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EXTREME AND EXPOSED SYMMETRIC BILINEAR FORMS ON THE SPACE $\mathcal{L}_s({}^2l_\infty^2)$

KIM SUNG GUEN

We classify extreme points and exposed points of the unit ball of the space of bilinear symmetric forms on the real Banach space of bilinear symmetric forms on l_∞^2. It is shown that for this case, the set of extreme points is equal to the set of exposed points.

Key words and phrases: extreme point, exposed point.

Department of Mathematics, Kyungpook National University, 41566, Daegu, South Korea
E-mail: `sgk317@knu.ac.kr`

INTRODUCTION

Throughout the paper, we let $n \in \mathbb{N}, n \geq 2$. We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. We denote by $\operatorname{ext} B_E$ the set of all extreme points of B_E. An element $x \in B_E$ is called an *exposed point* of B_E if there is a functional $f \in E^*$ such that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $\exp B_E$ the set of exposed points of B_E. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}({}^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}({}^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \ldots, x_n)|$. $\mathcal{L}_s({}^nE)$ denotes the closed subspace of all continuous symmetric n-linear forms on E. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer to [8].

Let us introduce the history of classification problems of extreme and exposed points of the unit ball of continuous n-homogeneous polynomials on a Banach space. We let $l_p^n = \mathbb{R}^n$ for every $1 \leq p \leq \infty$ equipped with the l_p-norm. Choi et al. ([3, 4]) initiated and classified $\operatorname{ext} B_{\mathcal{P}({}^2l_p^2)}$ for $p = 1, 2$. Choi and Kim [7] classified $\operatorname{ext} B_{\mathcal{P}({}^2l_p^2)}$ for $p = 1, 2, \infty$. Later, B. Grecu [12] classified the sets $\operatorname{ext} B_{\mathcal{P}({}^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. Kim et al. [37] showed that if E is a separable real Hilbert space with $\dim(E) \geq 2$, then, $\operatorname{ext} B_{\mathcal{P}({}^2E)}$ is equal to $\exp B_{\mathcal{P}({}^2E)}$. Kim [16] classified $\exp B_{\mathcal{P}({}^2l_p^2)}$ for every $1 \leq p \leq \infty$. Kim [18] characterized $\operatorname{ext} B_{\mathcal{P}({}^2d_*(1,w)^2)}$, where $d_*(1,w)^2$ denotes $\mathbb{R}^2$ equipped with the octagonal norm

$$\|(x,y)\|_w = \max\left\{|x|, |y|, \frac{|x|+|y|}{1+w}\right\}$$

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for $0 < w < 1$. Kim [25] classified $\exp B_{\mathcal{P}(^2d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}(^2d_*(1,w)^2)}$ is a proper subset of $\text{ext}\, B_{\mathcal{P}(^2d_*(1,w)^2)}$. Recently, Kim ([30, 33]) classified $\text{ext}\, B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ and $\exp B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$, where $\mathbb{R}^2_{h(\frac{1}{2})}$ denotes $\mathbb{R}^2$ endowed with a hexagonal norm

$$\|(x,y)\|_{h(\frac{1}{2})} = \max\left\{|y|, |x| + \frac{1}{2}|y|\right\}.$$

Parallel to the classification problems of $\text{ext}B_{\mathcal{P}(^nE)}$ and $\text{exp}B_{\mathcal{P}(^nE)}$, it seems to be very natural to study the classification problems of extreme and exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space. Kim [17] initiated and classified $\text{ext}\, B_{\mathcal{L}_s(^2l_\infty^2)}$ and $\exp B_{\mathcal{L}_s(^2l_\infty^2)}$. Kim ([19, 21, 22, 24]) classified $\text{ext}\, B_{\mathcal{L}_s(^2d_*(1,w)^2)}$, $\text{ext}\, B_{\mathcal{L}(^2d_*(1,w)^2)}$, $\exp B_{\mathcal{L}_s(^2d_*(1,w)^2)}$, and $\exp B_{\mathcal{L}(^2d_*(1,w)^2)}$. Kim ([28, 29]) also classified $\text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$ and $\text{ext}\, B_{\mathcal{L}_s(^3l_\infty^2)}$. It was shown that $\text{ext}\, B_{\mathcal{L}_s(^2l_\infty^3)}$ and $\text{ext}\, B_{\mathcal{L}_s(^3l_\infty^2)}$ are equal to $\exp B_{\mathcal{L}_s(^2l_\infty^3)}$ and $\exp B_{\mathcal{L}_s(^3l_\infty^2)}$, respectively. Kim [32] classified $\text{ext}\, B_{\mathcal{L}(^2l_\infty^n)}$ and $\text{ext}\, B_{\mathcal{L}_s(^2l_\infty^n)}$. Kim [34] characterized $\text{ext}\, B_{\mathcal{L}(^nl_\infty^2)}$, $\text{ext}\, B_{\mathcal{L}(^nl_\infty^2)}$, $\text{ext}\, B_{\mathcal{L}_s(^nl_\infty^2)}$, $\exp B_{\mathcal{L}_s(^nl_\infty^2)}$ and showed that $\exp B_{\mathcal{L}(^nl_\infty^2)}$ and $\exp B_{\mathcal{L}_s(^nl_\infty^2)}$ are equal to $\text{ext}\, B_{\mathcal{L}(^nl_\infty^2)}$ and $\text{ext}\, B_{\mathcal{L}_s(^nl_\infty^2)}$, respectively. Recently, Kim [35] characterized for $m \geq 2$, $\text{ext}\, B_{\mathcal{L}(^nl_\infty^m)}$, $\text{ext}\, B_{\mathcal{L}(^nl_\infty^m)}$, $\text{ext}\, B_{\mathcal{L}_s(^nl_\infty^m)}$, $\exp B_{\mathcal{L}_s(^nl_\infty^m)}$ and showed that $\exp B_{\mathcal{L}(^nl_\infty^m)}$ and $\exp B_{\mathcal{L}_s(^nl_\infty^m)}$ are equal to $\text{ext}\, B_{\mathcal{L}(^nl_\infty^m)}$ and $\text{ext}\, B_{\mathcal{L}_s(^nl_\infty^m)}$, respectively.

We refer to [1,2,5,6,9–11,13–15,20,23,26,27,31,36,38–47] for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces.

In this paper, we classify $\text{ext}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$ and $\exp B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$. It is shown that

$$\text{ext}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))} = \exp B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}.$$

1 RESULTS

Throughout the paper, $\mathbb{R}^6_{\mathcal{L}_s(^2l_\infty^2)}$ denotes $\mathbb{R}^6$ with the $\mathcal{L}_s(^2l_\infty^2)$-norm

$$\begin{aligned}\Big\|(a,b,c,d,e,f)\Big\|_{\mathcal{L}_s(^2l_\infty^2)} :&= \max\Big\{|a|,\ |b|,\ |d|,\ \frac{1}{2}\Big(|a-d|+|e|\Big),\ \frac{1}{2}\Big(|b-d|+|f|\Big),\\ &\quad \frac{1}{4}\Big(|a+b-2d|+|c|\Big),\ \frac{1}{4}\Big|\ |a+b-2d|-|c|\ \Big|+\frac{1}{2}|e-f|\Big\}.\end{aligned}$$

Notice that if $(a,b,c,d,e,f) \in \mathbb{R}^6_{\mathcal{L}_s(^2l_\infty^2)}$ with $\Big\|(a,b,c,d,e,f)\Big\|_{\mathcal{L}_s(^2l_\infty^2)} = 1$, then $|a| \leq 1$, $|b| \leq 1$, $|d| \leq 1$, $|c| \leq 4$, $|e| \leq 2$, $|f| \leq 2$. Notice that

$$\begin{aligned}\Big\|(a,b,c,d,e,f)\Big\|_{\mathcal{L}_s(^2l_\infty^2)} &= \Big\|(b,a,c,d,f,e)\Big\|_{\mathcal{L}_s(^2l_\infty^2)} = \Big\|(a,b,-c,d,e,f)\Big\|_{\mathcal{L}_s(^2l_\infty^2)}\\ &= \Big\|(a,b,c,d,-e,-f)\Big\|_{\mathcal{L}_s(^2l_\infty^2)} = \Big\|(-a,-b,c,-d,e,f)\Big\|_{\mathcal{L}_s(^2l_\infty^2)}.\end{aligned}$$

Therefore, without loss of generality we may assume that $a \geq |b|,\ c \geq 0$ and $e \geq 0$.

In [36] it was shown that the space $\mathbb{R}^6_{\mathcal{L}_s(^2l_\infty^2)}$ is isometrically isomorphic to the space $\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))$.

Theorem 1. *Let* $(a,b,c,d,e,f) \in \mathbb{R}^6$. *Then, the following statements are equivalent:*

(1) $(a,b,c,d,e,f) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}}$;

(2) $(b,a,c,d,f,e) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}}$;

(3) $(a,b,-c,d,e,f) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}}$;

(4) $(a,b,c,d,-e,-f) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}}$;

(5) $(-a,-b,c,-d,e,f) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}}$.

Proof. It is obvious. □

Lemma 1. *Let* $a,b \in \mathbb{R}$ *be such that* $|a|+|b|=1$. *Then the following are equivalent:*

(1) $(|a|=1, b=0)$ *or* $(a=0, |b|=1)$;

(2) *if* $\varepsilon, \delta \in \mathbb{R}$ *satisfies* $|a+\varepsilon|+|b+\delta| \leq 1$ *and* $|a-\varepsilon|+|b-\delta| \leq 1$, *then* $\varepsilon=\delta=0$.

Proof. By symmetry, we may assume that $|a| \geq |b|$.

(1) $\Rightarrow$ *(2).* Suppose that $|a|=1, b=0$ and let $\varepsilon,\delta \in \mathbb{R}$ be such that $|a+\varepsilon|+|b+\delta| \leq 1$ and $|a-\varepsilon|+|b-\delta| \leq 1$. Then $|a+\varepsilon|+|\delta| \leq 1$ and $|a-\varepsilon|+|\delta| \leq 1$, which shows that $1 \geq |a|+|\varepsilon|+|\delta| = 1+|\varepsilon|+|\delta|$. Therefore, $\varepsilon=\delta=0$.

(2) $\Rightarrow$ *(1).* Assume otherwise. Then $0<|b| \leq |a| < 1$. Let $t>0$ be such that $t|a|<|b|$. Let $\varepsilon := t|a|\text{sign}(a)$ and $\delta := -t|a|\text{sign}(b)$. Notice that $\varepsilon \neq 0$ and $\delta \neq 0$. It follows that

$$|a+\varepsilon|+|b+\delta| = (|a|+t|a|)+(|b|-t|a|) = |a|+|b| = 1$$

and

$$|a-\varepsilon|+|b-\delta| = (|a|-t|a|)+(|b|+t|a|) = |a|+|b| = 1.$$

This is a contradiction. Therefore, *(2)* $\Rightarrow$ *(1)* is true. □

We are in position to classify the extreme points of $B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}}$.

Theorem 2.

$$\begin{aligned}\text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}} = \Big\{ &\pm(1,1,\pm4,1,2,2),\ \pm(1,1,\pm4,1,-2,-2),\ \pm(1,-1,\pm4,0,1,1),\\ &\pm(1,-1,\pm4,0,-1,-1),\ \pm(1,-1,\pm2,1,2,0),\ \pm(1,-1,\pm2,1,-2,0),\\ &\pm(1,-1,\pm2,-1,0,2),\ \pm(1,-1,\pm2,-1,0,-2),\ \pm(1,1,\pm2,0,1,-1),\\ &\pm(1,1,\pm2,0,-1,1),\ \pm(1,1,0,1,\pm2,0),\ \pm(1,1,0,1,0,\pm2),\\ &\pm(1,1,0,-1,0,0)\Big\}.\end{aligned}$$

Proof. Let $T=(a,b,c,d,e,f) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}}$. Without loss of generality we may assume that $a \geq |b|$, $c \geq 0$ and $e \geq 0$.

Claim: $a=1$.

Assume otherwise. Then, $a<1$. We claim that $|d|<1$. Assume that $|d|=1$. Since $T=(a,b,c,d,e,f) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2l^2_\infty)}}$, by Lemma 1,

$$\frac{1}{2}\Big(|a-d|+e\Big) = \frac{1}{2}\Big(|b-d|+|f|\Big) = \frac{1}{4}\Big(|a+b-2d|+c\Big) = 1,\ |a+b-2d|=c,\ |e-f|=2.$$

Hence, $c = 2$. Since $2 = |2d| = 2 + a + b \geq 2$, $a + b = 0$, so $a = b = 0$. Hence,

$$1 = \frac{1}{2}\Big(|a-d| + e\Big) = \frac{1}{2}(1+e),\ 1 = \frac{1}{2}\Big(|b-d| + |f|\Big) = \frac{1}{2}(1+|f|),$$

which shows that $e = |f| = 1$. Since $|e - f| = 2$, $e = -f = 1$. Hence, $T = (0, 0, 2, \pm 1, 1, -1)$. We will show that T is not extreme. Notice that for $n \in \mathbb{N}$,

$$(0,0,2,1,1,-1) = \frac{1}{2}\Big(\big(\frac{1}{n}, -\frac{1}{n}, 2, 1, 1+\frac{1}{n}, -1+\frac{1}{n}\big) + \big(-\frac{1}{n}, +\frac{1}{n}, 2, 1, 1-\frac{1}{n}, -1-\frac{1}{n}\big)\Big)$$

and $\|(\pm\frac{1}{n}, \mp\frac{1}{n}, 2, 1, 1 \pm \frac{1}{n}, -1 \pm \frac{1}{n})\|_{\mathcal{L}_s({}^2l_\infty^2)} = 1$. Notice that for $n \in \mathbb{N}$,

$$(0,0,2,-1,1,-1) = \frac{1}{2}\Big(\big(\frac{1}{n}, -\frac{1}{n}, 2, -1, 1-\frac{1}{n}, -1-\frac{1}{n}\big) + \big(-\frac{1}{n}, +\frac{1}{n}, 2, -1, 1+\frac{1}{n}, -1+\frac{1}{n}\big)\Big)$$

and $\|(\pm\frac{1}{n}, \mp\frac{1}{n}, 2, -1, 1 \mp \frac{1}{n}, -1 \mp \frac{1}{n})\|_{\mathcal{L}_s({}^2l_\infty^2)} = 1$. This is a contradiction. Therefore, $|d| < 1$. Since $|b| \leq a < 1, |d| < 1$, choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \min\{1 - a,\ 1 - |d|\}.$$

Then,

$$\Big\|\Big(a \pm \frac{1}{N}, b \pm \frac{1}{N}, c, d \pm \frac{1}{N}, e, f\Big)\Big\|_{\mathcal{L}_s({}^2l_\infty^2)} = 1$$

and

$$T = \frac{1}{2}\Big(\Big(a + \frac{1}{N}, b + \frac{1}{N}, c, d + \frac{1}{N}, e, f\Big) + \Big(a - \frac{1}{N}, b - \frac{1}{N}, c, d - \frac{1}{N}, e, f\Big)\Big),$$

which shows that T is not extreme. This is a contradiction. Therefore, the claim holds.

Claim: $c = 0$ or 2 or 4.

Assume otherwise. Then, $0 < c < 2$ or $2 < c < 4$. We will reach to a contradiction.

Suppose that $0 < c < 2$. Let $|d| < 1$. Notice that if $b = 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1-d+e) = \frac{1}{2}(1-d+|f|) = \frac{1}{4}(2-2d+c) = \frac{1}{4}|2-2d+c| + \frac{1}{2}|e-f|,$$

so, $d = 0$ and $c = 2 + d = 2$, which is a contradiction. Notice that if $b = -1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1-d+e) = \frac{1}{2}(1+d+|f|) = \frac{1}{4}(2|d|+c) = \frac{1}{4}|2|d|-c| + \frac{1}{2}|e-f|,$$

so, $c = 4 - 2|d| > 2$, which is a contradiction. Let $|b| < 1$. Notice that if $\frac{1}{2}(1 - d + e) = 1$, then, by Lemma 1,

$$b - d = 0,\ |f| = 2,\ |1 + b - 2d| = c,\ |e - f| = 2,$$

which shows that $e = 0$ and $d = -1$, which is a contradiction. Let us note that if $\frac{1}{4}(|1 + b - 2d| + c) = 1$, then, by Lemma 1,

$$b - d = 0,\ |f| = 2,\ |1 + b - 2d| = c,\ |e - f| = 2,$$

which shows that $c = 2$, which is a contradiction. Let us note that if $\frac{1}{2}(1 - d + e) = \frac{1}{4}(|1 + b - 2d| + c) = 1$, then, by Lemma 1,

$$b - d = 0,\ |f| = 2,\ \frac{1}{4}||1 + b - 2d| - c| + \frac{1}{2}|e - f| = 1,$$

which shows that $c = 3 + d > 2$, which is a contradiction. Suppose that $\frac{1}{2}(1 - d + e) = \frac{1}{4}(|1 + b - 2d| + c) = 1$. If $b - d = 0$, $|f| = 2$, $\frac{1}{4}||1 + b - 2d| - c| + \frac{1}{2}|e - f| = 1$, then $c = 3 + d > 2$, which is a contradiction. If $|1 + b - 2d| = c$, $|e - f| = 2$, $\frac{1}{2}(1 - d + |f|) = 1$, then $c = 2$, which is a contradiction.

Let $d = 1$. Suppose $e < 2$. If $|b| < 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1 - b + |f|) = \frac{1}{4}(1 - b + c),\ 1 - b - c = 0,\ |e - f| = 2,$$

which shows that $c = 3 + b > 2$, which is a contradiction. If $b = 1$, then, by Lemma 1,

$$|f| = 2,\ \frac{1}{4}c + \frac{1}{2}|e - f| = 1,$$

so $T = (1, 1, c, 1, \frac{1}{2}c, 2)$ or $(1, 1, c, 1, -\frac{1}{2}c, -2)$ for $0 < c < 2$. Hence, T is not extreme. This is a contradiction. If $b = -1$, then, by Lemma 1,

$$f = 0,\ \frac{1}{4}(2 - c) + \frac{1}{2}|e - f| = 1,$$

which shows that $e = 2 + \frac{1}{2}c$. Hence, $c = 0$, which is a contradiction. Suppose $e = 2$. If $|b| < 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}(1 - b + c)$$

or

$$1 = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}|1 - b - c| + \frac{1}{2}(2 - f)$$

or

$$1 = \frac{1}{4}(1 - b + c) = \frac{1}{4}|1 - b - c| + \frac{1}{2}(2 - f).$$

If

$$1 = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}(1 - b + c) \quad \text{or} \quad 1 = \frac{1}{4}(1 - b + c) = \frac{1}{4}|1 - b - c| + \frac{1}{2}(2 - f),$$

then $c = 3 + b > 2$, which is a contradiction. If

$$1 = \frac{1}{2}(1 - d + |f|) = \frac{1}{4}|1 - b - c| + \frac{1}{2}(2 - f),$$

then $T = (1, b, -(1 + 3b), 1, 2, 1 + b)$ for $-1 < b < -\frac{1}{3}$. Hence, T is not extreme. This is a contradiction. If $b = 1$, then $f = 2$ or $\frac{1}{4}c + \frac{1}{2}(2 - f) = 1$. If $f = 2$, then $T = (1, 1, c, 1, 2, 2)$ for $0 < c < 2$. Hence, T is not extreme. This is a contradiction. If $\frac{1}{4}c + \frac{1}{2}(2 - f) = 1$, then $T = (1, 1, c, 1, 2, \frac{1}{2}c)$ for $0 < c < 2$. Hence, T is not extreme. This is a contradiction. If $b = -1$, then $f = 0$ and $c \geq 2$, which is a contradiction. Let $d = -1$. If $|b| < 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1 + b + |f|) = \frac{1}{4}(3 + b + c)$$

or

$$1 = \frac{1}{2}(1 + d + |f|) = \frac{1}{4}(3 + b - c) + \frac{1}{2}|f|$$

or

$$1 = \frac{1}{4}(3 + b + c) = \frac{1}{4}(3 + b - c) + \frac{1}{2}|f|.$$

Hence, $T = (1, b,\ 1-b,\ -1, 0,\ \pm(1-b))$ for $-1 < b < 1$. Hence, T is not extreme. This is a contradiction. If $b = 1$, then $f = 0$ and $1 \geq \frac{1}{4}(|a+b-2d| + c) = 1 + \frac{c}{4}$. Hence, $c = 0$, which is a contradiction. If $b = -1$, then $f = 0$ and $\frac{1}{4}(2-c) + \frac{1}{2}|f| = 1$. Hence, $T = (1, -1,\ c,\ -1, 0,\ \pm(1+\frac{c}{2}))$ for $0 < c < 2$. Hence, T is not extreme. This is a contradiction. We have shown that if $0 < c < 2$, then T is not extreme.

Suppose that $2 < c < 4$. Let $|d| < 1$. If $|b| < 1$, then, by Lemma 1,

$$b - d,\ |f| = 2,\ |1+b-2d| = c,\ |e-f| = 2.$$

If $\frac{1}{2}(1-d+2) = 1$, then $e = 0$ and $d = -1$, which is a contradiction. If $\frac{1}{4}(|1+b-2d| + c) = 1$, then $c = 1 - d < 2$, which is a contradiction. If $b = -1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1-d+e) = \frac{1}{2}(1+d+|f|) = \frac{1}{4}(2d+c) = \frac{1}{4}|2d-c| + \frac{1}{2}|e-f|.$$

Hence, $T = (1, -1, c,\ 2 - \frac{1}{2}c,\ 3 - \frac{1}{2}c,\ -1 + \frac{1}{2}c)$ for $2 < c < 4$. Hence, T is not extreme. This is a contradiction. If $b = 1$, then

$$1 = \frac{1}{2}(1-d+e) = \frac{1}{2}(1-d+|f|) = \frac{1}{4}(2-2d+c) = \frac{1}{4}|2-2d-c| + \frac{1}{2}|e-f|.$$

Hence, $d = \frac{c-2}{2}$, $e = \frac{1}{2}c = |f|$. If $f = \frac{1}{2}c$, then $1 = \frac{1}{4}|2-2d-c| = \frac{c}{2} - 1$, so $c = 4$. This is a contradiction. If $f = -\frac{1}{2}c$, then $1 = c - 1$, so $c = 2$. This is a contradiction. Let $|d| = 1$. Suppose that $e < 2$. If $|b| < 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1-b+|f|) = \frac{1}{4}(1-b+c),\ 1-b-c = 0,\ |e-f| = 2.$$

Hence $b = -1$. This is a contradiction. If $b = 1$, then $T = (1, 1, c,\ \pm 1,\ \pm\frac{1}{2}c,\ \pm 2)$ for $2 < c < 4$. Hence, T is not extreme. This is a contradiction. If $b = -1$, then $f = 0$ and $c \leq 2$. This is a contradiction. Suppose that $e = 2$. If $|b| < 1$, then, by Lemma 1,

$$1 = \frac{1}{2}(1-b+|f|) = \frac{1}{4}(1-b+c) \text{ or } 1-b-c = f = 0.$$

If $1 = \frac{1}{2}(1-b+|f|) = \frac{1}{4}(1-b+c)$, then $T = (1, b, 3+b,\ 1, 2,\ \pm(1+b))$ for $-1 < b < 1$. Hence, T is not extreme. This is a contradiction. If $1-b-c = f = 0$, then $c = 1-b < 2$. This is a contradiction. Let $d = 1$. If $b = 1$, then, by Lemma 1,

$$f = 0 \quad \text{or} \quad \frac{1}{4}c + \frac{1}{2}(2-f) = 1.$$

If $f = 0$, then $T = (1, 1, c, 1, 2, 0)$ for $2 < c < 4$. Hence, T is not extreme. This is a contradiction. If $\frac{1}{4}c + \frac{1}{2}(2-f) = 1$, then $T = (1, 1, c, 1, 2, \frac{1}{2}c)$ for $2 < c < 4$. Hence, T is not extreme. This is a contradiction.

Let $d = -1$. If $|b| < 1$, then we reach to a contradiction as in the proof of the case $d = 1$. If $b = 1$, then, by Lemma 1, $f = 0$ and $1 \geq \frac{1}{4}(|a+b-2d| + |c) = \frac{1}{4}(4+c)$, so $c = 0$. This is a contradiction. If $b = -1$, then, by Lemma 1,

$$\frac{1}{2} + \frac{1}{4}c = \frac{1}{4}(c-2) + \frac{1}{2}|f| = 1,$$

so $c = 2$. This is a contradiction. We have shown that if $2 < c < 4$, then T is not extreme.

Case 1: $c = 0$.

Claim: $|b| = |d| = 1$.

Assume otherwise. Then, $(|b| < 1, |d| < 1)$ or $(|b| = 1, |d| < 1)$ or $(|b| < 1, |d| = 1)$. Assume that $|b| < 1$ and $|d| < 1$. By Lemma 1,

$$\frac{1}{2}(1-d+e) = \frac{1}{2}(|b-d|+|f|) = 1,\ 1+b-2d = 0,\ |e-f| = 2.$$

Hence, $b = -1$, which is a contradiction. Assume that $|b| = 1$ and $|d| < 1$. If $b = 1$, then, by Lemma 1,

$$\frac{1}{2}(1-d+e) = \frac{1}{2}(1-d+|f|) = \frac{1}{2}(1-d+|e-f|) = 1.$$

Hence, $d = -1$, which is a contradiction. If $b = -1$, then, by Lemma 1,

$$\frac{1}{2}(1-d+e) = \frac{1}{2}(1+d+|f|) = 1,\ d = 0,\ |e-f| = 2.$$

Hence, $T = (1, -1, 0, 0, 1, -1)$. Notice that T is not extreme since

$$T = \frac{1}{2}\Big(\big(1, -1, \frac{2}{n}, \frac{1}{n}, 1+\frac{1}{n}, -1+\frac{1}{n}\big) + \big(1, -1, -\frac{2}{n}, -\frac{1}{n}, 1-\frac{1}{n}, -1-\frac{1}{n}\big)\Big)$$

and $\|(1, -1, \pm\frac{2}{n}, \pm\frac{1}{n}, 1\pm\frac{1}{n}, -1\pm\frac{1}{n})\|_{\mathcal{L}_s(^2 l_\infty^2)} = 1$ for every $n \in \mathbb{N}$. Assume that $|b| < 1$ and $|d| = 1$. If $d = 1$, then, by Lemma 1,

$$e = 2,\ \frac{1}{2}(1-b+|f|) = \frac{1}{4}(1-b) + \frac{1}{2}|e-f| = 1.$$

Hence, $T = (1, -\frac{1}{3}, 0, 1, 2, \frac{2}{3})$. Notice that T is not extreme since

$$T = \frac{1}{2}\Big(\big(1, -\frac{1}{3}+\frac{1}{n}, -\frac{3}{n}, 1, 2, \frac{2}{3}+\frac{1}{n}\big) + \big(1, -\frac{1}{3}-\frac{1}{n}, \frac{3}{n}, 1, 2, \frac{2}{3}-\frac{1}{n}\big)\Big)$$

and

$$\big\|\big(1, -\frac{1}{3}\pm\frac{1}{n}, \mp\frac{3}{n}, 1, 2, \frac{2}{3}\pm\frac{1}{n}\big)\big\|_{\mathcal{L}_s(^2 l_\infty^2)} = 1$$

for every $n > 3$. If $d = -1$, then, by Lemma 1,

$$e = 0,\ \frac{1}{2}(1+b+|f|) = \frac{1}{4}(3+b) + \frac{1}{2}|f| = 1.$$

Hence, $b = 1$, which is a contradiction. We have shown that the claim holds.

Suppose that $b = d = 1$. By Lemma 1,

$$(e = |f| = 2) \quad \text{or} \quad (e = |e-f| = 2) \quad \text{or} \quad (|f| = |e-f| = 2).$$

If $e = |f| = 2$, then $T = (1, 1, 0, 1, 2, 2)$. Notice that T is not extreme since

$$T = \frac{1}{2}\Big(\big(1, 1, \frac{1}{n}, 1, 2, 2\big) + \big(1, 1, -\frac{1}{n}, 1, 2, 2\big)\Big)$$

and $\|(1, 1, \pm\frac{1}{n}, 1, 2, 2)\|_{\mathcal{L}_s(^2 l_\infty^2)} = 1$ for every $n \in \mathbb{N}$. This is a contradiction. If $e = |e - f| = 2$, then $T = (1, 1, 0, 1, 2, 0) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l_\infty^2)}}$. Indeed, let $T_1 := T + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ and

$T_2 := T - (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ for some $\varepsilon_j, \beta_j \in \mathbb{R}$ (j =1, 2, 3). Obviously, $\varepsilon_1 = \varepsilon_2 = \delta_1 = 0$. Since $|2 \pm \delta_2| \leq 2$, we have $\delta_2 = 0$. Since

$$\frac{1}{4}|\varepsilon_3| + \frac{1}{2}|2 - \delta_3| \leq 2, \ \frac{1}{4}|-\varepsilon_3| + \frac{1}{2}|2 + \delta_3| \leq 2,$$

we have $\delta_3 = \varepsilon_3 = 0$. Therefore, $T_1 = T_2 = T$. Hence, T is extreme. If $|f| = |e - f| = 2$, then $T = (1, 1, 0, 1, 0, \pm 2)$. By Theorem 1, T is extreme. If $b = -d = 1$, then, by Lemma 1, $T = (1, 1, 0, -1, 0, 0)$. We claim that T is extreme. Let $T_1 := T + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ and $T_2 := T - (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ for some $\varepsilon_j, \beta_j \in \mathbb{R}$ (j =1, 2, 3). Obviously, $\varepsilon_1 = \varepsilon_2 = \delta_1 = 0$. Since $|2 \pm \delta_2| \leq 2$, $|2 \pm \delta_3| \leq 2$, we have $\delta_2 = \delta_3 = 0$. Since $\frac{1}{4}(4 + |\varepsilon_3|) \leq 1$, we have $\varepsilon_3 = 0$. Therefore, $T_1 = T_2 = T$. Hence, T is extreme. Notice that $(1, 1, 0, -1, 0, 0)$. If $-b = -d = 1$, then $|f| = 1$ and $T = (1, -1, 0, -1, 0, \pm 1)$. Notice that T is not extreme since

$$T = \frac{1}{2}\Big((1, -1, \frac{2}{n}, -1, 0, 1 + \frac{1}{n}) + (1, -1, -\frac{2}{n}, -1, 0, 1 - \frac{1}{n})\Big)$$

and $\|(1, -1, \pm\frac{2}{n}, -1, 0, 1 \pm \frac{1}{n})\|_{\mathcal{L}_s(^2l_\infty^2)} = 1$ for every $n \in \mathbb{N}$. This is a contradiction. If $-b = d = 1$, then $c = 2$. This is a contradiction.

Case 2: $c = 2$.

Claim: $|d| = 0$ or 1.

Assume that $0 < |d| < 1$. If $b = d$, by Lemma 1,

$$|f| = 2, \ \frac{1}{2}(1 - d + e) = \frac{1}{4}(1 - d) + \frac{1}{2} = 1.$$

Hence, $d = -1$, which is a contradiction. Assume that $b \neq d$. If $|b| < 1$, by Lemma 1,

$$\frac{1}{2}(1 - d + e) = \frac{1}{2}(|b - d| + |f|) = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1, \ ||1 + b - 2d| - 2| = 4, \ e - f = 0$$

or

$$\frac{1}{2}(1 - d + e) = \frac{1}{2}(|b - d| + |f|) = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1, \ |1 + b - 2d| = 2, \ |e - f| = 2.$$

If $\frac{1}{2}(1 - d + e) = \frac{1}{2}(|b - d| + |f|) = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1$, $|1 + b - 2d| = 2, |e - f| = 2$, then $b = -1$, which is a contradiction. If $\frac{1}{2}(1 - d + e) = \frac{1}{2}(|b - d| + |f|) = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1$, $||1 + b - 2d| - 2| = 4$, $e - f = 0$, then $|d| = 2$, which is a contradiction. If $|b| = 1$, then, by Lemma 1,

$$\frac{1}{2}(1 - d + e) = \frac{1}{4}(1 + b - 2d) + \frac{1}{2} = 1, \ |1 + b - 2d| = |e - f| = 2.$$

If $b = 1$, then $d = 0$, which is a contradiction. If $b = -1$, then $d = 1$, which is a contradiction. Therefore, we have shown that $|d| = 0$ or 1.

Suppose that $d = 0$. If $|b| < 1$, then, by Lemma 1,

$$e = 1, \ \frac{1}{2}(|b| + |f|) = \frac{1}{4}(1 + b) + \frac{1}{2} = 1.$$

Hence, $b = 1$, which is a contradiction. Let $|b| = 1$. Suppose that $\frac{1}{2} + \frac{1}{2}e = 1$. Then, $e = 1$ and $T = (1, 1, 2, 0, 1, -1)$ or $(1, 1, 2, 0, -1, 1)$. We claim that $(1, 1, 2, 0, 1, -1)$ is extreme. Indeed, let

$T_1 := T + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ and $T_2 := T - (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ for some $\varepsilon_j,\ \beta_j \in \mathbb{R}$ (j =1, 2, 3). Obviously, $\varepsilon_1 = \varepsilon_2 = 0$. Since

$$|1 \mp \delta_1| + |1 \pm \delta_2| \leq 2, \quad |1 \pm \delta_1| + |1 \pm \delta_3| \leq 2, \quad |2 \mp 2\delta_1| + |2 \pm \varepsilon_3| \leq 4,$$

we have $\delta_1 = \delta_2 = -\delta_3 = \frac{1}{2}\varepsilon_3$. Since

$$\frac{3}{4}|\pm \delta_1| + |1 \pm \delta_1| \leq 1,$$

we have $\delta_1 = \delta_2 = -\delta_3 = \frac{1}{2}\varepsilon_3 = 0$. Therefore, $T_1 = T_2 = T$. Hence, T is extreme. By Theorem 1, $(1, 1, 2, 0, -1, 1)$ is extreme.

Suppose that $\frac{1}{2} + \frac{1}{2}e < 1$. By Lemma 1, $|f| = 1$. If $f = 1$, then $T = (1, 1, 2, 0, e, 1)$ for $0 \leq e < 1$. Notice that such $(1, 1, 2, 0, e, 1)$ is not extreme. If $f = -1$, then $T = (1, 1, 2, 0, 0, -1)$. Notice that $(1, 1, 2, 0, 0, -1)$ is not extreme since

$$T = \frac{1}{2}\Big(\big(1, 1, 2 + \frac{2}{n}, \frac{1}{n}, \frac{1}{n}, -1 + \frac{1}{n}\big) + \big(1, 1, 2 - \frac{2}{n}, -\frac{1}{n}, -\frac{1}{n}, -1 - \frac{1}{n}\big)\Big)$$

and $\|(1, 1, 2 \pm \frac{2}{n}, \pm\frac{1}{n}, \pm\frac{1}{n}, -1 \pm \frac{1}{n})\|_{\mathcal{L}_s(^2 l_\infty^2)} = 1$ for every $n > 2$. This is a contradiction.

Suppose that $|d| = 1$. We claim that $|b| = 1$. Assume that $|b| < 1$. If $d = 1$, then, by Lemma 1,

$$\frac{1}{2}(1 - b + |f|) = \frac{1}{4}(1 - b) + \frac{1}{2} = 1$$

or

$$\frac{1}{4}(1 - b) + \frac{1}{2} = \frac{1}{4}(1 - b) + \frac{1}{2} = 1$$

or

$$\frac{1}{2}(1 - b + |f|) = \frac{1}{4}(1 + b) + \frac{1}{2}|e - f| = 1.$$

Hence, $b = -1$, which is a contradiction. If $d = -1$, then, by Lemma 1,

$$e = 0,\ \frac{1}{2}(1 + b + |f|) = \frac{1}{4}(3 + b) + \frac{1}{2} = 1.$$

Hence, $b = -1$, which is a contradiction. Therefore, $|b| = 1$. Suppose that $b = d = 1$. If $\frac{1}{2} + \frac{1}{2}|e - f| < 1$, then $T = (1, 1, 2, 1, 2, \pm 2)$. Notice that $(1, 1, 2, 1, 2, \pm 2)$ is not extreme since

$$T = \frac{1}{2}\Big(\big(1, 1, 2 + \frac{1}{n}, 1, 2, 2\big) + \big(1, 1, 2 - \frac{1}{n}, 1, 2, 2\big)\Big)$$

and $\|(1, 1, 2 \pm \frac{1}{n}, 1, 2, 2)\|_{\mathcal{L}_s(^2 l_\infty^2)} = 1$ for every $n > 2$. This is a contradiction. Suppose that $\frac{1}{2} + \frac{1}{2}|e - f| = 1$. If $e = 2$, then $T = (1, 1, 2, 1, 2, 0)$. Notice that $(1, 1, 2, 1, 2, 0)$ is not extreme since

$$T = \frac{1}{2}\Big(\big(1, 1, 2 + \frac{1}{n}, 1, 2, \frac{1}{2n}\big) + \big(1, 1, 2 - \frac{1}{n}, 1, 2, -\frac{1}{2n}\big)\Big)$$

and $\|(1, 1, 2 + \frac{1}{n}, 1, 2, \frac{1}{2n})\|_{\mathcal{L}_s(^2 l_\infty^2)} = 1$ for every $n \in \mathbb{N}$. This is a contradiction.

If $|f| = 2$, then $T = (1, 1, 2, 1, 0, 2)$. By Theorem 1, $(1, 1, 2, 1, 0, 2)$ is not extreme. Suppose that $-b = d = 1$. Then $T = (1, -1, 2, 1, e, 0)$ for $0 \leq e \leq 2$. Since T is extreme, $e = 0$ or 2. Notice that $(1, -1, 2, 1, 0, 0)$ is not extreme. We claim that $T = (1, -1, 2, 1, 2, 0)$ is extreme. Let $T_1 := T + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ and $T_2 := T - (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ for some $\varepsilon_j,\ \beta_j \in \mathbb{R}$ (j =1, 2, 3). Obviously, $\varepsilon_1 = \varepsilon_2 = \delta_1 = \delta_2 = \delta_3 = 0$. Since $2 + |2 \pm \varepsilon_3| \leq 4$, we have $\varepsilon_3 = 0$. Therefore, $T_1 = T_2 = T$. Hence, T is extreme.

Suppose that $b = d = -1$. Then $T = (1, -1, 2, -1, 0, f)$ for $-2 \leq f \leq 2$. Since T is extreme, $f = \pm 2$. By Theorem 1, $T = (1, -1, 2, -1, o, \pm 2)$ is extreme. Suppose that $b = -d = 1$. Then,

$$1 \geq \frac{1}{4}(|1 + b - 2d| + c) = \frac{3}{2},$$

which is a contradiction.

Case 3: $c = 4$.

Claim: $|b| = 1$.

Assume that $|b| < 1$. By Lemma 1, we have $0 < d < 1$, $\frac{1}{2}(1 - d + e) = 1$. Hence, $T = (1, 2d - 1, 4, d, 1 + d, 1 + d)$ for $0 < d < 1$. Hence, T is not extreme. This is a contradiction. Therefore, $|b| = 1$. If $b = 1$, then $T = (1, 1, 4, 1, e, e)$ for $0 \leq e \leq 2$. Since T is extreme, $e = 0$ or 2. We claim that $(1, 1, 4, 1, 2, 2)$ is extreme. Let $T_1 := T + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ and $T_2 := T - (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$ for some ε_j, $\beta_j \in \mathbb{R}$ (j =1, 2, 3). Obviously, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \delta_1 = 0$, $\delta_3 = \delta_2$. Since $|2 \pm \delta_2| \leq 2$, we have $\delta_2 = 0$. Therefore, $T_1 = T_2 = T$. Hence, T is extreme.

Notice that $(1, 1, 4, 1, 0, 0)$ is not extreme since

$$T = \frac{1}{2}\Big((1, 1, 4, 1, \frac{1}{n}, \frac{1}{n}) + (1, 1, 4, 1, -\frac{1}{n}, -\frac{1}{n})\Big)$$

and $\|(1, 1, 4, 1, \pm\frac{1}{n}, \pm\frac{1}{n})\|_{\mathcal{L}_s({}^2l^2_\infty)} = 1$ for every $n \in \mathbb{N}$. This is a contradiction.

If $b = -1$, then $d = 0$, $e = f$, $0 \leq e \leq 1$. Hence, $T = (1, -1, 4, 0, e, e)$ for $0 \leq e \leq 1$. Since T is extreme, e =0 or 1. Notice that $(1, -1, 4, 0, 0, 0)$ is not extreme since

$$(1, -1, 4, 0, 0, 0) = \frac{1}{2}\Big((1, -1, 4, 0, \frac{1}{n}, \frac{1}{n}) + (1, -1, 4, 0, -\frac{1}{n}, -\frac{1}{n})\Big)$$

and $\|(1, -1, 4, 0, \pm\frac{1}{n}, \pm\frac{1}{n})\|_{\mathcal{L}_s({}^2l^2_\infty)} = 1$ for every $n \in \mathbb{N}$. This is a contradiction. We claim that $T = (1, -1, 4, 0, 1, 1)$ is extreme. Let

$$T_1 := T + (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3) \quad \text{and} \quad T_2 := T - (\varepsilon_1, \varepsilon_2, \varepsilon_3, \delta_1, \delta_2, \delta_3)$$

for some ε_j, $\beta_j \in \mathbb{R}$ (j =1, 2, 3). Obviously, $\varepsilon_j = 0$ for $j = 1, 2, 3$. Since

$$2|\delta_1| + 4 \leq 4,\ 1 + |1 \pm \delta_2| \leq 2,\ 1 + |1 \pm \delta_3| \leq 2,$$

we have $\delta_j = 0$ for $j = 1, 2, 3$. Therefore, $T_1 = T_2 = T$. Hence, T is extreme.

Therefore, we complete the proof. □

Theorem 3 ([22])**.** *Let E be a real Banach space such that $\text{ext}\, B_E$ is finite. Suppose that $x \in \text{ext}\, B_E$ satisfies that there exists $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in \text{ext}\, B_E \backslash \{\pm x\}$. Then, $x \in \exp B_E$.*

The following theorem gives the explicit formula for the norm of every linear functional on $\mathbb{R}^6_{\mathcal{L}_s({}^2l^2_\infty)}$.

Theorem 4. *Let $f \in (\mathbb{R}^6_{\mathcal{L}_s({}^2l^2_\infty)})^*$. Let $\alpha_1 := f(e_1)$, $\alpha_2 := f(e_2)$, $\alpha_3 := f(e_4)$, $\beta := f(e_3)$, $\gamma_1 := f(e_5)$, $\gamma_2 := f(e_6)$. Then,*

$$\begin{aligned}\|f\| = \Big\{&|\alpha_1 + \alpha_2 + \alpha_3| + 4|\beta| + 2|\gamma_1 + \gamma_2|,\ |\alpha_1 - \alpha_2| + 4|\beta| + |\gamma_1 + \gamma_2|,\\ &|\alpha_1 - \alpha_2 + \alpha_3| + 2|\beta| + 2|\gamma_1|,\ |\alpha_1 - \alpha_2 - \alpha_3| + 2|\beta| + 2|\gamma_2|,\\ &|\alpha_1 + \alpha_2| + 2|\beta| + |\gamma_1 - \gamma_2|,\ |\alpha_1 + \alpha_2 + \alpha_3| + 2|\gamma_1|,\ |\alpha_1 + \alpha_2 - \alpha_3|\Big\}.\end{aligned}$$

Proof. It follows from the Krein-Milman Theorem and the fact that

$$\|f\| = \sup_{T \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}}} |f(T)|.$$

□

Notice that if $f \in (\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)})^*$ and $\|f\| = 1$, then $|\alpha_j| \le 1$, $|\beta| \le \frac{1}{4}$, $|\gamma_k| \le \frac{1}{2}$ for $j = 1,2,3$ and $k = 1,2$.

Theorem 5. $\text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}} = \exp B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}}$.

Proof. It is enough to show that if $T = (a,b,c,d,e,f) \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}}$, then T is exposed.

Claim: $T = (1,1,4,1,2,2)$ is exposed.

Let $f \in (\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)})^*$ be such that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\beta = \gamma_1 = \gamma_2 = \frac{1}{8}$. By Theorem 4, $f(T) = \|f\| = 1$ and $|f(R)| < 1$ for every $R \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}} \backslash \{\pm T\}$. By Theorem 3, T is exposed. By Theorem 1, $\pm(1,1,-4,1,2,2)$, $\pm(1,1,\pm 4,1,-2,-2)$ are exposed.

Claim: $T = (1,-1,4,0,1,1)$ is exposed.

Let $f \in (\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)})^*$ be such that $\alpha_1 = -\alpha_2 = \frac{1}{8}$, $\alpha_3 = 0$, $\beta = \frac{3}{16}$, $\gamma_1 = \gamma_2 = 0$. By Theorem 4, $f(T) = \|f\| = 1$ and $|f(R)| < 1$ for every $R \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}} \backslash \{\pm T\}$. By Theorem 3, T is exposed. By Theorem 1, $\pm(1,1,-4,0,1,1)$, $\pm(1,1,\pm 4,0,-1,-1)$ are exposed.

Claim: $T = (1,-1,2,1,2,0)$ is exposed.

Let $f \in (\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)})^*$ be such that $\alpha_1 = -\alpha_2 = \alpha_3 = \frac{1}{3}$, $\beta = \gamma_1 = \gamma_2 = 0$. By Theorem 4, $f(T) = \|f\| = 1$ and $|f(R)| < 1$ for every $R \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}} \backslash \{\pm T\}$. By Theorem 3, T is exposed. By Theorem 1, $\pm(1,-1,-2,1,\pm 2,0)$, $\pm(1,-1,-2,1,-2,0)$, $\pm(1,-1,-2,-1,0,\pm 2)$ are exposed

Claim: $T = (1,1,2,0,1,-1)$ is exposed.

Let $f \in (\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)})^*$ be such that $\alpha_1 = \alpha_2 = -\alpha_3 = \frac{1}{6}$, $\beta = 0$, $\gamma_1 = -\gamma_2 = \frac{1}{3}$. By Theorem 4, $f(T) = \|f\| = 1$ and $|f(R)| < 1$ for every $R \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}} \backslash \{\pm T\}$. By Theorem 3, T is exposed. By Theorem 1, $\pm(1,1,-2,0,1,-1)$, $\pm(1,1,\pm 2,0,-1,1)$ are exposed

Claim: $T = (1,1,0,1,2,0)$ is exposed.

Let $f \in (\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)})^*$ be such that $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{6}$, $\beta = 0$, $\gamma_1 = -\gamma_2 = \frac{1}{4}$. By Theorem 4, $f(T) = \|f\| = 1$ and $|f(R)| < 1$ for every $R \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}} \backslash \{\pm T\}$. By Theorem 3, T is exposed. By Theorem 1, $\pm(1,1,0,1,\pm 2,0)$, $\pm(1,1,0,1,0,\pm 2)$ are exposed

Claim: $T = (1,1,0,-1,0,0)$ is exposed.

Let $f \in (\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)})^*$ be such that $\alpha_1 = \alpha_2 = -\alpha_3 = \frac{1}{3}$, $\beta = \gamma_1 = \gamma_2 = 0$. By Theorem 4, $f(T) = \|f\| = 1$ and $|f(R)| < 1$ for every $R \in \text{ext}\, B_{\mathbb{R}^6_{\mathcal{L}_s(^2 l^2_\infty)}} \backslash \{\pm T\}$. By Theorem 3, T is exposed. By Theorem 1, $-(1,1,0,-1,0,0)$ is exposed. Therefore, we complete the proof. □

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Кім С.Ґ. *Екстремальні та виставлені симетричні білінійні форми на просторі* $\mathcal{L}_s(^2l^2_\infty)$ // Карпатські матем. публ. — 2020. — Т.12, №2. — С. 340–352.

Класифіковано екстремальні точки та виставлені точки одиничної кулі простору білінійних симетричних форм на дійсному банаховому просторі білінійних симетричних форм на l^2_∞. Показано, що в цьому випадку множина екстремальних точок дорівнює множині виставлених точок.

Ключові слова і фрази: екстремальна точка, виставлена точка.