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Gradient almost Ricci solitons on multiply warped product manifolds

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In this paper, we investigate multiply warped product manifold

$$M = B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} \ldots \times_{b_m} F_m$$

as a gradient almost Ricci soliton. Taking $b_i = b$ for $1 \leq i \leq m$ lets us to deduce that potential field depends on B. With this idea we also get a rigidity result and show that base is a generalized quasi-Einstein manifold if ∇b is conformal.

Key words and phrases: multiply warped product, gradient almost Ricci soliton, generalized quasi-Einstein manifold, conformal vector field.

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Introduction

The term of warped products was first introduced by R.L. Bishop and B. O'Neill, who used it to give a class of complete Riemannian manifolds with negative sectional curvature [5]. The notion of warped product plays very important role not only in geometry but also in mathematical physics, especially in general relativity.

In order to give an answer to A.L. Besse [4, p. 265], D.-S. Kim and Y.H. Kim [17] shown that there does not exist a compact Einstein warped product with non-constant warping function if the scalar curvature is non-positive. Furthermore, the authors found a necessary condition for a warped product to be an Einstein manifold, which is the base of the product manifold $B^n \times F^m$ to be a m-quasi-Einstein manifold, i.e. the Riemannian manifold B is endowed with Bakry-Emery Ricci tensor such that

$$\mathrm{Ric} + \nabla^2 h - \frac{1}{m} dh \otimes dh = \lambda g, \tag{1}$$

where h is a smooth function on B and λ, m are constants. For more detailed work on the m-quasi-Einstein manifolds, see [7,17,22]. We also need to mention generalized quasi-Einstein manifolds, i.e. (1) is satisfed where the parameter λ is also a smooth function defined on B, which was studied in [8]. The m-quasi-Einstein manifolds are generalizations of Ricci solitons. The concept of Ricci solitons emerges from the Ricci flow, introduced by R.S. Hamilton [14,15]. A complete Riemannian manifold (M^n, g) is said to be a Ricci soliton if there exists a vector

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field X satisfying the equation

$$\mathrm{Ric} + \frac{1}{2}\mathscr{L}_X g = \lambda g,$$

where λ is a constant. There are many studies involving Ricci solitons, some of which are [6, 18, 19]. S. Pigola et al. [20] modified the definition of the Ricci solitons by taking λ as a function and introduced another generalization of Ricci solitons: the almost Ricci solitons. Indeed, (M, g) is said to be a gradient almost Ricci soliton if

$$\mathrm{Ric} + \nabla^2\varphi = \lambda g \tag{2}$$

is satisfied for $\varphi, \lambda : M \to \mathbb{R}$, where $\nabla^2\varphi$ is the Hessian of the potential function φ. In [2, 3], some characterizations of the almost Ricci solitons are given and rigidity of such solitons are studied. In particular, from (4) of Propositon 1 in [3], we know that the equation

$$\nabla(R + |\nabla\varphi|^2 - 2(n-1)\lambda) = 2\lambda\nabla\varphi \tag{3}$$

holds for a gradient almost Ricci soliton. Taking trace of (2) and plugging in (3), we get

$$-2\lambda\nabla\varphi + \nabla((2-n)\lambda + |\nabla\varphi|^2 - \Delta\varphi) = 0. \tag{4}$$

Likewise, we require a similar equation to hold in our theory to reach some new results.

We are inspired at first by [11], where the authors constructed a gradient Ricci soliton $(M, g, \nabla\tilde{\varphi}, \lambda)$ as a warped product manifold $M = B \times_f F$, where the potential $\tilde{\varphi}$ of gradient Ricci soliton as the lift of a function defined on the base B. In [9, 13], the authors studied gradient Ricci solitons as warped products. In [1, 16], the gradient Ricci soliton multiply warped product manifols are examined. In [12], the authors also studied the case of gradient almost Ricci solitons. It comes to our attention that what happens when the multiply warped product manifold is a gradient almost Ricci soliton. In this paper, we investigate gradient almost Ricci solitons as multiply warped product manifolds and reach a rigidity result.

1 Preliminaries

In this section, we give a brief summary of multiply warped products defined in [21].

Definition 1 ([21]). *Let (B, g_B) and (F_i, g_{F_i}) be r and s_i dimensional Riemannian manifolds, where $i \in \{1, 2, \ldots, m\}$ and also $M = B \times F_1 \times F_2 \times \ldots \times F_m$ be an n-dimensional Riemannian manifold, where $n = r + \sum_{i=1}^{m} s_i$. Let $b_i : B \to \mathbb{R}^+$ be smooth functions for $i \in \{1, 2, \ldots, m\}$. The multiply warped product is the product manifold $B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} \ldots \times_{b_m} F_m$ furnished with the metric tensor $g = g_B \oplus b_1^2 g_{F_1} \oplus \ldots \oplus b_m^2 g_{F_m}$ defined by*

$$g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus \ldots \oplus (b_m \circ \pi)^2 \sigma_m^*(g_{F_m}),$$

where π and σ_i are the natural projections on B and F_i, respectively. The functions b_i are called the warping functions for $i \in \{1, 2, \ldots, m\}$. If $m = 1$, then we obtain a singly warped product. If all $b_i \equiv 1$, then we have a product manifold.

We denote ∇, ${}^B\nabla$ and ${}^{F_i}\nabla$; Ric, ${}^B\mathrm{Ric}$ and ${}^{F_i}\mathrm{Ric}$ the Levi-Civita connections and Ricci curvatures of the M, B and F_i, respectively. R stands for the scalar curvature.

The lift of X to M is the unique element of $\mathfrak{X}(M)$ that is π-related to X and σ_i-related to zero vector field on B. Similarly, $V_i \in \mathfrak{X}(F_i)$ can be lifted to M and the set of all such lifts is denoted by $\mathfrak{L}(F_i)$. We will use the same notation for a vector field and its lift. In the same way, functions defined on B can be lifted to M. Let h be a smooth function on B. The lift of h to M is the function $\tilde{h} = h \circ \pi$.

Now, we recall the covariant derivative formulas for multiply warped products.

Lemma 1 ([10]). *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} \ldots \times_{b_m} F_m$ be a multiply warped product manifold. For $X, Y \in \mathfrak{L}(B)$, $V \in \mathfrak{L}(F_i)$ and $W \in \mathfrak{L}(F_j)$,*

(i) $\nabla_X Y$ *is the lift of* ${}^B\nabla_X Y$ *on* B,

(ii) $\nabla_X V = \nabla_V X = (X(b_i)/b_i)V$,

(iii)

$$\nabla_V W = \begin{cases} 0, & \text{if } i \neq j, \\ {}^{F_i}\nabla_V W - (g(V,W)/b_i)\mathrm{grad}_B(b_i), & \text{if } i = j. \end{cases}$$

Proposition 1. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} \ldots \times_{b_m} F_m$ be a multiply warped product manifold and $\varphi : B \to \mathbb{R}$ be a smooth function. Then*

1) $\nabla\tilde{\varphi} = {}^B\nabla\varphi$,

2) $\Delta\varphi = \Delta_B\varphi + \sum_{i=1}^m s_i(g_B({}^B\nabla\varphi, {}^B\nabla b_i)/b_i)$.

If all of the warping functions $b_i = b$, then the following result can be stated.

Corollary 1. *Let $M = B \times_b F_1 \times_b F_2 \times_b \ldots \times_b F_m$ be a multiply warped product and $\varphi : B \to \mathbb{R}$ be a smooth function. Then*

$$\Delta\tilde{\varphi} = \Delta_B\varphi + \left(\sum_{i=1}^m s_i\right)\frac{{}^B\nabla\varphi(b)}{b}.$$

Lemma 2 ([10]). *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} \ldots \times_{b_m} F_m$ be a multiply warped product manifold. For $X, Y \in \mathfrak{L}(B)$, $V \in \mathfrak{L}(F_i)$ and $W \in \mathfrak{L}(F_j)$,*

(i) $\mathrm{Ric}(X,Y) = {}^B\mathrm{Ric}(X,Y) - \sum_{i=1}^m (s_i/bi)\mathrm{Hess}_B^{b_i}(X,Y)$,

(ii) $\mathrm{Ric}(X,V) = 0$,

(iii) *for* $i \neq j$, $\mathrm{Ric}(V,W) = 0$,

(iv) *for* $i = j$,

$$\mathrm{Ric}(V,W) = {}^{F_i}\mathrm{Ric}(V,W) - \left[\frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{|\mathrm{grad}_B b_i|^2}{b_i^2} + \sum_{\substack{i=1 \\ k\neq i}}^m s_k \frac{g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)}{b_i b_k}\right] g(V,W).$$

The following proposition helps us to deduce that the potential function depends on B.

Proposition 2. *Let $M = B^r \times_b F_1^{s_1} \times_b F_2^{s_2} \times_b \ldots \times_b F_m^{s_m}$ be a multiply warped product, and $\psi : M \to \mathbb{R}$ and $\lambda : B \to \mathbb{R}$ be two smooth functions. If $(M, g, \nabla\psi, \tilde{\lambda})$ is a gradient almost Ricci soliton, then $\psi = \tilde{\varphi}$ for some function $\varphi : B \to \mathbb{R}$. Moreover, the equation*

$$-2\lambda {}^B\nabla\varphi + \nabla\left(\left(2 - r - \sum_{i=1}^m s_i\right)\lambda + |{}^B\nabla\varphi|^2 - \Delta_B\varphi - \left(\sum_{i=1}^m s_i\right)\frac{{}^B\nabla\varphi(b)}{b}\right) = 0 \tag{5}$$

is satisfied.

Proof. Assume that $(M, g, \nabla\psi, \tilde{\lambda})$ is a gradient almost Ricci soliton. Let $X \in \mathfrak{L}(B)$ and $V_j \in \mathfrak{L}(F_j)$ for $1 \leq j \leq m$. From Lemma 2, we have $\mathrm{Ric}(X, V_j) = 0$ for all $1 \leq j \leq m$. It is clear that $\nabla\psi$ can be written as $\nabla\psi = (\nabla\psi)_B + (\nabla\psi)_{F_1} + \ldots + (\nabla\psi)_{F_m}$, where $(\nabla\psi)_B \in \mathfrak{L}(B)$ and $(\nabla\psi)_{F_i} \in \mathfrak{L}(F_i)$, $1 \leq i \leq m$. Now, we consider the equation (2),

$$\mathrm{Ric}\left(X, V_j\right) + \nabla^2\psi\left(X, V_j\right) = \tilde{\lambda} g\left(X, V_j\right),$$

which becomes

$$\begin{aligned} 0 = \nabla^2\psi(X, V_j) &= g(\nabla_X \nabla\psi, V_j) = g(\nabla_X (\nabla\psi)_B, V_j) + g\left(\nabla_X\left(\sum_{i=1}^{m} (\nabla\psi)_{F_i}\right), V_j\right) \\ &= g(\nabla_X (\nabla\psi)_{F_1}, V_j) + \ldots + g(\nabla_X (\nabla\psi)_{F_m}, V_j) \\ &= \frac{Xb}{b} g((\nabla\psi)_{F_1}, V_j) + \ldots + \frac{Xb}{b} g((\nabla\psi)_{F_m}, V_j) = \frac{Xb}{b} g((\nabla\psi)_{F_j}, V_j). \end{aligned}$$

We observe that $(\nabla\psi)_{F_j} = 0$ for all $1 \leq j \leq m$ and hence $\nabla\psi = (\nabla\psi)_B \in \mathfrak{L}(B)$. Thus, there exists a smooth function $\varphi : B \to \mathbb{R}$ such that $\psi = \tilde{\varphi}$ because of the uniqueness of the lift. From (4) we have

$$-2\tilde{\lambda}\nabla\tilde{\varphi} + \nabla\left(\left(2 - r - \sum_{i=1}^{m} s_i\right)\tilde{\lambda} + |\nabla\tilde{\varphi}|^2 - \Delta\tilde{\varphi}\right) = 0.$$

Using Lemma 1 and Corollary 1 in the above equation, we arrive

$$-2\lambda^B\nabla\varphi + \nabla\left(\left(2 - r - \sum_{i=1}^{m} s_i\right)\lambda + |^B\nabla\varphi|^2 - \Delta_B\varphi - \left(\sum_{i=1}^{m} s_i\right)\frac{^B\nabla\varphi(b)}{b}\right) = 0.$$

□

Using Lemma 2, we obtain some results for the components of the gradient almost Ricci soliton multiply warped product.

Proposition 3. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} \ldots \times_{b_m} F_m$ be a multiply warped product, and $\varphi : B \to \mathbb{R}$ and $\lambda : B \to \mathbb{R}$ be two smooth functions so that $(M, g, \nabla\tilde{\varphi}, \tilde{\lambda})$ is a gradient almost Ricci soliton. Then*

$$^B\mathrm{Ric} + \mathrm{Hess}_B^{\varphi} = \lambda g_B + \sum_{i=1}^{m} \frac{s_i}{b_i}\mathrm{Hess}_B^{b_i} \tag{6}$$

and $^{F_i}\mathrm{Ric} = \mu\sum_{i=1}^{m} b_i^2 g_{F_i}$ *with μ satisfying*

$$\mu = \tilde{\lambda} - \frac{^B\nabla\varphi(b_i)}{b_i} + \frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{|^B\nabla\, b_i|^2}{b_i^2} + \sum_{\substack{i=1 \\ k\neq i}}^{m} s_k \frac{^B\nabla b_i(b_k)}{b_i b_k}.$$

Proof. Suppose that $(M, g, \nabla\tilde{\varphi}, \tilde{\lambda})$ is a gradient almost Ricci soliton. From *(i)* of Lemma 1 and (2), we have

$$\tilde{\lambda} g_B(X, Y) - \nabla^2\tilde{\varphi}(X, Y) = {}^B\mathrm{Ric}(X, Y) - \sum_{i=1}^{m} \frac{s_i}{bi}\mathrm{Hess}_B^{b_i}(X, Y)$$

for $X, Y \in \mathfrak{L}(B)$. Since $\nabla^2\tilde{\varphi}(X, Y) = \mathrm{Hess}_B^{\varphi}(X, Y)$, we find

$$\lambda g_B(X, Y) - \mathrm{Hess}_B^{\varphi}(X, Y) = {}^B\mathrm{Ric}(X, Y) - \sum_{i=1}^{m} \frac{s_i}{bi}\mathrm{Hess}_B^{b_i}(X, Y),$$

which is (6). To prove the second assertion, we use again (2) and *(iv)* of Lemma 2, which yields

$$\tilde{\lambda} g(V,W) - \nabla^2 \tilde{\varphi}(V,W) = {}^{F_i}\mathrm{Ric}(V,W) - \left[\frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{|{}^B\nabla b_i|^2}{b_i^2} + \sum_{\substack{i=1 \\ k \neq i}}^{m} s_k \frac{g_B({}^B\nabla b_i, {}^B\nabla b_k)}{b_i b_k} \right] g(V,W)$$

for $V, W \in \mathfrak{L}(F_i)$. Since

$$\nabla^2 \tilde{\varphi}(V,W) = g(\nabla_V \nabla \tilde{\varphi}, W) = \frac{\nabla \tilde{\varphi}(b_i)}{b_i} g(V,W) = \frac{{}^B\nabla \varphi(b_i)}{b_i} \sum_{i=1}^{m} b_i^2 g_{F_i}(V,W),$$

we may write

$${}^{F_i}\mathrm{Ric}(V,W) = \left[\tilde{\lambda} - \frac{{}^B\nabla \varphi(b_i)}{b_i} + \frac{\Delta_B b_i}{b_i} + (s_i - 1)\frac{|{}^B\nabla b_i|^2}{b_i^2} + \sum_{\substack{i=1 \\ k \neq i}}^{m} s_k \frac{g_B({}^B\nabla b_i, {}^B\nabla b_k)}{b_i b_k} \right] \sum_{i=1}^{m} b_i^2 g_{F_i}(V,W),$$

which completes the proof. □

If all of the warping functions $b_i = b$, then we have the following corollary.

Corollary 2. *Let $M = B \times_b F_1 \times_b F_2 \times_b \cdots \times_b F_m$ be a multiply warped product, and $\varphi : B \to \mathbb{R}$ and $\lambda : B \to \mathbb{R}$ be two smooth functions so that $(M, g, \nabla\tilde{\varphi}, \tilde{\lambda})$ is a gradient almost Ricci soliton. Then*

$${}^B\mathrm{Ric} + \mathrm{Hess}_B^{\varphi} = \lambda g_B + \frac{\sum_{i=1}^m s_i}{b} \mathrm{Hess}_B^{b} \tag{7}$$

and ${}^{F_i}\mathrm{Ric} = \mu \sum_{i=1}^{m} g_{F_i}$ with μ satisfying

$$\mu = \tilde{\lambda} b^2 + b\Delta_B b + \left(\sum_{i=1}^{m} s_i - 1 \right) |{}^B\nabla b|^2 - b\,{}^B\nabla \varphi(b). \tag{8}$$

The next proposition plays an important role to prove our results in the next section.

Proposition 4. *Let (B^r, g) be a Riemannian manifold with smooth functions $b > 0$, φ and λ satisfying*

$$\mathrm{Ric} + \nabla^2 \varphi = \lambda g + \frac{\sum_{i=1}^m s_i}{b} \nabla^2 b \tag{9}$$

and assume that the equation (5) holds for some constants $s_i > 0$. Then b, φ and λ satisfy

$$\mu = \lambda b^2 + b\Delta b + \left(\sum_{i=1}^{m} s_i - 1 \right) |\nabla b|^2 - b \nabla \varphi(b) \tag{10}$$

for a constant $\mu \in \mathbb{R}$.

Before proving the proposition, we remind that a $(0,2)$-tensor field T on a Riemannian manifold (M, g) can be considered as a $(1,1)$-tensor field by $g(T(X), Y) = T(X,Y)$ for all $X, Y \in X(M)$. The formula $\mathrm{div}(fT) = f \mathrm{div} T + T(\nabla f, \cdot)$ is valid for a smooth function on M. We know that the second contracted Bianchi identity

$$\mathrm{divRic} = \frac{1}{2} \nabla R \tag{11}$$

holds. Now, we can prove the Proposition 4.

Proof of Proposition 4. Taking trace of the equation (9), we have

$$R = r\lambda + \frac{\sum_{i=1}^m s_i}{b}\Delta b - \Delta\varphi. \tag{12}$$

The covariant derivative of the equation (12) is

$$\nabla R = r\nabla\lambda - \frac{\sum_{i=1}^m s_i}{b^2}\Delta b\nabla b + \frac{\sum_{i=1}^m s_i}{b}\nabla\Delta b - \nabla\Delta\varphi. \tag{13}$$

On the other hand, taking the divergence of (9), we have

$$\mathrm{divRic} = \mathrm{div}(\lambda g) + \Big(\sum_{i=1}^m s_i\Big)\mathrm{div}\Big(\frac{1}{b}\nabla^2 b\Big) - \mathrm{div}\nabla^2\varphi. \tag{14}$$

After some calculation, we get

$$\mathrm{div}\,(\lambda g) = \lambda \mathrm{div} g + g(\nabla\lambda, \cdot) = \nabla\lambda, \tag{15}$$

$$\begin{aligned}\mathrm{div}\Big(\frac{1}{b}\nabla^2 b\Big) &= \frac{1}{b}\mathrm{div}\nabla^2 b - \frac{1}{b^2}\nabla^2 b(\nabla b, \cdot) = \frac{1}{b}\mathrm{div}\nabla^2 b - \frac{1}{2b^2}\nabla|\nabla b|^2 \\ &= \frac{1}{b}(\mathrm{Ric}(\nabla b) + \nabla\Delta b) - \frac{1}{2b^2}\nabla|\nabla b|^2 \\ &= \frac{1}{b}\Big[\lambda\nabla b + \frac{\sum_{i=1}^m s_i}{2b}\nabla|\nabla b|^2 - \nabla^2\varphi(\nabla b, \cdot)\Big] + \frac{1}{b}\nabla\Delta b - \frac{1}{2b^2}\nabla|\nabla b|^2 \\ &= \frac{1}{b}\lambda\nabla b + \frac{(\sum_{i=1}^m s_i - 1)}{2b^2}\nabla|\nabla b|^2 - \frac{1}{b}\nabla^2\varphi(\nabla b, \cdot) + \frac{1}{b}\nabla\Delta b,\end{aligned} \tag{16}$$

and

$$\mathrm{div}(\nabla^2\varphi) = \lambda\nabla\varphi + \frac{\sum_{i=1}^m s_i}{b}\nabla^2 b(\nabla\varphi, \cdot) - \frac{1}{2}\nabla|\nabla\varphi|^2 + \nabla\Delta\varphi. \tag{17}$$

Plugging in (15), (16) and (17) in the equation (14) we have

$$\begin{aligned}\mathrm{divRic} = \nabla\lambda &+ \frac{\sum_{i=1}^m s_i}{b}\lambda\nabla b + \frac{(\sum_{i=1}^m s_i)(\sum_{i=1}^m s_i - 1)}{2b^2}\nabla|\nabla b|^2 - \frac{\sum_{i=1}^m s_i}{b}\nabla^2\varphi(\nabla b, \cdot) \\ &+ \frac{\sum_{i=1}^m s_i}{b}\nabla\Delta b - \lambda\nabla\varphi - \frac{\sum_{i=1}^m s_i}{b}\nabla^2 b(\nabla\varphi, \cdot) + \frac{1}{2}\nabla|\nabla\varphi|^2 - \nabla\Delta\varphi.\end{aligned} \tag{18}$$

Using (13) and (18) in (11), we have

$$\begin{aligned}0 = -\frac{1}{2}\nabla R + \mathrm{divRic} &= -\frac{r}{2}\nabla\lambda + \frac{\sum_{i=1}^m s_i}{2b^2}\Delta b\nabla b - \frac{\sum_{i=1}^m s_i}{2b}\nabla\Delta b + \frac{1}{2}\nabla\Delta\varphi + \nabla\lambda \\ &+ \frac{\sum_{i=1}^m s_i}{b}\lambda\nabla b + \frac{(\sum_{i=1}^m s_i)(\sum_{i=1}^m s_i - 1)}{2b^2}\nabla|\nabla b|^2 - \frac{\sum_{i=1}^m s_i}{b}\nabla^2\varphi(\nabla b, \cdot) + \frac{\sum_{i=1}^m s_i}{b}\nabla\Delta b \\ &- \lambda\nabla\varphi - \frac{\sum_{i=1}^m s_i}{b}\nabla^2 b(\nabla\varphi, \cdot) + \frac{1}{2}\nabla|\nabla\varphi|^2 - \nabla\Delta\varphi \\ &= \frac{2-r}{2}\nabla\lambda + \frac{\sum_{i=1}^m s_i}{2b^2}\Delta b\nabla b - \frac{1}{2}\nabla\Delta\varphi + \frac{\sum_{i=1}^m s_i}{2b}\nabla\Delta b + \frac{\sum_{i=1}^m s_i}{b}\lambda\nabla b - \lambda\nabla\varphi + \frac{1}{2}\nabla|\nabla\varphi|^2 \\ &+ \frac{(\sum_{i=1}^m s_i)(\sum_{i=1}^m s_i - 1)}{2b^2}\nabla|\nabla b|^2 - \frac{\sum_{i=1}^m s_i}{b}(\nabla^2\varphi(\nabla b, \cdot) + \nabla^2 b(\nabla\varphi, \cdot)) \\ &= \frac{2-r}{2}\nabla\lambda + \frac{\sum_{i=1}^m s_i}{2b^2}\Delta b\nabla b - \frac{1}{2}\nabla\Delta\varphi + \frac{h}{2b}\nabla\Delta b + \frac{\sum_{i=1}^m s_i}{b}\lambda\nabla b - \lambda\nabla\varphi + \frac{1}{2}\nabla|\nabla\varphi|^2 \\ &+ \frac{(\sum_{i=1}^m s_i)(\sum_{i=1}^m s_i - 1)}{2b^2}\nabla|\nabla b|^2 - \frac{\sum_{i=1}^m s_i}{b}\nabla(\nabla\varphi(b)).\end{aligned}$$

Plugging (5) in the last row and multiplying by $2b^2/\sum_{i=1}^m s_i$, we obtain

$$0 = \nabla\Big(\lambda b^2 + b\Delta b + \Big(\sum_{i=1}^m s_i - 1\Big)|\nabla b|^2 - b\nabla\varphi(b)\Big),$$

i.e. $\lambda b^2 + b\Delta b + (\sum_{i=1}^m s_i - 1)\,|\nabla b|^2 - b\nabla\varphi(b)$ is constant. □

2 Main Results

In this section, we consider a gradient almost Ricci soliton multiply warped product M, whose potential function is the lift of a smooth function φ defined on B.

Theorem 1. *Let (B^r, g_B) be a Riemannian manifold with smooth functions $b > 0$, φ, and λ satisfying (7) and (8). Assume that for $1 \leq i \leq m$, $(F_i^{s_i}, g_{F_i})$ are complete Riemann manifolds with Ricci tensor satisfying ${}^{F_i}\mathrm{Ric} = \mu \sum_{i=1}^m g_{F_i}$ for a constant μ as in the equation (10). Then $(M = B \times_b F_1 \times_b \ldots \times_b F_m, g, \nabla\tilde{\varphi}, \tilde{\lambda})$ is a gradient almost Ricci soliton.*

Proof. Let $M = B \times_b F_1 \times_b F_2 \times_b \ldots \times_b F_m$ be a multiply warped product with the metric tensor $g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus \ldots \oplus (b_m \circ \pi)^2 \sigma_m^*(g_{F_m})$. For $X, Y \in \mathfrak{L}(B)$, we know that $\nabla^2\tilde{\varphi}(X,Y) = \mathrm{Hess}_B^{\varphi}(X,Y)$ and $\nabla^2\tilde{b}(X,Y) = \mathrm{Hess}_B^{b}(X,Y)$. From (7), we have

$$ {}^B\mathrm{Ric}(X,Y) + \nabla^2\tilde{\varphi}(X,Y) = \tilde{\lambda} g_B(X,Y) + \frac{\sum_{i=1}^m s_i}{b} \nabla^2\tilde{b}(X,Y). $$

Using this in *(i)* of Lemma 1, we conclude that $\mathrm{Ric}(X,Y) + \nabla^2\tilde{\varphi}(X,Y) = \tilde{\lambda} g_B(X,Y)$ is satisfied. For $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F_i)$, we have $\nabla^2\tilde{\varphi}(X,V) = g(\nabla_X \nabla\tilde{\varphi}, V) = 0$. Using *(ii)* of Lemma 2, we assure that the equation (2) is satisfied. For $i \neq j$, $V \in \mathfrak{L}(F_i)$ and $W \in \mathfrak{L}(F_j)$, we have $\nabla^2\tilde{\varphi}(V,W) = g((\nabla\tilde{\varphi}(b)/b)V, W) = 0$ and by using *(iii)* of Lemma 2 we find that the equation (2) is satisfied. Lastly, for $V, W \in \mathfrak{L}(F_i)$, we have $\nabla^2\tilde{\varphi}(V,W) = g((\nabla\tilde{\varphi}(b)/b)V, W)$ and by using *(iv)* of Lemma 2, (8) and the hypothesis ${}^{F_i}\mathrm{Ric} = \mu \sum_{i=1}^m g_{F_i}$, we get

$$ \begin{aligned} \mathrm{Ric}(V,W) &= \left[\mu - b\Delta_B b - (s_i - 1)|{}^B\nabla b|^2 - \sum_{\substack{i=1 \\ k \neq i}}^m s_k |{}^B\nabla b|^2 \right] \sum_{i=1}^m g_{F_i}(V,W) \\ &= \left[\mu - b\Delta_B b - \left(\sum_{i=1}^m s_i - 1 \right) |{}^B\nabla b|^2 \right] \sum_{i=1}^m g_{F_i}(V,W) \\ &= \left[\tilde{\lambda} - \frac{\nabla\tilde{\varphi}(b)}{b} \right] g(V,W) = \tilde{\lambda} g(V,W) - \nabla^2\tilde{\varphi}(V,W), \end{aligned} $$

which satisfies the equation (2). Hence, the multiply warped product manifold M is a gradient almost Ricci soliton. ☐

Theorem 2. *A multiply warped product gradient almost Ricci soliton*

$$ (M = B \times_b F_1 \times_b \ldots \times_b F_m, g, \nabla\tilde{\varphi}, \tilde{\lambda}) $$

is a Ricci soliton and usual Riemannian product if one of the following conditions hold:

(i) b reaches a minimum and $\tilde{\lambda} \geq \mu/b^2$ (or b reaches a maximum and $\tilde{\lambda} \leq \mu/b^2$),

(ii) $\tilde{\lambda} \leq 0$ and $\tilde{\lambda}(p) \leq \tilde{\lambda}(q)$, where p and q are maximum and minimum points of b, respectively.

Proof. From Corollary 2, the equation (7) and ${}^{F_i}\mathrm{Ric} = \mu \sum_{i=1}^m g_{F_i}$ is satisfied, where μ is

$$ \mu = \tilde{\lambda} b^2 + b\Delta_B b + \left(\sum_{i=1}^m s_i - 1 \right) |{}^B\nabla b|^2 - b^B\nabla\varphi(b). \tag{19} $$

By Proposition 2, we know that the equation (5) is also satisfied. Thus, by Proposition 1, μ is constant. If the function b reaches a minimum and $\tilde{\lambda} \geq \mu/b^2$, then we have $\Delta_B b \leq 0$. Hence, b is constant from the maximum principle.

Let p and q be maximum and minimum points of b, respectively. So we have $\nabla b(p) = \nabla b(q) = 0$ and $\Delta b(p) \leq 0 \leq \Delta b(q)$. Since $b > 0$, we get

$$0 \geq b(p)\Delta b(p) = \mu - \tilde{\lambda}(p)b^2(p) \geq \mu - \tilde{\lambda}(q)b^2(q) = b(q)\Delta b(q) \geq 0$$

and hence $\mu - \tilde{\lambda}(p)b^2(p) = 0 = \mu - \tilde{\lambda}(q)b^2(q)$. If $\lambda(p) = 0$, then $\mu = 0$ and $\tilde{\lambda}(q) = 0$. Thus, the equation (19) becomes

$$-\tilde{\lambda}b = \Delta_B b + \left(\sum_{i=1}^{m} s_i - 1\right)\frac{|{}^B\nabla b|^2}{b} - {}^B\nabla\varphi(b) \geq 0,$$

so we arrive that b is constant. If $\tilde{\lambda}(p) \neq 0$, then $\tilde{\lambda}(q) \neq 0$ and since $\tilde{\lambda}(p) \leq \tilde{\lambda}(p) < 0$,

$$b^2(p) = \Big(\frac{\lambda(q)}{\lambda(p)}\Big)b^2(q) \leq b^2(q) \leq b^2(p).$$

Thus, $b(p) = b(q)$, i.e. b is constant. In both cases, M is a Riemannian product. □

Theorem 3. *Let $M = B \times_b F_1 \times_b \ldots \times_b F_m$ be a multiply warped product and $\tilde{\varphi}$ be the lift of a smooth function $\varphi : B^r \to \mathbb{R}$ to M such that $(M, g, \nabla\tilde{\varphi}, \tilde{\lambda})$ is a gradient almost Ricci soliton. If $\nabla\varphi$ is a conformal vector field on B, then B is a generalized quasi-Einstein manifold.*

Proof. Assume that $\nabla\varphi$ is a conformal vector field on B, i.e. $\mathrm{Hess}_B^{\varphi} = \alpha g_B$, where $\alpha : B^r \to \mathbb{R}$ is a smooth function. From the equation (7), we have

$$^B\mathrm{Ric} + \alpha g_B = \lambda g_B + \frac{\sum_{i=1}^m s_i}{b}\mathrm{Hess}_B^b. \tag{20}$$

Letting $\gamma = \lambda - \alpha$, $\gamma : B \to \mathbb{R}$ and rearranging (20), we arrive

$$^B\mathrm{Ric} - \frac{\sum_{i=1}^m s_i}{b}\mathrm{Hess}_B^b = \gamma g_B.$$

For simplicity, we take $k = \sum_{i=1}^m s_i$. Let $f = e^{-u/k}$ so that $\nabla^2 u - (1/k)du \otimes du = -(k/f)\nabla^2 f$, we get $^B\mathrm{Ric} + \nabla^2 u - (1/k)du \otimes du = \gamma g_B$, i.e. $(B, g_B, \nabla u, \gamma)$ is a generalized quasi-Einstein manifold. □

Remark 1. *In [11], it is shown that the gradient Ricci soliton warped products consist a class of quasi-Einstein metrics. The argument in the above theorem concludes a parallel result, i.e. the gradient almost Ricci solitons on multiply warped products lead to the generalized quasi-Einstein metrics.*

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Ґюнсен С., Онат Л. *Градієнтні майже солітони Річчі на многовидах кратних викривлених добутків* // Карпатські матем. публ. — 2011. — Т.13, №2. — С. 386–394.

У цій роботі ми досліджуємо многовид кратних викривлених добутків

$$M = B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} \ldots \times_{b_m} F_m$$

як градієнтний майже солітон Річчі. Покладаючи $b_i = b$ для $1 \leq i \leq m$, доходимо висновку, що потенційне поле залежить від B. За допомогою цієї ідеї ми також отримуємо результат жорсткості і показуємо, що база є узагальненим квазі-Айнштайновим многовидом, якщо ∇b конформне.

Ключові слова і фрази: кратний викривлений добуток, градієнтний майже солітон Річчі, узагальнений квазі-Айнштайновий многовид, конформне векторне поле.