y(\sigma), F) e^{(\sigma - y(\sigma))\lambda_n}.$$

This and (17) imply that

$$\frac{|a_n| e^{\sigma\lambda_n}}{\mu(\sigma, F)} \le e^{(y(\sigma)-\sigma)(\overline{\Phi}^{-1}(\sigma) - \lambda_n)}, \quad n \ge 0. \tag{18}$$

Since $\lambda_{\nu(\sigma,F')} > \overline{\Phi}^{-1}(\sigma)$, from (18) it follows that

$$\frac{\mu(\sigma, F')}{\mu(\sigma, F)} \le \sup_{\lambda_n > \overline{\Phi}^{-1}(\sigma)} \lambda_n e^{(y(\sigma)-\sigma)(\overline{\Phi}^{-1}(\sigma) - \lambda_n)}. \tag{19}$$

Let us consider the function

$$h(t) = t e^{(y(\sigma)-\sigma)(\overline{\Phi}^{-1}(\sigma) - t)}, \quad t \in \mathbb{R}.$$

It is easy to check that this function is descending on the interval $[t_0, +\infty)$, where

$$t_0 = \frac{1}{y(\sigma) - \sigma}.$$

Using Lemma 2, we have

$$t_0 = \frac{\overline{\Phi}^{-1}(\sigma)}{\Phi(y(\sigma))} \le \overline{\Phi}^{-1}(\sigma),$$

and so from (19) it follows that

$$\frac{\mu(\sigma, F')}{\mu(\sigma, F)} \le h(\overline{\Phi}^{-1}(\sigma)) = \overline{\Phi}^{-1}(\sigma).$$

Theorem 3 is proved. □

Proof of Theorem 4. Suppose that $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and prove that there exists a Dirichlet series of the form (1) such that for an infinite set E of positive integers we have $a_n = e^{-\widetilde{\Phi}(\lambda_n)}$ when $n \in E$, $a_n = 0$ when $n \notin E$, and this series satisfies (12).

Let $\varphi(x) = \widetilde{\Phi}'_+(x)$ for all $x \in \mathbb{R}$, and $x_0 = \inf\{x > 0 : \Phi(\varphi(x)) > 0\}$.

Since, by Lemma 1, $\widetilde{\Phi}$ is convex on $\mathbb{R}$, we have

$$\frac{\widetilde{\Phi}(x) - \widetilde{\Phi}(b)}{x - b} \le \varphi(x), \quad x > b. \tag{20}$$

In addition, $\overline{\Phi}$ is increasing on $(x_0, +\infty)$. Therefore, if $x > b > x_0$, then $\overline{\Phi}(x) > \overline{\Phi}(b)$. This implies that

$$\overline{\Phi}(x) < \frac{\widetilde{\Phi}(x) - \widetilde{\Phi}(b)}{x - b}, \quad x > b > x_0. \tag{21}$$

Now we show that

$$\overline{\Phi}(x) - \overline{\Phi}(b) = o(\Phi(\varphi(x))), \quad x \to +\infty. \tag{22}$$

Since, by Lemma 1, $\overline{\Phi}(x) \to A$ and $\Phi(\varphi(x)) \to +\infty$ as $x \to +\infty$, relation (22) is obvious in the case $A < +\infty$. If $A = +\infty$, then we get

$$\overline{\Phi}(x) < \varphi(x) = o(\Phi(\varphi(x)))$$

as $x \to +\infty$, and this also implies (22).

It follows from the above that there exists a sequence (n_k) of positive integers such that we have $\lambda_{n_0} > x_0$ and also

$$\lambda_{n_k} = o(\lambda_{n_{k+1}}), \quad k \to \infty; \tag{23}$$

$$\overline{\Phi}(\lambda_{n_{k+1}}) > \varphi(\lambda_{n_k}), \quad k \ge 0; \tag{24}$$

$$\lambda_{n_k}(\overline{\Phi}(\lambda_{n_{k+1}}) - \overline{\Phi}(\lambda_{n_k})) = o(\Phi(\varphi(\lambda_{n_{k+1}}))), \quad k \to \infty. \tag{25}$$

For each $k \ge 0$ we set

$$\sigma_k = \overline{\Phi}(\lambda_{n_{k+1}}), \quad \varkappa_k = \frac{\widetilde{\Phi}(\lambda_{n_{k+1}}) - \widetilde{\Phi}(\lambda_{n_k})}{\lambda_{n_{k+1}} - \lambda_{n_k}}.$$

Using (21) and (20) with $x = \lambda_{n_{k+1}}$ and $b = \lambda_{n_k}$, as well as (24), we obtain

$$\sigma_k = \overline{\Phi}(\lambda_{n_{k+1}}) < \varkappa_k \le \varphi(\lambda_{n_{k+1}}) < \overline{\Phi}(\lambda_{n_{k+2}}) = \sigma_{k+1}, \quad k \ge 0.$$

This implies that $(\varkappa_k)$ is a sequence increasing to A.

Let $\sigma_0 = \overline{\Phi}(x_0 + 0)$, and $y(\sigma) = \varphi(\overline{\Phi}^{-1}(\sigma))$ for all $\sigma \in (\sigma_0, A)$. Using (23) and (25), we have

$$\varkappa_k = \overline{\Phi}(\lambda_{n_{k+1}}) + \frac{\lambda_{n_k}(\overline{\Phi}(\lambda_{n_{k+1}}) - \overline{\Phi}(\lambda_{n_k}))}{\lambda_{n_{k+1}} - \lambda_{n_k}} = \sigma_k + \frac{o(\Phi(\varphi(\lambda_{n_{k+1}})))}{\lambda_{n_{k+1}}} = \sigma_k + \frac{o(\Phi(y(\sigma_k)))}{\overline{\Phi}^{-1}(\sigma_k)}$$

as $k \to \infty$. From this and from Lemma 3 we see that

$$\overline{\Phi}^{-1}(\varkappa_k) \sim \overline{\Phi}^{-1}(\sigma_k), \quad k \to \infty. \tag{26}$$

Put $a_{n_k} = e^{-\widetilde{\Phi}(\lambda_{n_k})}$ for all $k \ge 0$, and let $a_n = 0$ if $n \ne n_k$ for every $k \ge 0$, i.e., $E = \{n_0, n_1, \dots\}$. Consider series (1) with such coefficients a_n. Since

$$\varkappa_k = \frac{\ln a_{n_k} - \ln a_{n_{k+1}}}{\lambda_{n_{k+1}} - \lambda_{n_k}} \uparrow A, \quad k \to \infty,$$

for this series by Lemma 4 we have $\lambda_{\nu(\varkappa_k,F)} = \lambda_{n_{k+1}}$, $k \geq 0$. Therefore, using (26), we get

$$\varlimsup_{\sigma\uparrow A} \frac{\lambda_{\nu(\sigma,F)}}{\overline{\Phi}^{-1}(\sigma)} \geq \varlimsup_{k\to\infty} \frac{\lambda_{\nu(\varkappa_k,F)}}{\overline{\Phi}^{-1}(\varkappa_k)} = \varlimsup_{k\to\infty} \frac{\lambda_{n_{k+1}}}{\overline{\Phi}^{-1}(\sigma_k)} = 1.$$

Theorem 4 is proved. □

Proof of Theorem 1. Let $A \in (-\infty, +\infty]$ and $\Phi \in \Omega_A$.

(i) Suppose that a Dirichlet series of the form (1) with $\beta(F) = A$ satisfies the condition $t_\Phi(F) \leq 1$. Let $q > 1$ be an arbitrary fixed number, and let $\Psi(\sigma) = q\Phi(\sigma)$ for all $\sigma \in D_\Phi$. Then, as it is easy to see, $\overline{\Psi}^{-1}(\sigma) = q\overline{\Phi}^{-1}(\sigma)$ for each $\sigma \in (A_0, A)$. From the condition $t_\Phi(F) \leq 1$ it follows that $\ln \mu(\sigma, F) \leq \Psi(\sigma)$, $\sigma \in [\sigma_1, A)$. Therefore, by Theorem 3 we have

$$\frac{\mu(\sigma, F')}{\mu(\sigma, F)} \leq \overline{\Psi}^{-1}(\sigma) = q\overline{\Phi}^{-1}(\sigma), \quad \sigma \in [\sigma_2, A).$$

Since $q > 1$ is arbitrary, this implies estimate (7).

(ii) By Theorem 4 there exists a Dirichlet series of the form (1) such that for an infinite set E of nonnegative integers we have $a_n = e^{-\widetilde{\Phi}(\lambda_n)}$ if $n \in E$ and $a_n = 0$ if $n \notin E$, and this series satisfies (12). Then $\beta(F) = A$ (see Remark 4). Using (13), for this series we obtain $t_\Phi(F) = 1$, and hence, by the first part of our theorem, we have (7). From (7) and (12), due to the left of inequalities (10), we immediately obtain (9).

Theorem 1 is proved. □

Proof of Theorem 2. Suppose that $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and Γ is defined by (14). First, let us prove that there exists a function $\Theta \in \Omega_A$ such that $\overline{\Theta}(x) = \Gamma(x)$ for all $x \in [\alpha, +\infty)$.

Let $\varphi(x) = \widetilde{\Phi}'_+(x)$, $x \in \mathbb{R}$. Put

$$\theta(x) = \varphi(x) - \frac{1}{x}, \quad x \in [\alpha, +\infty).$$

Since $\alpha > e$ (see above), the function θ is increasing and continuous from the right on $[\alpha, +\infty)$, and also $\lim_{x\uparrow+\infty} \theta(x) = A$. Consider the function

$$\theta^{-1}(\sigma) = \inf\{x \in [\alpha, +\infty) : \theta(x) > \sigma\}, \quad \sigma \in [a, A),$$

where $a = \theta(\alpha)$. By Lemma 5, the function θ^{-1} is nondecreasing continuous on $[a, A)$, and also

$$\theta(x) = \max\{\sigma \in [a, A) : \theta^{-1}(\sigma) \leq x\}, \quad x \in [\alpha, +\infty).$$

Put

$$\Theta_0(\sigma) = \int_a^\sigma \theta^{-1}(t)dt, \quad \sigma \in [a, A).$$

Let $A < +\infty$, and let $\eta(x) = A - \frac{1}{x}$, $x \in [\alpha, +\infty)$. Since $\varphi(x) < A$ for all $x \in \mathbb{R}$, we have $\theta(x) < \eta(x)$ for each $x \in [\alpha, +\infty)$. Then

$$\theta^{-1}(\sigma) \geq \eta^{-1}(\sigma) = \frac{1}{A - \sigma}, \quad \sigma \in [a, A),$$

and hence for all $\in [a, A)$ we get

$$\Theta_0(\sigma) \geq \int_a^\sigma \frac{dt}{A - \sigma} = \ln \frac{A - a}{A - \sigma}.$$

This implies that $\Theta_0(\sigma) \to +\infty$ as $\sigma \to A-0$. In the case $A=+\infty$, for all sufficiently large σ we have

$$\Theta_0(\sigma) \ge \int_{\sigma/2}^{\sigma} \theta^{-1}(t)dt \ge \frac{\sigma}{2}\theta^{-1}\left(\frac{\sigma}{2}\right).$$

This implies that $\Theta_0(\sigma)/\sigma \to +\infty$ as $\sigma \to \infty$. Therefore, $\Theta_0 \in \Omega_A$ always.

Let $x \in [\alpha, +\infty)$ be an arbitrary fixed number. Consider the function

$$h(\sigma) = x\sigma - \Theta_0(\sigma), \quad \sigma \in [a, A).$$

Since $h'(\sigma) = x - \theta^{-1}(\sigma)$, the function h assumes its maximum value on $[a, A)$ at the point $\sigma = \theta(x)$, and this point is maximal among all possible maximum points of h.

Therefore, from Lemma 1 we can see that for all $x \in [\alpha, +\infty)$ the function $\theta(x)$ is defined by Θ as well as $\varphi(x)$ by Φ, and hence $\theta(x) = \widetilde{\Theta}'_+(x)$. Put $C = -\widetilde{\Phi}(\alpha) + \ln\alpha + \widetilde{\Theta}(\alpha)$ and let $\Theta(\sigma) = \Theta_0(\sigma) + C$ for all $\sigma \in [a, A)$. Then $\Theta \in \Omega_A$ and for every $x \in [\alpha, +\infty)$ we have

$$\begin{aligned}\widetilde{\Theta}(x) = \widetilde{\Theta}_0(x) - C = \int_\alpha^x \theta(t)dt + \widetilde{\Theta}_0(\alpha) - C = \int_\alpha^x \left(\varphi(t) - \frac{1}{t}\right)dt + \widetilde{\Theta}_0(\alpha) - C \\ = \widetilde{\Phi}(x) - \ln x - \widetilde{\Phi}(\alpha) + \ln\alpha + \widetilde{\Theta}_0(\alpha) - C = \widetilde{\Phi}(x) - \ln x = x\Gamma(x),\end{aligned}$$

and hence $\overline{\Theta}(x) = \Gamma(x)$.

(i) Suppose that a Dirichlet series of the form (1) with $\beta(F) = A$ satisfies the condition $t_\Phi(F) \le 1$. Let $q > 1$ be an arbitrary fixed number. Then $t_\Phi(F) < q$, and therefore from (13) for all $n \ge n_1$ we obtain the inequality

$$\lambda_n \le q\overline{\Phi}^{-1}\left(\frac{1}{\lambda_n}\ln\frac{1}{|a_n|}\right),$$

which, as is easy to see, is equivalent to the inequality

$$\ln|a_n| \le -q\widetilde{\Phi}\left(\frac{\lambda_n}{q}\right).$$

Hence, for all $n \ge n_2$ we have

$$\ln|\lambda_n a_n| \le \ln\lambda_n - q\widetilde{\Phi}\left(\frac{\lambda_n}{q}\right) = -q\widetilde{\Theta}\left(\frac{\lambda_n}{q}\right) + (1-q)\ln\lambda_n + q\ln q \le -q\widetilde{\Theta}\left(\frac{\lambda_n}{q}\right),$$

which implies that

$$\lambda_n \le q\overline{\Theta}^{-1}\left(\frac{1}{\lambda_n}\ln\frac{1}{|\lambda_n a_n|}\right).$$

Therefore, using (13) with F' and Θ instead of F and Φ respectively, we obtain $t_\Theta(F') \le q$. Since $q > 1$ is arbitrary, this implies that $t_\Theta(F') \le 1$.

Recalling that $\overline{\Theta}(x) = \Gamma(x)$ for all $x \in [\alpha, +\infty)$, and using Theorem 1 with F' and Θ instead of F and Φ respectively, we have

$$\varlimsup_{\sigma\uparrow A}\frac{\lambda_{\nu(\sigma,F')}}{\Gamma^{-1}(\sigma)} = \varlimsup_{\sigma\uparrow A}\frac{\lambda_{\nu(\sigma,F')}}{\overline{\Theta}^{-1}(\sigma)} \le \varlimsup_{\sigma\uparrow A}\frac{\mu(\sigma, F'')}{\mu(\sigma, F')\overline{\Theta}^{-1}(\sigma)} \le 1.$$

(ii) Since $\Theta \in \Omega_A$, by Theorem 4 there exists a Dirichlet series of the form

$$G(s) = \sum_{n=0}^{\infty} b_n e^{s\lambda_n}$$

such that for an infinite set E of positive integers we have $b_n = e^{-\widetilde{\Theta}(\lambda_n)}$ if $n \in E$ and $b_n = 0$ if $n \notin E$, and this series satisfies the equation

$$\varlimsup_{\sigma\uparrow A} \frac{\lambda_{\nu(\sigma,G)}}{\overline{\Theta}^{-1}(\sigma)} = 1. \tag{27}$$

We note also that $\beta(G) = A$ (see Remark 4).

Put $a_n = e^{-\widetilde{\Phi}(\lambda_n)} = b_n/\lambda_n$ if $n \in E$ and $a_n = 0$ if $n \notin E$, and consider series (1) with such coefficients a_n. For this series we have $F' = G$, and hence $\beta(F) = \beta(G) = A$. By (13) we obtain $t_\Phi(F) = 1$. In addition, for this series equality (15) holds, because this equality coincides with (27). Theorem 2 is proved. ☐

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Нехай $A \in (-\infty, +\infty]$, $\Phi : [a, A) \to \mathbb{R}$ — довільна неперервна функція така, що $x\sigma - \Phi(\sigma) \to -\infty$, $\sigma \uparrow A$, для кожного $x \in \mathbb{R}$, $\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma) : \sigma \in [a, A)\}$ — функція, спряжена з Φ за Юнгом, $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ і $\Gamma(x) = (\widetilde{\Phi}(x) - \ln x)/x$ для всіх достатньо великих x, (λ_n) — невід'ємна зростаюча до $+\infty$ послідовність, а $F(s) = \sum\limits_{n=0}^{\infty} a_n e^{s\lambda_n}$ — ряд Діріхле, максимальний член $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geq 0\}$ та центральний індекс $\nu(\sigma, F) = \max\{n \geq 0 : |a_n|e^{\sigma\lambda_n} = \mu(\sigma, F)\}$ якого визначені для всіх $\sigma < A$. Доведено, що якщо $\ln \mu(\sigma, F) \leq (1 + o(1))\Phi(\sigma)$, $\sigma \uparrow A$, то виконуються нерівності

$$\varlimsup_{\sigma\uparrow A} \frac{\mu(\sigma, F')}{\mu(\sigma, F)\overline{\Phi}^{-1}(\sigma)} \leq 1, \qquad \varlimsup_{\sigma\uparrow A} \frac{\lambda_{\nu(\sigma,F')}}{\Gamma^{-1}(\sigma)} \leq 1,$$

і ці нерівності є точними.

Ключові слова і фрази: ряд Діріхле, максимальний член, центральний індекс, центральний показник, спряжена за Юнгом функція.