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Smooth symmetric bilinear forms on $\mathcal{L}_s(^2 l_\infty^2)$

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In [Carpathian Math. Publ. 2020, **12** (2), 340–352], the author classified the extreme points and exposed points of the unit ball of the space of symmetric bilinear forms on the space $\mathcal{L}_s(^2 l_\infty^2)$, where $\mathcal{L}_s(^2 l_\infty^2)$ is the space of symmetric bilinear forms on the plane with the supremum norm. Motivated by this paper, we classify the smooth points of the unit ball of the space of symmetric bilinear forms on $\mathcal{L}_s(^2 l_\infty^2)$.

Key words and phrases: smooth point, space of symmetric bilinear forms.

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Introduction

Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2$. We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is an $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. A point $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $\operatorname{ext} B_E, \exp B_E$ and $\operatorname{sm} B_E$ the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \ldots, x_n)|$. An n-linear form T is symmetric if $T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for every permutation σ on $\{1, \ldots, n\}$. $\mathcal{L}_s(^n E)$ denotes the closed subspace of all continuous symmetric n-linear forms on E. Notice that $\mathcal{L}(^n E)$ is identified with the dual of n-fold projective tensor product $\hat{\bigotimes}_{\pi,n} E$. With this identification, the action of a continuous n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{\pi,n} E$ is given by

$$\Big\langle \sum_{i=1}^{k} x^{(1),i} \otimes \cdots \otimes x^{(n),i},\, T \Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i}, \cdots, x^{(n),i}\Big).$$

Notice also that $\mathcal{L}_s(^n E)$ is identified with the dual of n-fold symmetric projective tensor product $\hat{\bigotimes}_{s,\pi,n} E$. With this identification, the action of a continuous symmetric n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{s,\pi,n} E$ is given by

$$\Big\langle \sum_{i=1}^{k} \frac{1}{n!} \Big(\sum_{\sigma} x^{\sigma(1),i} \otimes \cdots \otimes x^{\sigma(n),i} \Big),\, T \Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i}, \cdots, x^{(n),i}\Big),$$

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where σ goes over all permutations on $\{1,\dots,n\}$. A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a unique $T \in \mathcal{L}_s({}^nE)$ such that $P(x) = T(x,\dots,x)$ for every $x \in E$. We denote by $\mathcal{P}({}^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [3].

The main result about smooth points is known as "the Mazur density theorem". Recall that the Mazur density theorem (see [5, p. 171]) says that the set of all the smooth points of a solid closed convex subset of a separable Banach space is a residual subset of its boundary. Y.S. Choi and S.G. Kim [1, 2] initiated and characterized the smooth points of the unit balls of $\mathcal{P}({}^2l_1^2)$ and $\mathcal{P}({}^2l_2^2)$. B.C. Grecu [4] characterized the smooth 2-homogeneous polynomials on Hilbert spaces. S.G. Kim [7] classified the smooth points of the unit ball of $\mathcal{P}({}^2d_*(1,w)^2)$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm $\|(x,y)\|_w = \max\Big\{|x|, |y|, \frac{|x|+|y|}{1+w}\Big\}$ for $0 < w < 1$. S.G. Kim [6, 8, 9] classified the smooth points of the unit balls of $\mathcal{L}_s({}^2l_\infty^2)$, $\mathcal{L}_s({}^3l_\infty^2)$ and $\mathcal{L}_s({}^nl_\infty^2)$. Let $l_\infty^m = \mathbb{R}^m$ with the supremum norm. S.G. Kim [12] characterized the smooth points of the unit balls of $\mathcal{L}({}^nl_\infty^m)$ and $\mathcal{L}_s({}^nl_\infty^m)$ for $n, m \geq 2$. In [13], S.G. Kim classified the smooth points of the unit ball of $\mathcal{P}({}^2\mathbb{R}^2_{h(\frac{1}{2})})$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with the hexagonal norm $\|(x,y)\|_{h(\frac{1}{2})} = \max\Big\{|y|, |x| + \frac{1}{2}|y|\Big\}$, and in [10], he classified the extreme and exposed points of the unit ball of $\mathcal{L}_s({}^2\mathcal{L}_s({}^2l_\infty^2))$. Recently, S.G. Kim [11] classified the smooth points of the unit ball of $\mathcal{L}_s({}^nl_1)$ and $\mathcal{L}({}^nl_1)$ for $n \geq 2$.

In this paper we classify the smooth points of the unit ball of $\mathcal{L}_s({}^2\mathcal{L}_s({}^2l_\infty^2))$.

1 Results

Let $n \geq 2$. For $j = 0,\dots,n$, we let

$$F_j\Big((x_1,y_1),\dots,(x_n,y_n)\Big) = \sum_{\{l_1,\dots,l_j,k_1,\dots,k_{n-j}\}=\{1,\dots,n\}} x_{l_1}\cdots x_{l_j} y_{k_1}\cdots y_{k_{n-j}}.$$

Then,

$$\Big\{F_0\Big((x_1,y_1),\dots,(x_n,y_n)\Big),\dots,F_n\Big((x_1,y_1),\dots,(x_n,y_n)\Big)\Big\}$$

is a basis for $\mathcal{L}_s({}^nl_\infty^2)$. Hence, $\dim(\mathcal{L}_s({}^nl_\infty^2)) = n+1$. If $S \in \mathcal{L}_s({}^nl_\infty^2)$, then

$$S\Big((x_1,y_1),\dots,(x_n,y_n)\Big) = \sum_{j=0}^n a_jF_j\Big((x_1,y_1),\dots,(x_n,y_n)\Big)$$

for some $a_0,\dots,a_n \in \mathbb{R}$. By simplicity we denote $S = (a_0,\dots,a_n)^t$.

Let $\mathbb{R}^{n+1}_{\mathcal{L}_s({}^nl_\infty^2)} := \mathbb{R}^{n+1}$ with the $\mathcal{L}_s({}^nl_\infty^2)$-norm

$$\Big\|(a_0,\dots,a_n)\Big\|_{\mathcal{L}_s({}^nl_\infty^2)} : = \sup_{\|(x_k,y_k)\|_\infty=1,\ k=1,\dots,n} \Big|\sum_{j=0}^n a_jF_j((x_1,y_1),\dots,(x_n,y_n))\Big| = \Big\|(a_0,\dots,a_n)^t\Big\|.$$

We have the following identification.

Theorem 1. *For $m, n \geq 2$, the equality $\mathcal{L}_s({}^m\mathcal{L}_s({}^nl_\infty^2)) = \mathcal{L}_s({}^m\mathbb{R}^{n+1}_{\mathcal{L}_s({}^nl_\infty^2)})$ holds.*

Notice that $\operatorname{ext} B_{\mathcal{L}_s(^m\mathcal{L}_s(^n l_\infty^2))} = \operatorname{ext} B_{\mathcal{L}_s(^m\mathbb{R}^{n+1}_{\mathcal{L}_s(^n l_\infty^2)})}$, $\exp B_{\mathcal{L}_s(^m\mathcal{L}_s(^n l_\infty^2))} = \exp B_{\mathcal{L}_s(^m\mathbb{R}^{n+1}_{\mathcal{L}_s(^n l_\infty^2)})}$ and $\operatorname{sm} B_{\mathcal{L}_s(^m\mathcal{L}_s(^n l_\infty^2))} = \operatorname{sm} B_{\mathcal{L}_s(^m\mathbb{R}^{n+1}_{\mathcal{L}_s(^n l_\infty^2)})}$.

S.G. Kim [10] classified the extreme and exposed points of the unit ball of the space $\mathbb{R}^6_{\mathcal{L}_s(^2 l_\infty^2)} = \mathcal{L}_s(^2\mathbb{R}^3_{\mathcal{L}_s(^2 l_\infty^2)}) = \mathcal{L}_s(^2\mathcal{L}_s(^2 l_\infty^2))$.

Let $T \in \mathcal{L}_s(^2\mathcal{L}_s(^2 l_\infty^2))$. Then, for $(t_1,t_2,t_3),(s_1,s_2,s_3) \in \mathbb{R}^3$,

$$T((t_1,t_2,t_3),(s_1,s_2,s_3)) = at_1s_1 + bt_2s_2 + ct_3s_3 + d(t_1s_2+t_2s_1) + e(t_1s_3+t_3s_1) + f(t_2s_3+t_3s_2),$$

where

$$a = T(x_1y_1,x_1y_1),\ b = T(x_2y_2,x_2y_2),\ c = T(x_1y_2+x_2y_1,x_1y_2+x_2y_1),$$
$$d = T(x_1y_1,x_2y_2),\ e = T(x_1y_1,x_1y_2+x_2y_1),\ f = T(x_2y_2,x_1y_2+x_2y_1).$$

Notice that $\{t_1s_1,\ t_2s_2,\ t_3s_3,\ t_1s_2+t_2s_1,\ t_1s_3+t_3s_1,\ t_2s_3+t_3s_2\}$ is a basis for $\mathcal{L}_s(^2\mathcal{L}_s(^2 l_\infty^2))$ and $\dim(\mathcal{L}_s(^2\mathcal{L}_s(^2 l_\infty^2))) = 6$. For simplicity we denote $T = (a,b,c,d,e,f)^t$.

Let S be a non-empty subset of a real Banach space E. Let

$$\operatorname{conv}(S) := \Big\{ \sum_{j=1}^{k} t_j a_j : 0 \le t_j \le 1, t_1 + \cdots + t_k = 1, a_j \in S \text{ for } k \in \mathbb{N} \text{ and } 1 \le j \le k \Big\}.$$

We call $\operatorname{conv}(S)$ the convex hull of S. Recall that the Krein-Milman Theorem [14] states that every nonempty compact convex subset of a Hausdorff locally convex space is the closed convex hull of its set of extreme points. Hence, the unit ball of $\mathcal{L}_s(^2 l_\infty^2)$ is the closed convex hull of $\operatorname{ext} B_{\mathcal{L}_s(^2 l_\infty^2)}$.

The geometrical structures of the unit ball of the space $\mathcal{L}_s(^2 l_\infty^2)$ were investigated in [6]. In particular, it was shown that

$$\operatorname{ext} B_{\mathcal{L}_s(^2 l_\infty^2)} = \exp B_{\mathcal{L}_s(^2 l_\infty^2)} = \Big\{ \pm(1,0,0)^t, \pm(0,1,0)^t, \pm\Big(\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}\Big)^t \Big\}.$$

The following presents an explicit formulae for the norm of $T \in \mathcal{L}_s(^2\mathcal{L}_s(^2 l_\infty^2))$.

Theorem 2. *Let $T \in \mathcal{L}_s(^2\mathcal{L}_s(^2 l_\infty^2))$ with $T = (a,b,c,d,e,f)^t$ for some $a,b,c,d,e,f \in \mathbb{R}$. Then,*

$$\begin{aligned}\|T\| = \max\Big\{&|a|, |b|, |d|, \frac{1}{2}(|a-d|+|e|),\ \frac{1}{2}(|b-d|+|f|),\\ &\frac{1}{4}|a+b-c-2d|,\ \frac{1}{4}(|a+b+c-2d|+2|e-f|)\Big\}.\end{aligned}$$

Proof. In [6] it was shown that

$$\operatorname{ext} B_{\mathcal{L}_s(^2 l_\infty^2)} = \Big\{ \pm(1,0,0)^t, \pm(0,1,0)^t, \pm\Big(\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}\Big)^t \Big\}.$$

Let

$$\begin{aligned}
U_1 : &= \Big((1,0,0)^t, (1,0,0)^t\Big),\ U_2 := \Big((0,1,0)^t, (0,1,0)^t\Big),\\
U_3 : &= \Big((1,0,0)^t, (0,1,0)^t\Big), U_4 := \Big((1,0,0)^t, \Big(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big)^t\Big),\\
U_5 : &= \Big((1,0,0)^t, \Big(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\Big)^t\Big), U_6 := \Big((0,1,0)^t, \Big(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big)^t\Big),\\
U_7 : &= \Big((0,1,0)^t, \Big(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\Big)^t\Big), U_8 := \Big(\Big(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big)^t, \Big(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big)^t\Big),\\
U_9 : &= \Big(\Big(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big)^t, \Big(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\Big)^t\Big), U_{10} := \Big(\Big(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\Big)^t, \Big(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big)^t\Big).
\end{aligned}$$

Notice that $T(U_1) = a, T(U_2) = b$, $T(U_3) = d$, $T(U_4) = \frac{1}{2}(a-d+e)$, $T(U_5) = \frac{1}{2}(a-d-e)$, $T(U_6) = \frac{1}{2}(-b+d+f)$, $T(U_7) = \frac{1}{2}(-b+d-f)$, $T(U_8) = \frac{1}{4}(a+b+c)+\frac{1}{2}(-d+e-f)$, $T(U_9) = \frac{1}{4}(a+b-c)-\frac{1}{2}d$, $T(U_{10}) = \frac{1}{4}(a+b+c)+\frac{1}{2}(-d-e+f)$.

By the bilinearity of T and the Krein-Milman theorem,

$$\|T\| = \sup\left\{|T(L_1, L_2)| : L_1, L_2 \in \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)}\right\} = \sup_{1\le j\le 10} |T(U_j)|.$$

□

For $z_1, \cdots, z_6 \in \mathbb{R}$, we let

$$\begin{aligned}
Y_1(z_1,\cdots,z_6) &:= z_1, & X_1 &:= [1,0,0,0,0,0],\\
Y_2(z_1,\cdots,z_6) &:= z_2, & X_2 &:= [0,1,0,0,0,0],\\
Y_3(z_1,\cdots,z_6) &:= z_4, & X_3 &:= [0,0,0,1,0,0],\\
Y_4(z_1,\cdots,z_6) &:= \frac{1}{2}(z_1-z_4+z_5), & X_4 &:= \left[\frac{1}{2},0,0,-\frac{1}{2},\frac{1}{2},0\right],\\
Y_5(z_1,\cdots,z_6) &:= \frac{1}{2}(z_1-z_4-z_5), & X_5 &:= \left[\frac{1}{2},0,0,-\frac{1}{2},-\frac{1}{2},0\right],\\
Y_6(z_1,\cdots,z_6) &:= \frac{1}{2}(-z_2+z_4+z_6), & X_6 &:= \left[0,-\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2}\right],\\
Y_7(z_1,\cdots,z_6) &:= \frac{1}{2}(-z_2+z_4-z_6), & X_7 &:= \left[0,-\frac{1}{2},0,\frac{1}{2},0,-\frac{1}{2}\right],\\
Y_8(z_1,\cdots,z_6) &:= \frac{1}{4}(z_1+z_2+z_3)+\frac{1}{2}(-z_4+z_5+z_6), & X_8 &:= \left[\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right],\\
Y_9(z_1,\cdots,z_6) &:= \frac{1}{4}(z_1+z_2-z_3)-\frac{1}{2}z_4, & X_9 &:= \left[\frac{1}{4},\frac{1}{4},-\frac{1}{4},-\frac{1}{2},0,0\right],\\
Y_{10}(z_1,\cdots,z_6) &:= \frac{1}{4}(z_1+z_2+z_3)+\frac{1}{2}(-z_4-z_5+z_6), & X_{10} &:= \left[\frac{1}{4},\frac{1}{4},\frac{1}{4},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right].
\end{aligned}$$

Notice that for $j = 1, \ldots, 10$,

$$X_j \cdot T = T(U_j) = Y_j(a,b,c,d,e,f).$$

Lemma 1. *Let $T = (a,b,c,d,e,f)^t \in \mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))$ and $H \in \mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))^*$ with $\|T\| = \|H\| = |H(T)| = 1$. Suppose that $\left\|T \pm (\delta_1,\ldots,\delta_6)^t\right\| \le 1$ for some $\delta_1,\ldots,\delta_6 \in \mathbb{R}$. Then,*

$$\sum_{j=1}^{6} \delta_j v_j = 0,$$

where

$$\begin{aligned}
v_1 &:= H(t_1s_1), & v_2 &:= H(t_2s_2), & v_3 &:= H(t_3s_3)\\
v_4 &:= H(t_1s_2+t_2s_1), & v_5 &:= H(t_1s_3+t_3s_1), & v_6 &:= H(t_2s_3+t_3s_2).
\end{aligned}$$

Proof. It follows that

$$\begin{aligned}
1 &\ge \max\left\{\left|H(T+(\delta_1,\ldots,\delta_6)^t)\right|, \left|H(T-(\delta_1,\ldots,\delta_6)^t)\right|\right\}\\
&= \max\left\{\left|H(T)+H((\delta_1,\ldots,\delta_6)^t)\right|, \left|H(T)-H((\delta_1,\ldots,\delta_6)^t)\right|\right\}\\
&= 1+\left|H((\delta_1,\ldots,\delta_6)^t)\right| = 1+\left|\sum_{j=1}^{6}\delta_j v_j\right|,
\end{aligned}$$

which implies that $\sum_{j=1}^{6}\delta_j v_j = 0$. □

By Lemma 1, we can characterize $\mathrm{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Theorem 3. *Let $T = (a,b,c,d,e,f)^t \in \mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))$ with $\|T\| = 1$. Then $T \in \mathrm{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$ if and only if there is $j_0 \in \{1,\ldots,10\}$ such that*

$$|T(U_{j_0})| = 1 \ \text{ and } \ |T(U_k)| < 1 \ \text{ for all } \ k \neq j_0.$$

Proof. Necessity. Suppose that $T \in \mathrm{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$. We claim show that there is $j_0 \in \{1,\ldots,10\}$ such that $|T(U_{j_0})| = 1$ and $|T(U_k)| < 1$ for all $k \neq j_0$.

Otherwise. There are $j_1 \neq j_2 \in \{1,\ldots,10\}$ such that $|T(U_{j_1})| = |T(U_{j_2})| = 1$. For $k = 1,2$, let $H_k = \mathrm{sign}(T(U_{j_k}))\delta_{U_{j_k}}$, where $\delta_{U_{j_k}} \in \mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))^*$ is defined by $\delta_{U_{j_k}}(L) = L(U_{j_k})$ for all $L \in \mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))$. Then,

$$H_1 \neq H_2, \|H_k\| = 1 = H_k(T), \ \forall k = 1,2.$$

Hence, $T \notin \mathrm{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Sufficiency. Let $H \in \mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))^*$ be such that $H(T) = 1 = \|H\|$ with $v_1 := H(t_1s_1)$, $v_2 := H(t_2s_2)$, $v_3 := H(t_3s_3)$, $v_4 := H(t_1s_2 + t_2s_1)$, $v_5 := H(t_1s_3 + t_3s_1)$, $v_6 := H(t_2s_3 + t_3s_2)$. For simplicity we denote $H := [v_1, v_2, v_3, v_4, v_5, v_6]$.

Claim. $H = \mathrm{sign}(T(U_{j_0}))X_{j_0}$.

We divide into ten cases as follows.

Case 1. $j_0 = 1$.

It follows that

$$1 = \mathrm{sign}(T(U_1))a.$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\left\|T \pm \left(0, \frac{1}{N}, 0,0,0,0,0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0,0,\frac{1}{N},0,0,0\right)^t\right\| \leq 1,$$
$$\left\|T \pm \left(0,0,0,\frac{1}{N},0,0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0,0,0,0,\frac{1}{N},0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0,0,0,0,0,\frac{1}{N}\right)^t\right\| \leq 1.$$

By Lemma 1, $v_2 = v_3 = v_4 = v_5 = v_6 = 0$. It follows that

$$1 = av_1 = \mathrm{sign}(T(U_1))v_1,$$

which shows that $v_1 = \mathrm{sign}(T(U_1))$ and $H = \mathrm{sign}(T(U_1))X_1$. Therefore, $T \in \mathrm{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Case 2. $j_0 = 2$.

It follows that

$$1 = \mathrm{sign}(T(U_2))b.$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\left\|T \pm \left(\frac{1}{N}, 0,0,0,0,0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0,0,\frac{1}{N},0,0,0\right)^t\right\| \leq 1,$$
$$\left\|T \pm \left(0,0,0,\frac{1}{N},0,0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0,0,0,0,\frac{1}{N},0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0,0,0,0,0,\frac{1}{N}\right)^t\right\| \leq 1.$$

By Lemma 1, $v_1 = v_3 = v_4 = v_5 = v_6 = 0$. It follows that

$$1 = bv_2 = \mathrm{sign}(T(U_2))v_2,$$

which shows that $v_2 = \text{sign}(T(U_2))$ and $H = \text{sign}(T(U_2))X_2$. Therefore, $T \in \text{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Case 3. $j_0 = 3$.

It follows that
$$1 = \text{sign}(T(U_3))d.$$

By Theorem 2, there is $N \in \mathbb{N}$ such that
$$\Big\|T \pm \Big(\frac{1}{N}, 0, 0, 0, 0, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(0, \frac{1}{N}, 0, 0, 0, 0\Big)^t\Big\| \le 1,$$
$$\Big\|T \pm \Big(0, 0, \frac{1}{N}, 0, 0, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(0, 0, 0, 0, \frac{1}{N}, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(0, 0, 0, 0, 0, \frac{1}{N}\Big)^t\Big\| \le 1.$$

By Lemma 1, $v_1 = v_2 = v_3 = v_5 = v_6 = 0$. It follows that
$$1 = dv_4 = \text{sign}(T(U_3))v_4,$$

which shows that $v_4 = \text{sign}(T(U_3))$ and $H = \text{sign}(T(U_3))X_3$. Therefore, $T \in \text{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Case 4. $j_0 = 4$.

It follows that
$$1 = \text{sign}(T(U_4))\frac{1}{2}(a - d + e).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that
$$\Big\|T \pm \Big(\frac{1}{N}, 0, 0, \frac{1}{N}, 0, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(\frac{1}{N}, 0, 0, 0, -\frac{1}{N}, 0\Big)^t\Big\| \le 1,$$
$$\Big\|T \pm \Big(0, \frac{1}{N}, 0, 0, 0, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(0, 0, \frac{1}{N}, 0, 0, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(0, 0, 0, 0, 0, \frac{1}{N}\Big)^t\Big\| \le 1.$$

By Lemma 1, $v_1 = -v_4 = v_5, v_2 = v_3 = v_6 = 0$. It follows that
$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_1(a - d + e) = 2\,\text{sign}(T(U_4))v_1,$$

which shows that $v_1 = \frac{\text{sign}(T(U_4))}{2}$ and $H = \text{sign}(T(U_4))X_4$. Therefore, $T \in \text{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Case 5. $j_0 = 5$.

It follows that
$$1 = \text{sign}(T(U_5))\frac{1}{2}(a - d - e).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that
$$\Big\|T \pm \Big(\frac{1}{N}, 0, 0, \frac{1}{N}, 0, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(\frac{1}{N}, 0, 0, 0, \frac{1}{N}, 0\Big)^t\Big\| \le 1,$$
$$\Big\|T \pm \Big(0, \frac{1}{N}, 0, 0, 0, 0\Big)^t\Big\| \le 1, \quad \Big|T \pm \Big(0, 0, \frac{1}{N}, 0, 0, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(0, 0, 0, 0, 0, \frac{1}{N}\Big)^t\Big\| \le 1.$$

By Lemma 1, $v_1 = -v_4 = -v_5, v_2 = v_3 = v_6 = 0$. It follows that
$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_1(a - d - e) = 2\,\text{sign}(T(U_5))v_1,$$

which shows that $v_1 = \frac{\text{sign}(T(U_5))}{2}$ and $H = \text{sign}(T(U_5))X_5$. Therefore, $T \in \text{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Case 6. $j_0 = 6$.
It follows that

$$1 = \text{sign}(T(U_6))\frac{1}{2}(-b+d+f).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\left\|T \pm \left(0, \frac{1}{N}, 0, \frac{1}{N}, 0, 0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0, \frac{1}{N}, 0, 0, 0, \frac{1}{N}\right)^t\right\| \leq 1,$$
$$\left\|T \pm \left(\frac{1}{N}, 0, 0, 0, 0, 0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0, 0, \frac{1}{N}, 0, 0, 0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0, 0, 0, 0, \frac{1}{N}, 0\right)^t\right\| \leq 1.$$

By Lemma 1, $v_2 = -v_4 = -v_6, v_1 = v_3 = v_5 = 0$. It follows that

$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_2(b-d-f) = -2\,\text{sign}(T(U_6))v_2,$$

which shows that $v_2 = -\frac{\text{sign}(T(U_6))}{2}$ and $H = \text{sign}(T(U_6))X_6$. Therefore, $T \in \text{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l^2_\infty))}$.

Case 7. $j_0 = 7$.
It follows that

$$1 = \text{sign}(T(U_7))\frac{1}{2}(-b+d-f).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\left\|T \pm \left(0, \frac{1}{N}, 0, \frac{1}{N}, 0, 0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0, \frac{1}{N}, 0, 0, 0, -\frac{1}{N}\right)^t\right\| \leq 1,$$
$$\left\|T \pm \left(\frac{1}{N}, 0, 0, 0, 0, \right)^t\right\| \leq 1, \quad \left\|T \pm \left(0, 0, \frac{1}{N}, 0, 0, 0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(0, 0, 0, 0, \frac{1}{N}, 0\right)^t\right\| \leq 1.$$

By Lemma 1,

$$v_2 = -v_4 = v_6, v_1 = v_3 = v_5 = 0.$$

It follows that

$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_2(b-d+f) = -2\,\text{sign}(T(U_7))v_2,$$

which shows that $v_2 = -\frac{\text{sign}(T(U_7))}{2}$ and $H = \text{sign}(T(U_7))X_7$.
Therefore, $T \in \text{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l^2_\infty))}$.

Case 8. $j_0 = 8$.
It follows that

$$1 = \text{sign}(T(U_8))\Big(\frac{1}{4}(a+b+c) + \frac{1}{2}(-d+e-f)\Big).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\left\|T \pm \left(\frac{1}{N}, -\frac{1}{N}, 0, 0, 0, 0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(\frac{1}{N}, 0, -\frac{1}{N}, 0, 0, 0\right)^t\right\| \leq 1,$$
$$\left\|T \pm \left(\frac{1}{N}, 0, 0, \frac{1}{2N}, 0, 0\right)^t\right\| \leq 1, \quad \left\|T \pm \left(\frac{1}{N}, 0, 0, 0, -\frac{1}{2N}, 0\right)^t\right\| \leq 1,$$
$$\left\|T \pm \left(\frac{1}{N}, 0, 0, 0, 0, \frac{1}{2N}\right)^t\right\| \leq 1.$$

By Lemma 1, $v_2 = v_1 = v_3 = -\frac{1}{2}v_4 = \frac{1}{2}v_5 = -\frac{1}{2}v_6$. It follows that

$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_1\Big(a+b+c+2(-d+e-f)\Big) = 4\,\text{sign}(T(U_8))v_1,$$

which shows that $v_1 = \frac{\mathrm{sign}(T(U_8))}{4}$ and $H = \mathrm{sign}(T(U_8))X_8$. Therefore, $T \in \mathrm{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Case 9. $j_0 = 9$.

It follows that

$$1 = \mathrm{sign}(T(U_9))\Big(\frac{1}{4}(a+b-c) - \frac{1}{2}d\Big).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\Big\|T \pm \Big(\frac{1}{N}, -\frac{1}{N}, 0,0,0,0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(\frac{1}{N}, 0, \frac{1}{N}, 0,0,0\Big)^t\Big\| \le 1,$$
$$\Big\|T \pm \Big(\frac{1}{N}, 0,0,\frac{1}{2N}, 0,0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(0,0,0,0,\frac{1}{N}, 0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(0,0,0,0,0,\frac{1}{N}\Big)^t\Big\| \le 1.$$

By Lemma 1, $v_2 = v_1 = -v_3 = -\frac{1}{2}v_4, v_5 = v_6 = 0$. It follows that

$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_1(a+b-c-2d) = 4\,\mathrm{sign}(T(U_9))v_1,$$

which shows that $v_1 = \frac{\mathrm{sign}(T(U_9))}{4}$ and $H = \mathrm{sign}(T(U_9))X_9$. Therefore, $T \in \mathrm{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

Case 10. $j_0 = 10$.

It follows that

$$1 = \mathrm{sign}(T(U_{10}))\Big(\frac{1}{4}(a+b+c) + \frac{1}{2}(-d-e+f)\Big).$$

By Theorem 2, there is $N \in \mathbb{N}$ such that

$$\Big\|T \pm \Big(\frac{1}{N}, -\frac{1}{N}, 0,0,0,0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(\frac{1}{N}, 0, -\frac{1}{N}, 0,0,0\Big)^t\Big\| \le 1,$$
$$\Big\|T \pm \Big(\frac{1}{N}, 0,0,\frac{1}{2N}, 0,0\Big)^t\Big\| \le 1, \quad \Big\|T \pm \Big(\frac{1}{N}, 0,0,0,\frac{1}{2N}, 0\Big)^t\Big\| \le 1,$$
$$\Big\|T \pm \Big(\frac{1}{N}, 0,0,0,0,-\frac{1}{2N}\Big)^t\Big\| \le 1.$$

By Lemma 1, $v_2 = v_1 = v_3 = -\frac{1}{2}v_4 = -\frac{1}{2}v_5 = \frac{1}{2}v_6$. It follows that

$$1 = av_1 + bv_2 + cv_3 + dv_4 + ev_5 + fv_6 = v_1(a+b+c-2d-2e+2f) = 4\,\mathrm{sign}(T(U_{10}))v_1,$$

which shows that $v_1 = \frac{\mathrm{sign}(T(U_{10}))}{4}$ and $H = \mathrm{sign}(T(U_{10}))X_{10}$. Therefore, $T \in \mathrm{sm}\, B_{\mathcal{L}_s(^2\mathcal{L}_s(^2l_\infty^2))}$.

We complete the proof. $\square$

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Кім С.Ґ. *Гладкі симетричні білінійні форми на* $\mathcal{L}_s(^2l_\infty^2)$ // Карпатські матем. публ. — 2022. — Т.14, №1. — C. 20–28.

У статті [Carpathian Math. Publ. 2020, **12** (2), 340–352] автор класифікував екстремальні та виставлені точки одиничної кулі простору симетричних білінійних форм на просторі $\mathcal{L}_s(^2l_\infty^2)$, де $\mathcal{L}_s(^2l_\infty^2)$ — це простір симетричних білінійних форм на площині із супремум нормою. У продовження результатів згаданої статті ми класифікуємо неперервні точки одиничної кулі простору симетричних білінійних форм на $\mathcal{L}_s(^2l_\infty^2)$.

Ключові слова і фрази: гладка точка, простір симетричних білінійних форм.