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On Hadamard composition of Gelfond-Leont'ev derivatives of entire and analytic functions in the unit disk

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For an entire function and an analytic in the unit disk function the growth of the Hadamard composition of their Gelfond-Leont'ev derivatives is investigated in terms of generalized orders. A relationship between the behaviors of the maximal terms of Hadamard composition of Gelfond-Leont'ev derivatives and of the Gelfond-Leont'ev derivative of Hadamard composition is established.

Key words and phrases: analytic function, Hadamard composition, Gelfond-Leont'ev derivative, maximal term.

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Introduction

For power series

$$f(z)=\sum_{k=0}^{\infty} f_k z^k,\quad g(z)=\sum_{k=0}^{\infty} g_k z^k \tag{1}$$

with the convergence radii $R[f]$ and $R[g]$, the series $(f*g)(z)=\sum\limits_{k=0}^{\infty} f_k g_k z^k$ is called the Hadamard composition. It is well known [1,2], that $R[f*g]\geq R[f]R[g]$.

Obtained by J. Hadamard properties of this composition have the applications [2,3] in the theory of the analytic continuation of the functions represented by power series. We remark also, that singular points of the Hadamard composition are investigated in the article [4].

For $0\leq r<R[f]$, let $M(r,f)=\max\{|f(z)|:|z|=r\}$, $\mu(r,f)=\max\{|f_k|r^k:k\geq 0\}$ be the maximal term and $\nu(r,f)=\max\{k:|f_k|r^k=\mu(r,f)\}$ be the central index of the power expansion of f.

For a power series of the form (1) with the convergence radius $R[f]\in[0,+\infty]$ and a power series $l(z)=\sum\limits_{k=0}^{\infty} l_k z^k$ with the convergence radius $R[l]\in[0,+\infty]$ and coefficients $l_k>0$ for all $k\geq 0$ the power series

$$D_l^{(n)}f(z)=\sum_{k=0}^{\infty}\frac{l_k}{l_{k+n}}f_{k+n}z^k$$

is called [5] Gelfond-Leont'ev derivative of n-th order. If $l(z)=e^z$ then $D_l^{(n)}f(z)=f^{(n)}(z)$ is the usual derivative of n-th order. The properties of Hadamard compositions of Gelfond-Leont'ev derivatives of analytic functions f and g in cases when either $R[f]=R[g]=+\infty$

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or $R[f] = R[g] = 1$ are investigated in [6]. For example, for entire functions f and g it is proved [6], that if

$$0 < \varliminf_{k\to\infty} \frac{l_k}{(k+1)l_{k+1}} \le \varlimsup_{k\to\infty} \frac{l_k}{(k+1)l_{k+1}} < +\infty, \tag{2}$$

then

$$\varlimsup_{r\to+\infty} \frac{1}{\ln r} \ln \frac{\mu(r, D_l^{(n)} f * D_l^{(n)} g)}{\mu(r, D_l^{(n)}(f*g))} = n\varrho[f*g]$$

and

$$\varliminf_{r\to+\infty} \frac{1}{\ln r} \ln \frac{\mu(r, D_l^{(n)} f * D_l^{(n)} g)}{\mu(r, D_l^{(n)}(f*g))} = n\lambda[f*g],$$

where $\varrho[f]$ is the order and $\lambda[f]$ is the lower order of the function f.

If $R[f] = R[g] = R[f*g] = 1$ and (2) holds, then [6]

$$n\varrho^{(1)}[f*g] \le \varlimsup_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln^+ \frac{\mu(r, D_l^{(n)} f * D_l^{(n)} g)}{\mu(r, D_l^{(n)}(f*g))} \le n(\varrho^{(1)}[f*g]+1)$$

and

$$n\lambda^{(1)}[f*g] \le \varliminf_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln^+ \frac{\mu(r, D_l^{(n)} f * D_l^{(n)} g)}{\mu(r, D_l^{(n)}(f*g))} \le n(\lambda^{(1)}[f*g]+1),$$

where $\varrho^{(1)}[f]$ is the order and $\lambda^{(1)}[f]$ is the lower order of the analytic function f in the unit disk.

The question about the similar properties of the Hadamard composition naturally arises for the case $R[f] \ne R[g]$. Here, we restrict ourselves to the case $R[f] = +\infty$ and $R[g] = 1$. We will conduct researches in terms of generalized orders.

1 Analyticity and growth

Since $R[f] = +\infty$, $R[g] = 1$ and $R[f*g] \ge R[f]R[g]$, we have $R[f*g] = +\infty$. In [6], it is proved that for an arbitrary series of the form (1) the equalities $R[f] = +\infty$ and $R[D_l^{(n)} f] = +\infty$ are equivalent if and only if

$$0 < q = \varliminf_{k\to\infty} \sqrt[k]{l_k/l_{k+1}} \le \varlimsup_{k\to\infty} \sqrt[k]{l_k/l_{k+1}} = Q < +\infty. \tag{3}$$

Hence, it follows that $R[D_l^{(n)}(f*g)] = +\infty$ for each $n \ge 0$. Finally, if (3) holds, then $R[D_l^{(n)} f * D_l^{(m)} g] = +\infty$ for each $n \ge 0$ and $m \ge 0$. Indeed,

$$\frac{1}{R[D_l^{(n)} f * D_l^{(m)} g]} = \varlimsup_{k\to\infty} \sqrt[k]{\frac{l_k}{l_{k+n}}|f_{k+n}| \frac{l_k}{l_{k+m}}|g_{k+m}|} \le Q^{n+m} \varlimsup_{k\to\infty} \sqrt[k]{|f_{k+n}|} \varlimsup_{k\to\infty} \sqrt[k]{|g_{k+m}|} = 0.$$

As in [7], let L be a class of continuous nonnegative on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \ge 0$ for $x \le x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \le x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each fixed $c \in (0, +\infty)$, i.e. α is a slowly increasing function. Clearly, $L_{si} \subset L^0$.

For $\alpha \in L$, $\beta \in L$ and an entire transcendental function (1) the quantities

$$\varrho_{\alpha,\beta}[f] := \varrho_{\alpha,\beta}[\ln M, f] = \varlimsup_{r\to+\infty} \frac{\alpha(\ln M(r,f))}{\beta(\ln r)}$$

and

$$\lambda_{\alpha,\beta}[f] := \lambda_{\alpha,\beta}[\ln M, f] = \varliminf_{r\to+\infty} \frac{\alpha(\ln M(r,f))}{\beta(\ln r)}$$

are called the generalized order and the lower generalized order, respectively. If here we substitute $\ln \mu(r,f)$ or $\nu(r,f)$ instead of $\ln M(r,f)$, then we obtain the definitions of the quantities $\varrho_{\alpha,\beta}[\ln \mu, f]$, $\lambda_{\alpha,\beta}[\ln \mu, f]$ and $\varrho_{\alpha,\beta}[\nu, f]$, $\lambda_{\alpha,\beta}[\nu, f]$, respectively.

Lemma 1. *Let $\alpha \in L_{si}$, $\beta \in L^0$ and $\dfrac{d\beta^{-1}(c\alpha(x))}{d\ln x} = O(1)$ as $x \to +\infty$ for each $c \in (0, +\infty)$. Then*

$$\varrho_{\alpha,\beta}[f] = \varlimsup_{k\to\infty} \frac{\alpha(k)}{\beta\left(\dfrac{1}{k}\ln\dfrac{1}{|f_k|}\right)}. \tag{4}$$

If, moreover, $|f_k/f_{k+1}| \nearrow +\infty$ as $k_0 \le k \to \infty$, then

$$\lambda_{\alpha,\beta}[f] = \varliminf_{k\to\infty} \frac{\alpha(k)}{\beta\left(\dfrac{1}{k}\ln\dfrac{1}{|f_k|}\right)}. \tag{5}$$

Formula (4) was proved in [7], and formula (5) follows from the corresponding formula for entire Dirichlet series proved in [8].

Proposition 1. *Let the functions α and β satisfy the conditions of Lemma 1 and* (3) *holds. Then $\varrho_{\alpha,\beta}[f*g] \le \varrho_{\alpha,\beta}[f]$ and under condition*

$$\varliminf_{k\to\infty} \sqrt[k]{|g_k|} = h > 0 \tag{6}$$

the equality $\varrho_{\alpha,\beta}[f*g] = \varrho_{\alpha,\beta}[f]$ holds.

If, moreover, $|f_k|/|f_{k+1}| \nearrow +\infty$ and $|g_k/g_{k+1}| \nearrow 1$ as $k_0 \le k \to \infty$, then $\lambda_{\alpha,\beta}[f*g] \le \lambda_{\alpha,\beta}[f]$ and under condition (6) *the equality $\lambda_{\alpha,\beta}[f*g] = \lambda_{\alpha,\beta}[f]$ holds.*

Proof. Since $\varlimsup\limits_{k\to\infty} \sqrt[k]{|g_k|} = 1/R[g] = 1$, for every $h > 1$ and all $k \ge k_0(h)$ we have $\sqrt[k]{|g_k|} \le h$, that is $(1/k)\ln(1/|g_k|) \ge -\ln h$. Therefore, since $\beta(x+O(1)) = (1+o(1))\beta(x)$ as $x \to +\infty$, by Lemma 1 we get

$$\frac{1}{\varrho_{\alpha,\beta}[f*g]} = \varliminf_{k\to\infty} \frac{1}{\alpha(k)}\beta\left(\frac{1}{k}\ln\frac{1}{|f_k g_k|}\right)$$
$$= \varliminf_{k\to\infty} \frac{1}{\alpha(k)}\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|} + \frac{1}{k}\ln\frac{1}{|g_k|}\right) \ge \varliminf_{k\to\infty} \frac{1}{\alpha(k)}\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|} - \ln h\right) = \frac{1}{\varrho_{\alpha,\beta}[f]},$$

i.e. $\varrho_{\alpha,\beta}[f*g] \le \varrho_{\alpha,\beta}[f]$. If (6) holds, then as above we have

$$\frac{1}{\varrho_{\alpha,\beta}[f*g]} = \varliminf_{k\to\infty} \frac{1}{\alpha(k)}\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|} + \frac{1}{k}\ln\frac{1}{|g_k|}\right) \le \varliminf_{k\to\infty} \frac{1}{\alpha(k)}\beta\left(\frac{1}{k}\ln\frac{1}{|f_k|} - \ln(h/2)\right) = \frac{1}{\varrho_{\alpha,\beta}[f]},$$

i.e. $\varrho_{\alpha,\beta}[f*g] \ge \varrho_{\alpha,\beta}[f]$ and, thus, $\varrho_{\alpha,\beta}[f*g] = \varrho_{\alpha,\beta}[f]$.

The inequality $\lambda_{\alpha,\beta}[f*g] \le \lambda_{\alpha,\beta}[f]$ and the equality $\lambda_{\alpha,\beta}[f*g] = \lambda_{\alpha,\beta}[f]$ can be proved similarly. □

Remark 1. *In general, the equality* $\varrho_{\alpha,\beta}[f*g]=\varrho_{\alpha,\beta}[f]$ *may not hold. Indeed, let for example*

$$f(z)=\sum_{n=0}^{\infty}\exp\{-2n\beta^{-1}(\alpha(2n))\}z^{2n}\qquad \textit{and}\qquad g(z)=\sum_{n=0}^{\infty}z^{2n+1}.$$

Then by Lemma 1 we have $\varrho_{\alpha,\beta}[f*g]=0<1=\varrho_{\alpha,\beta}[f]$.

Lemma 2. *If* $\alpha\in L_{si}$ *and* $\beta\in L^0$*, then* $\varrho_{\alpha,\beta}[f]=\varrho_{\alpha,\beta}[\ln\mu,f]$ *and* $\lambda_{\alpha,\beta}[f]=\lambda_{\alpha,\beta}[\ln\mu,f]$*. If, moreover,* $\alpha(e^x)\in L_{si}$ *and* $\alpha(x)=o(\beta(x))$ *as* $x\to+\infty$*, then* $\varrho_{\alpha,\beta}[\ln\mu,f]=\varrho_{\alpha,\beta}[\nu,f]$ *and* $\lambda_{\alpha,\beta}[\ln\mu,f]=\lambda_{\alpha,\beta}[\nu,f]$.

Proof. In view of the conditions $\alpha\in L_{si}$ and $\beta\in L^0$ the equalities $\varrho_{\alpha,\beta}[f]=\varrho_{\alpha,\beta}[\ln\mu,f]$ and $\lambda_{\alpha,\beta}[f]=\lambda_{\alpha,\beta}[\ln\mu,f]$ follow from the estimates

$$\mu(r,f)\le M(r,f)\le\sum_{k=0}^{\infty}|f_k|r^k=\sum_{k=0}^{\infty}|f_k|(2r)^k2^{-k}\le 2\mu(2r,f).$$

It is well known [9, p. 13] that $\ln\mu(r,f)-\ln\mu(1,f)=\int_1^r\frac{\nu(t,f)}{t}dt$, whence

$$\nu(r/2,f)\ln 2\le\int\limits_{r/2}^{r}\frac{\nu(t,f)}{t}dt\le\ln\mu(r,f)-\ln\mu(1,f)\le\nu(r,f)\ln r, \tag{7}$$

and, therefore, in view of conditions $\alpha(e^x)\in L_{si}$, $\beta\in L^0$ and $\alpha(x)=o(\beta(x))$ as $x\to+\infty$ we have

$$\begin{aligned}(1+o(1))\frac{\alpha(\nu(r,f))}{\beta(\ln r)}&\le(1+o(1))\frac{\alpha(\ln\mu(r,f))}{\beta(\ln r)}\le\frac{\alpha(\exp\{\ln\nu(r,f)+\ln\ln r\})}{\beta(\ln r)}\\&\le\frac{\alpha(\exp\{2\max\{\ln\nu(r,f),\,\ln\ln r\}\})}{\beta(\ln r)}\\&=(1+o(1))\frac{\alpha(\exp\{\max\{\ln\nu(r,f),\,\ln\ln r\}\})}{\beta(\ln r)}\\&=(1+o(1))\frac{\max\{\alpha(\nu(r,f)),\,\alpha(\ln r)\}}{\beta(\ln r)}\le(1+o(1))\frac{\alpha(\nu(r,f))+\alpha(\ln r)}{\beta(\ln r)}\\&=(1+o(1))\frac{\alpha(\nu(r,f))}{\beta(\ln r)}+o(1),\quad r\to+\infty,\end{aligned}$$

and, thus, $\varrho_{\alpha,\beta}[\ln\mu,f]=\varrho_{\alpha,\beta}[\nu,f]$ and $\lambda_{\alpha,\beta}[\ln\mu,f]=\lambda_{\alpha,\beta}[\nu,f]$. The proof of Lemma 2 is completed. □

Remark 2. *From the condition* $\frac{d\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ *as* $x\to+\infty$ *for each* $c\in(0,+\infty)$ *it follows that* $\beta^{-1}(c\alpha(x))\le qx$ *for each* $c\in(0,+\infty)$ *and some* $q\in(0,+\infty)$*, that is* $\alpha(x)\le\beta(qx)/c=(1+o(1))\beta(x)/c$ *as* $x\to+\infty$*. Hence, in view of the arbitrariness of* c *we obtain* $\alpha(x)=o(\beta(x))$ *as* $x\to+\infty$.

The following proposition establishes a connection between the growth of entire function and its Gelfond-Leont'ev derivative.

Proposition 2. *Let $\alpha \in L^0$ and $\beta \in L^0$. If condition* (3) *holds and f is an entire function, then $\varrho_{\alpha,\beta}[D_l^{(n)} f] = \varrho_{\alpha,\beta}[f]$ and $\lambda_{\alpha,\beta}[D_l^{(n)} f] = \lambda_{\alpha,\beta}[f]$ for each $n \geq 1$.*

Proof. It is enough to consider the case $n = 1$. Condition (3) implies the existence of numbers $0 < q_1 \leq q_2 < +\infty$ such that $q_1^k \leq l_k/l_{k+1} \leq q_2^k$ for all $k \geq 0$. Therefore,

$$r\mu(r, D_l^{(1)} f) = \max\left\{\frac{l_k}{l_{k+1}}|f_{k+1}|r^{k+1} : k \geq 0\right\} \leq \frac{1}{q_2}\max\{|f_{k+1}|(q_2 r)^{k+1} : k \geq 0\} \leq \frac{\mu(q_2 r, f)}{q_2}$$

and by analogy $r\mu(r, D_l^{(1)} f) \geq \mu(q_1 r, f)/q_1$ for all r enough large. Since $\ln r = o(\ln \mu(r, f))$ as $r \to +\infty$ for each entire transcendental function, hence we get the asymptotic inequalities

$$(1 + o(1)) \ln \mu(q_1 r, f) \leq \ln \mu(r, D_l^{(1)} f) \leq (1 + o(1)) \ln \mu(q_2 r, f), \quad r \to +\infty,$$

whence by Proposition 1 the validity of Proposition 2 follows easy. □

Corollary 1. *Let $\alpha \in L_{si}$ and $\beta \in L^0$. If $R[f] = +\infty$, $R[g] = 1$ and* (3) *holds, then*

$$\lambda_{\alpha,\beta}[D_l^{(n)} f * D_l^{(n)} g] = \lambda_{\alpha,\beta}[D_l^{(n)}(f * g)] = \lambda_{\alpha,\beta}[f * g]$$

and

$$\varrho_{\alpha,\beta}[D_l^{(n)} f * D_l^{(n)} g] = \varrho_{\alpha,\beta}[D_l^{(n)}(f * g)] = \varrho_{\alpha,\beta}[f * g]$$

for all $n \geq 1$.

Indeed, the equalities $\lambda_{\alpha,\beta}[D_l^{(n)}(f * g)] = \lambda_{\alpha,\beta}[f * g]$ and $\varrho_{\alpha,\beta}[D_l^{(n)}(f * g)] = \varrho_{\alpha,\beta}[f * g]$ follow directly from Proposition 2. To prove the equalities $\lambda[D_l^{(n)} f * D_l^{(n)} g] = \lambda[D_l^{(n)}(f * g)]$ and $\varrho[D_l^{(n)} f * D_l^{(n)} g] = \varrho[D_l^{(n)}(f * g)]$ it is enough to notice that, as at proof of Proposition 2, it is possible to get inequality

$$q_1^n \mu(r, D_l^{(n)}(f * g)) \leq \mu(r, D_l^{(n)} f * D_l^{(n)} g) \leq q_2^n \mu(r, D_l^{(n)}(f * g)),$$

and use Lemma 2.

2 Relationship between the growth of the maximal terms of Hadamard composition of the derivatives and of the derivative of Hadamard composition

We start with two theorems close to the results cited in the introduction.

Theorem 1. *Let $n \in \mathbb{Z}_+$, $\alpha(e^x) \in L_{si}$, $\beta \in L^0$ and $\dfrac{d\beta^{-1}(c\alpha(x))}{d \ln x} = O(1)$ as $x \to +\infty$ for each $c \in (0, +\infty)$. Suppose that $R[f] = +\infty$, $R[g] = 1$ and conditions* (3) *with $q > 1$ and* (6) *hold. Then*

$$\varlimsup_{r \to +\infty} \frac{1}{\beta(\ln r)} \alpha\left(\ln \frac{\mu(r, D_l^{(n)} f * D_l^{(n)} g)}{\mu(r, D_l^{(n)}(f * g))}\right) = \varrho_{\alpha,\beta}[f]. \tag{8}$$

If, moreover, $l_k l_{k+2}/l_{k+1}^2 \nearrow 1$, $|f_k/f_{k+1}| \nearrow +\infty$ and $|g_k/g_{k+1}| \nearrow 1$ as $k_0 \leq k \to \infty$, then

$$\varliminf_{r \to +\infty} \frac{1}{\beta(\ln r)} \alpha\left(\ln \frac{\mu(r, D_l^{(n)} f * D_l^{(n)} g)}{\mu(r, D_l^{(n)}(f * g))}\right) = \lambda_{\alpha,\beta}[f]. \tag{9}$$

Proof. In [6], the following estimates are proved

$$\frac{l_{\nu(r,D_l^{(n)}(f*g))}}{l_{\nu(r,D_l^{(n)}(f*g))+n}} \le \frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))} \le \frac{l_{\nu(r,D_l^{(n)}f*D_l^{(n)}g)}}{l_{\nu(r,D_l^{(n)}f*D_l^{(n)}g)+n}}. \tag{10}$$

Condition (3) with $q > 1$ implies the existence of numbers $1 < q_1 \le q_2 < +\infty$ such that $q_1^{kn} \le l_k/l_{k+n} \le q_2^{kn}$ for all $k \ge k_0$. Therefore, from (10) we get

$$n\nu(r,D_l^{(n)}(f*g))\ln q_1 \le \ln\frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))} \le n\nu(r,D_l^{(n)}f*D_l^{(n)}g)\ln q_2, \quad r\ge r_0, \tag{11}$$

whence in view of the condition $\alpha\in L_{si}$ we obtain

$$\alpha(\nu(r,D_l^{(n)}(f*g))) \le (1+o(1))\alpha\left(\ln\frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}\right) \le \alpha(\nu(r,D_l^{(n)}f*D_l^{(n)}g))$$

as $r\to+\infty$. Thus,

$$\varrho_{\alpha,\beta}[\nu,D_l^{(n)}(f*g)] \le \varlimsup_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}\right) \le \varrho_{\alpha,\beta}[\nu,D_l^{(n)}f*D_l^{(n)}g] \tag{12}$$

and

$$\lambda_{\alpha,\beta}[\nu,D_l^{(n)}(f*g)] \le \varliminf_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}\right) \le \lambda_{\alpha,\beta}[\nu,D_l^{(n)}f*D_l^{(n)}g]. \tag{13}$$

By Lemma 2 we have $\varrho_{\alpha,\beta}[\nu,D_l^{(n)}(f*g)] = \varrho_{\alpha,\beta}[\ln\mu,D_l^{(n)}(f*g)] = \varrho_{\alpha,\beta}[D_l^{(n)}(f*g)]$ and $\varrho_{\alpha,\beta}[\nu,D_l^{(n)}f*D_l^{(n)}g] = \varrho_{\alpha,\beta}[\ln\mu,D_l^{(n)}f*D_l^{(n)}g] = \varrho_{\alpha,\beta}[D_l^{(n)}f*D_l^{(n)}g]$. On the other hand, by Corollary 1 we have $\varrho_{\alpha,\beta}[D_l^{(n)}(f*g)] = \varrho_{\alpha,\beta}[D_l^{(n)}f*D_l^{(n)}g] = \varrho_{\alpha,\beta}[f*g]$. Finally, by Proposition 1 we have $\varrho_{\alpha,\beta}[f*g] = \varrho_{\alpha,\beta}[f]$. Therefore, (12) implies (8). Also, by Lemma 2 we have $\lambda_{\alpha,\beta}[\nu,D_l^{(n)}(f*g)] = \lambda_{\alpha,\beta}[D_l^{(n)}(f*g)]$ and $\lambda_{\alpha,\beta}[\nu,D_l^{(n)}f*D_l^{(n)}g] = \lambda_{\alpha,\beta}[D_l^{(n)}f*D_l^{(n)}g]$. From Corollary 1 and Proposition 1 we obtain $\lambda_{\alpha,\beta}[D_l^{(n)}(f*g)] = \lambda_{\alpha,\beta}[f]$ and, thus, $\lambda_{\alpha,\beta}[\nu,D_l^{(n)}(f*g)] = \lambda_{\alpha,\beta}[f]$.

On the other hand, since $l_k l_{k+2}/l_{k+1}^2 \nearrow 1$ as $k_0\le k\to\infty$, we have

$$\frac{l_k l_{k+n+1}}{l_{k+1}l_{k+n}} = \prod_{j=0}^{n-1}\frac{l_{k+j}l_{k+j+2}}{l_{k+j+1}^2} \nearrow 1, \quad k_0\le k\to\infty,$$

and, thus,

$$\left(\frac{l_k|f_{k+n}|}{l_{k+n}}\right)\Big/\left(\frac{l_{k+1}|f_{k+n+1}|}{l_{k+n+1}}\right) \nearrow +\infty, \quad \left(\frac{l_k|g_{k+n}|}{l_{k+n}}\right)\Big/\left(\frac{l_{k+1}|g_{k+n+1}|}{l_{k+n+1}}\right) \nearrow 1 \tag{14}$$

as $k_0\le k\to\infty$. Therefore, by Propositions 1 and 2, $\lambda_{\alpha,\beta}[D_l^{(n)}f*D_l^{(n)}g] = \lambda_{\alpha,\beta}[D_l^{(n)}f] = \lambda_{\alpha,\beta}[f]$ and, thus, (13) implies (9). The proof of Theorem 1 is completed. ☐

Theorem 2. *Let $R[f]=+\infty$, $R[g]=1$ and (6) holds. Suppose that the functions α and β satisfy the conditions of Theorem 1, $n\in\mathbb{Z}_+$, $m\in\mathbb{Z}_+$ and $m\geq n$. If*

$$l_k/l_{k+1}\asymp k,\quad k\to\infty, \tag{15}$$

then

$$\varlimsup_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}(f*g))}\right)=\varrho_{\alpha,\beta}[f]. \tag{16}$$

If, moreover, $l_kl_{k+2}/l_{k+1}^2\nearrow 1$, $|f_k/f_{k+1}|\nearrow+\infty$ and $|g_k/g_{k+1}|\nearrow 1$ as $k_0\leq k\to\infty$, then

$$\varliminf_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}(f*g))}\right)=\lambda_{\alpha,\beta}[f]. \tag{17}$$

Proof. At first we show that

$$\begin{aligned}\frac{l_{\nu(r,D_l^{(n)}(f*g))+n}}{l_{\nu(r,D_l^{(n)}(f*g))}}\left(\frac{l_{\nu(r,D_l^{(n)}(f*g))+n-m}}{l_{\nu(r,D_l^{(n)}(f*g))+n}}\right)^2&\leq\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}(f*g))}\\&\leq\left(\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}\right)^2\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n}}.\end{aligned} \tag{18}$$

Indeed, from one side,

$$\begin{aligned}\mu(r,D_l^{(n)}(f*g))&=\frac{l_{\nu(r,D_l^{(n)}(f*g))}}{l_{\nu(r,D_l^{(n)}(f*g))+n}}|f_{\nu(r,D_l^{(n)}(f*g))+n}|\,|g_{\nu(r,D_l^{(n)}(f*g))+n}|r^{\nu(r,D_l^{(n)}(f*g))}\\&=\frac{l_{\nu(r,D_l^{(n)}(f*g))}}{l_{\nu(r,D_l^{(n)}(f*g))+n}}\left(\frac{l_{\nu(r,D_l^{(n)}(f*g))+n-m+m}}{l_{\nu(r,D_l^{(n)}(f*g))+n-m}}\right)^2\left(\frac{l_{\nu(r,D_l^{(n)}(f*g))+n-m}}{l_{\nu(r,D_l^{(n)}(f*g))+n-m+m}}\right)^2\\&\quad\times|f_{\nu(r,D_l^{(n)}(f*g))+n-m+m}|\,|g_{\nu(r,D_l^{(n)}(f*g))+n-m+m}|\,r^{\nu(r,D_l^{(n)}(f*g))+n-m}r^{m-n}\\&\leq\frac{l_{\nu(r,D_l^{(n)}(f*g))}}{l_{\nu(r,D_l^{(n)}(f*g))+n}}\left(\frac{l_{\nu(r,D_l^{(n)}(f*g))+n}}{l_{\nu(r,D_l^{(n)}(f*g))+n-m}}\right)^2\mu(r,D_l^{(m)}f*D_l^{(m)}g)r^{m-n}\end{aligned}$$

and, on the other hand,

$$\begin{aligned}&\mu(r,D_l^{(m)}f*D_l^{(m)}g)\\&\quad=\left(\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}\right)^2|f_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}||g_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}|r^{\nu(r,D_l^{(m)}f*D_l^{(m)}g)}\\&\quad=\left(\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}\right)^2\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n}}\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n+n}}\\&\quad\quad\times|f_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n+n}||g_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n+n}|\,r^{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n}r^{n-m}\\&\quad\leq\left(\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}\right)^2\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n}}\mu(r,D_l^{(n)}(f*g))r^{n-m},\end{aligned}$$

whence (18) follows.

Condition (15) implies the existence of numbers $0 < h_1 \le h_2 < +\infty$ such that

$$\frac{l_{\nu(r,D_l^{(n)}(f*g))+n}}{l_{\nu(r,D_l^{(n)}(f*g))}}\left(\frac{l_{\nu(r,D_l^{(n)}(f*g))+n-m}}{l_{\nu(r,D_l^{(n)}(f*g))+n}}\right)^2 \ge h_1\nu(r,D_l^{(n)}(f*g))^{2m-n}$$

and

$$\left(\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}\right)^2\frac{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m}}{l_{\nu(r,D_l^{(m)}f*D_l^{(m)}g)+m-n}} \le h_2\nu(r,D_l^{(m)}f*D_l^{(m)}g)^{2m-n}$$

and, therefore, (18) implies

$$h_1\nu(r,D_l^{(n)}(f*g))^{2m-n} \le \frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}(f*g))} \le h_2\nu(r,D_l^{(m)}f*D_l^{(m)}g)^{2m-n}. \tag{19}$$

Since $\alpha(e^x) \in L_{si}$, we have

$$\alpha(h_1\nu(r,D_l^{(n)}(f*g))^{2m-n}) = \alpha(\exp\{(2m-n)\ln\nu(r,D_l^{(n)}(f*g)) + \ln h_1\})$$
$$= (1+o(1))\alpha(\exp\{\ln\nu(r,D_l^{(n)}(f*g))\}) = (1+o(1))\alpha(\ln\nu(r,D_l^{(n)}(f*g)))$$

and similarly $\alpha(h_2\nu(r,D_l^{(m)}f*D_l^{(m)}g)^{2m-n}) = (1+o(1))\alpha(\nu(r,D_l^{(m)}f*D_l^{(m)}g))$ as $r\to+\infty$. Therefore, from (18) we obtain

$$\varrho_{\alpha,\beta}[\nu,D_l^{(n)}(f*g)] \le \varlimsup_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}(f*g))}\right) \le \varrho_{\alpha,\beta}[\nu,D_l^{(m)}f*D_l^{(m)}g] \tag{20}$$

and

$$\lambda_{\alpha,\beta}[\nu,D_l^{(n)}(f*g)] \le \varliminf_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}(f*g))}\right) \le \lambda_{\alpha,\beta}[\nu,D_l^{(m)}f*D_l^{(m)}g]. \tag{21}$$

From (20) and (21) as in the proof of Theorem 1 we get (16) and (17). Theorem 2 is proved. □

The following theorem indicates the relationship between

$$\mu(r,D_l^{(m)}(f*g)) \quad \text{and} \quad \mu(r,D_l^{(n)}(f*g)).$$

Theorem 3. *Let $R[f]=+\infty$, $R[g]=1$ and (6) holds. Suppose that the functions α and β satisfy the conditions of Theorem 1, $n\in\mathbb{Z}_+$, $m\in\mathbb{N}$ and $m>n$. If (3) holds with $q>1$, then*

$$\varlimsup_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{r^{m-n}\mu(r,D_l^{(m)}(f*g))}{\mu(r,D_l^{(n)}(f*g))}\right) = \varrho_{\alpha,\beta}[f] \tag{22}$$

and if, moreover, $|f_k/f_{k+1}|\nearrow+\infty$, $|g_k/g_{k+1}|\nearrow 1$ as $k_0\le k\to\infty$, then

$$\varliminf_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\ln\frac{r^{m-n}\mu(r,D_l^{(m)}(f*g))}{\mu(r,D_l^{(n)}(f*g))}\right) = \lambda_{\alpha,\beta}[f]. \tag{23}$$

If (15) *holds, then*

$$\overline{\lim_{r\to+\infty}} \frac{1}{\beta(\ln\, r)}\alpha\left(\frac{r^{m-n}\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}\right)=\varrho_{\alpha,\beta}[f] \tag{24}$$

and if, moreover, $|f_k/f_{k+1}|\nearrow+\infty$, $|g_k/g_{k+1}|\nearrow 1$ *as* $k_0\le k\to\infty$, *then*

$$\underline{\lim_{r\to+\infty}} \frac{1}{\beta(\ln\, r)}\alpha\left(\frac{r^{m-n}\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}\right)=\lambda_{\alpha,\beta}[f]. \tag{25}$$

Proof. Since [6]

$$\frac{l_{\nu(r,D_l^{(n)}(f*g))+n-m}}{l_{\nu(r,D_l^{(n)}(f*g))}}\le\frac{r^{m-n}\mu(r,D_l^{(m)}(f*g))}{\mu(r,D_l^{(n)}(f*g))}\le\frac{l_{\nu(r,D_l^{(m)}(f*g))}}{l_{\nu(r,D_l^{(m)}(f*g))+m-n}}, \tag{26}$$

if (3) holds with $q>1$, then as in the proof of Theorem 1 we have

$$\begin{aligned}(m-n)\nu(r,D_l^{(n)}(f*g))\ln\, q_1\le\ln\frac{r^{m-n}\mu(r,D_l^{(m)}(f*g))}{\mu(r,D_l^{(n)}(f*g))}&\\ \le(m-n)\nu(r,D_l^{(m)}(f*g))\ln\, q_2,\qquad 1<q_1<q_2<+\infty,&\end{aligned} \tag{27}$$

whence as above we get (22) and (23).

If (15) holds, then from (26) it follows that

$$h_1\nu(r,D_l^{(n)}(f*g))^{m-n}\le\frac{r^{m-n}\mu(r,D_l^{(m)}(f*g))}{\mu(r,D_l^{(n)}(f*g))}\le h_2\nu(r,D_l^{(m)}(f*g))^{m-n}. \tag{28}$$

The further proof of Theorem 3 is the same as the proof of Theorem 1. We only note that since $\nu(r,D_l^{(m)}f*D_l^{(m)}g)$ stands on the right-hand side of inequality (12) we need conditions (14), that is, the condition $l_kl_{k+2}/l_{k+1}^2\nearrow 1$ $(k\to\infty)$. Now this condition is not necessary, because on the right-hand side of (28) the term $\nu(r,D_l^{(m)}(f*g))$ stands and, therefore, from (28) we obtain (24) and (25). Theorem 3 is proved. □

Finally, the following theorem indicates the relationship between $\mu(r,D_l^{(m)}f*D_l^{(m)}g)$ and $\mu(r,D_l^{(n)}f*D_l^{(n)}g)$.

Theorem 4. *Let* $R[f]=+\infty$, $R[g]=1$ *and* (6) *holds. Suppose that the functions* α *and* β *satisfy the conditions of Theorem 1,* $n\in\mathbb{Z}_+$, $m\in\mathbb{N}$ *and* $m>n$. *If* (3) *holds with* $q>1$, *then*

$$\overline{\lim_{r\to+\infty}} \frac{1}{\beta(\ln\, r)}\alpha\left(\ln\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}\right)=\varrho_{\alpha,\beta}[f]$$

and if, moreover, $|f_k/f_{k+1}|\nearrow+\infty$, $|g_k/g_{k+1}|\nearrow 1$ *as* $k_0\le k\to\infty$, *then*

$$\underline{\lim_{r\to+\infty}} \frac{1}{\beta(\ln\, r)}\alpha\left(\ln\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}\right)=\lambda_{\alpha,\beta}[f].$$

If (15) *holds, then*

$$\overline{\lim_{r\to+\infty}}\frac{1}{\beta(\ln r)}\alpha\left(\sqrt{\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}}\right)=\varrho_{\alpha,\beta}[f]$$

and if, moreover, $|f_k/f_{k+1}|\nearrow+\infty$, $|g_k/g_{k+1}|\nearrow 1$ *as* $k_0\le k\to\infty$, *then*

$$\underline{\lim}_{r\to+\infty}\frac{1}{\beta(\ln r)}\alpha\left(\sqrt{\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}}\right)=\lambda_{\alpha,\beta}[f].$$

Indeed, since [6]

$$\frac{l_{\nu(r,D_l^{(n)}(f*g))+n-m}}{l_{\nu(r,D_l^{(n)}(f*g))}}\le\sqrt{\frac{r^{m-n}\mu(r,D_l^{(m)}f*D_l^{(m)}g)}{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}}\le\frac{l_{\nu(r,D_l^{(m)}(f*g))}}{l_{\nu(r,D_l^{(m)}(f*g))+m-n}},$$

the proof of this theorem is the same as the proof of Theorem 3.

3 Addition

Choosing $\alpha(x)=\ln^+ x$ and $\beta(x)=x^+$, from the definitions of $\varrho_{\alpha,\beta}[f]$ and $\lambda_{\alpha,\beta}[f]$ for entire function (1) we get the definitions of the order $\varrho[f]=\overline{\lim}_{r\to+\infty}\frac{\ln\ln M(r,f)}{\ln r}$ and the lower order $\lambda[f]=\underline{\lim}_{r+\infty}\frac{\ln\ln M(r,f)}{\ln r}$. The selected functions α and β satisfy all the conditions of Theorems 1–4 except of the condition $\alpha(e^x)\in L_{si}$ that arose as a result of the applying Lemma 2. But from (7) it follows, that $\varrho[\nu,f]=\varrho[\ln\mu,f]$, $\lambda[\nu,f]=\lambda[\ln\mu,f]$ and, thus, $\varrho[\nu,f]=\varrho[f]$, $\lambda[\nu,f]=\lambda[f]$. Therefore, from (11) we get

$$\varrho[D_l^{(n)}(f*g)]\le\overline{\lim_{r\to+\infty}}\frac{1}{\ln r}\ln\ln\frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}\le\varrho[D_l^{(n)}f*D_l^{(n)}g],$$

$$\lambda[D_l^{(n)}(f*g)]\le\underline{\lim}_{r\to+\infty}\frac{1}{\ln r}\ln\ln\frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}\le\lambda[D_l^{(n)}f*D_l^{(n)}g]$$

and repeating the proof of Theorem 1 we arrive at the following statement.

Proposition 3. *Let* $n\in\mathbb{Z}_+$, $R[f]=+\infty$, $R[g]=1$ *and conditions* (6) *and* (3) *with* $q>1$ *hold. Then*

$$\overline{\lim_{r\to+\infty}}\frac{1}{\ln r}\ln\ln\frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}=\varrho[f]$$

and if, moreover, $l_kl_{k+2}/l_{k+1}^2\nearrow 1$, $|f_k/f_{k+1}|\nearrow+\infty$ *and* $|g_k/g_{k+1}|\nearrow 1$ *as* $k_0\le k\to\infty$, *then*

$$\underline{\lim}_{r\to+\infty}\frac{1}{\ln r}\ln\ln\frac{\mu(r,D_l^{(n)}f*D_l^{(n)}g)}{\mu(r,D_l^{(n)}(f*g))}=\lambda[f].$$

If (15) holds, then from (19) we get

$$(2m-n)\varrho[D_l^{(n)}(f*g)] \le \overline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(m)}f * D_l^{(m)}g)}{\mu(r, D_l^{(n)}(f*g))} \le (2m-n)\varrho[D_l^{(m)}(f*g)],$$

$$(2m-n)\lambda[D_l^{(n)}(f*g)] \le \underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(m)}f * D_l^{(m)}g)}{\mu(r, D_l^{(n)}(f*g))} \le (2m-n)\lambda[D_l^{(m)}(f*g)]$$

and repeating the proof of Theorem 2 we arrive at the following statement.

Proposition 4. *Let $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $m > n$, $R[f] = +\infty$, $R[g] = 1$ and conditions (15) and (6) hold. Then*

$$\overline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(m)}f * D_l^{(m)}g)}{\mu(r, D_l^{(n)}(f*g))} = (2m-n)\varrho[f].$$

If, moreover, $l_k l_{k+2}/l_{k+1}^2 \nearrow 1$, $|f_k/f_{k+1}| \nearrow +\infty$ and $|g_k/g_{k+1}| \nearrow 1$ as $k_0 \le k \to \infty$, then

$$\underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(m)}f * D_l^{(m)}g)}{\mu(r, D_l^{(n)}(f*g))} = (2m-n)\lambda[f].$$

Using (27) and (28) similarly we prove the following statement.

Proposition 5. *Let $n \in \mathbb{Z}_+$, $m \in \mathbb{Z}_+$, $m > n$, $R[f] = +\infty$, $R[g] = 1$ and condition (6) holds. If (3) with $q > 1$ holds, then*

$$\overline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln \ln \frac{r^{m-n}\mu(r, D_l^{(m)}(f*g))}{\mu(r, D_l^{(n)}(f*g))} = \varrho[f]$$

and if, moreover, $|f_k/f_{k+1}| \nearrow +\infty$ and $|g_k/g_{k+1}| \nearrow 1$ as $k_0 \le k \to \infty$, then

$$\underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \ln \frac{r^{m-n}\mu(r, D_l^{(m)}(f*g))}{\mu(r, D_l^{(n)}(f*g))} = \lambda[f].$$

If (15) holds, then

$$\overline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(n)}f * D_l^{(n)}g)}{\mu(r, D_l^{(n)}(f*g))} = (m-n)\varrho[f],$$

and if, moreover, $|f_k/f_{k+1}| \nearrow +\infty$ and $|g_k/g_{k+1}| \nearrow 1$ as $k_0 \le k \to \infty$, then

$$\underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(n)}f * D_l^{(n)}g)}{\mu(r, D_l^{(n)}(f*g))} = (m-n)\lambda[f].$$

Finally, the following statement also is true.

Proposition 6. *Let $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $m > n$, $R[f] = +\infty$, $R[g] = 1$ and condition (6) holds. If (3) with $q > 1$ holds, then*

$$\overline{\lim_{r\to+\infty}} \frac{1}{\ln r} \ln \ln \frac{\mu(r, D_l^{(m)}(f*g))}{\mu(r, D_l^{(n)}(f*g))} = \varrho[f]$$

and if, moreover, $|f_k/f_{k+1}| \nearrow +\infty$ *and* $|g_k/g_{k+1}| \nearrow 1$ *as* $k_0 \le k \to \infty$, *then*

$$\underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r, D_l^{(m)}(f*g))}{\mu(r, D_l^{(n)}(f*g))} = \lambda[f].$$

If (15) *holds, then*

$$\overline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(n)}f * D_l^{(n)}g)}{\mu(r, D_l^{(n)}(f*g))} = 2(m-n)\varrho[f]$$

and if, moreover, $|f_k/f_{k+1}| \nearrow +\infty$ *and* $|g_k/g_{k+1}| \nearrow 1$ *as* $k_0 \le k \to \infty$, *then*

$$\underline{\lim}_{r\to+\infty} \frac{1}{\ln r} \ln \frac{r^{m-n}\mu(r, D_l^{(n)}f * D_l^{(n)}g)}{\mu(r, D_l^{(n)}(f*g))} = 2(m-n)\lambda[f].$$

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Для цілої функції і аналітичної в одиничному крузі функції у термінах узагальнених порядків досліджено зростання адамарової композиції їх похідних Гельфонда-Леонтьєва. Встановлено зв'язок між поводженнями максимальних членів адамарової композиції похідних Гельфонда-Леонтьєва та похідної Гельфонда-Леонтьєва адамарової композиції.

Ключові слова і фрази: аналітична функція, адамарова композиція, похідна Гельфонда-Леонтьєва, максимальний член.