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Asymptotic solutions of boundary value problem for singularly perturbed system of differential-algebraic equations

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This paper deals with the boundary value problem for a singularly perturbed system of differential algebraic equations of the second order. The case of simple roots of the characteristic equation is studied. The sufficient conditions for existence and uniqueness of a solution of the boundary value problem for system of differential algebraic equations are found. Technique of constructing the asymptotic solutions is developed.

Key words and phrases: boundary value problem, asymptotic solution, differential-algebraic system, singular perturbed system.

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Introduction

The boundary value problems for singularly perturbed differential equations of the second order began to be intensively studied in the middle of the last century. Note, that in R. Mises [17], O.A. Oleinik and A.I. Zhizhina [19], W. Wasow [26] theorems on the existence and uniqueness of solution $x = x(t, \varepsilon)$ of the next scalar two-pointed boundary value problem were proved

$$\varepsilon^2 x'' = f(x, x', t), \tag{1}$$

$$x(t_0, \varepsilon) = x_0, \quad x(t_1, \varepsilon) = x_1. \tag{2}$$

Moreover, it was found conditions under which $x(t, \varepsilon) \to x(t)$, $\varepsilon \to 0$, where $x(t)$ is the solution of the corresponding degenerated problem

$$f(x, x', t) = 0, \tag{3}$$

$$x(t_0, 0) = x_0, \quad x(t_1, 0) = x_1. \tag{4}$$

R.E. O'Malley [16] and J.W. Searl [22] used the method of multiple scales for construction of the asymptotic solution of the problem (1)–(2) in powers of parameter ε. An overview of the results of using this method for construction of the asymptotic solutions of the singularly perturbed differential-algebraic equations (DAEs) can be found in [14].

Another classical method of studying boundary value problems is the method of matched asymptotic expansions or the closely related method of boundary functions. According to this

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technique, the formal solution of the problem (1)–(2) can be found as a sum of a regular series and two boundary layer series [12, 24]. The presence of boundary layer series allows us to construct a uniform asymptotic solution of the problem (1)–(2) on the segment $[t_0; t_1]$. In order to prove the asymptotic properties of the constructed formal solutions the barrier function method was used [2, 13, 18, 25]. Note that under certain conditions imposed on the coefficients of the equation (1), the barrier function method can be used for the estimate the difference between the solution of the problem (1)–(2) and the solution of the corresponding degenerated problem (3)–(4), when the boundary function method is not applicable [3]. This idea allows us to study boundary value problems, for example, when their solutions are oscillating. The significant limitation of using the method of barrier functions is the impossibility of its effective application in researching of boundary value problems for systems of singularly perturbed equations [4, 6, 10].

This paper deals with the two-point boundary value problem

$$\varepsilon^2 A(t,\varepsilon)\frac{d^2x}{dt^2} = f(x,t,\varepsilon), \quad t \in [0;T], \tag{5}$$

$$x(0,\varepsilon) = x_0, \quad x(T,\varepsilon) = x_T, \tag{6}$$

where $x(t,\varepsilon)$ is an n-dimensional vector, $A(t,\varepsilon)$ is an $(n \times n)$-matrix, $f(x,t,\varepsilon)$, x_0, x_T are n-dimensional vectors with real or complex-valued elements, ε is a small parameter. Using the method of boundary functions, the formal solution of the problem (5)–(6) is constructed. Moreover, it is proved the asymptotic properties of the obtained solution.

Necessary and sufficient conditions for the existence and uniqueness of the solution of the boundary value problem for the DAEs (5) in case where $\varepsilon = 1$ was obtained in [1, 15]. The estimates for the exact solution were found as well. There was investigated the structure of the fundamental matrix in linear case [15, 20]. This result has been used for the construction of the solution of the given boundary value problem. Similar results for the DAEs with an irregular point one can found in [5].

The two-point boundary value problem for a singularly perturbed system of the first order with the identity matrix $A(t,\varepsilon)$ and special boundary conditions was considered by A.B. Vasil'eva and V.F. Butuzov [24]. Moreover, the boundary conditions for the components of the solution were agreed with the sign of the real parts of the eigenvalues of the matrix

$$\left(\frac{\partial f_i(\overline{x}_0(t),t,0)}{\partial x_j}\right)_{i,j=\overline{1,n}},$$

where $\overline{x}_0(t)$ is the solution of the equation $f(x,t,0) = 0$. We generalize the results, obtained by A.B. Vasil'eva and V.F. Butuzov to the case of the differential algebraic system (5).

The boundary value problem (5)–(6) in such form is considered for the first time.

1 Formal solutions

Assume that the following conditions are satisfied.

1. Elements of matrix $A(t,\varepsilon)$ are infinitely continuously differentiable functions with respect to variables t and ε ($A(t,\varepsilon) \in C^\infty(G)$) on the set

$$G = \{(t,\varepsilon):\ 0 \le t \le T,\ 0 \le \varepsilon \le \varepsilon_0\}.$$

2. Components of vector-function $f(x,t,\varepsilon)$ are infinitely continuously differentiable functions with respect to variables x, t and ε ($f(x,t,\varepsilon) \in C^{\infty}(K)$) on the set
$$K = \{(x,t,\varepsilon):\ \|x\| \le c,\ 0 \le t \le T,\ 0 \le \varepsilon \le \varepsilon_0\}, \quad \|x_0\| < c, \quad \|x_T\| < c.$$
3. Equation $f(x,t,0) = 0$ has the solution $x = \overline{x}_0(t)$, which satisfies the conditions:
   (i) $\overline{x}_0(t) \in C[0;T]$;
   (ii) the root $x = \overline{x}_0(t)$ is isolated on the segment $[0;T]$, that is, there is such $\eta > 0$, that $f(x,t,0) \neq 0$, when $0 < \|x - \overline{x}_0(t)\| < \eta$, $t \in [0;T]$.
4. $\det A(t,0) \equiv 0$, $t \in [0;T]$.
5. Pencil of matrices $f'_x(\overline{x}_0(t),t,0) - \lambda A(t,0)$, $t \in [0;T]$, where
$$f'_x(\overline{x}_0(t),t,0) = \left(\frac{\partial f_i(\overline{x}_0(t),t,0)}{\partial x_j}\right)_{i,j=\overline{1,n}}$$
is regular. Moreover, it has $n-1$ distinct eigenvalues.

Then there exist such nonsingular smooth matrices $P(t,\varepsilon)$, $Q(t,\varepsilon)$, that
$$P(t,\varepsilon)f'_x(\overline{x}_0(t),t,0)Q(t,\varepsilon) = \Omega(t,\varepsilon) \equiv \operatorname{diag}\{e(t,\varepsilon), W_{n-1}(t,\varepsilon)\},$$
$$P(t,\varepsilon)A(t,\varepsilon)Q(t,\varepsilon) = H(t,\varepsilon) \equiv \operatorname{diag}\{a(t,\varepsilon), I_{n-1}(t,\varepsilon)\},$$
where $e(t,0) = 1$, $W_{n-1}(t,\varepsilon) = \operatorname{diag}\{\lambda_1(t,\varepsilon), \lambda_2(t,\varepsilon), \ldots, \lambda_{n-1}(t,\varepsilon)\}$, $\lambda_i(t,\varepsilon)$, $i = \overline{1,n-1}$, are the roots of the characteristic equation
$$\det(f'_x(\overline{x}_0(t),t,0) - \lambda A(t,\varepsilon)) = 0;$$
$a(t,0) = 0$, $I_{n-1}(t,0) = I_{n-1}$, I_{n-1} is identity matrix of the $(n-1)$th order [21,23]. Without loss of generality [8,21], we can assume that
$$f'_x(\overline{x}_0(t),t,0) = \Omega(t,0), \quad A(t,0) = H(t,0).$$

Formal solution of the problem (5)–(6) we will find in the form
$$x(t,\varepsilon) = \overline{x}(t,\varepsilon) + \Pi x(\tau,\varepsilon) + Qx(\xi,\varepsilon), \tag{7}$$
where $\overline{x}(t,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \overline{x}_s(t)$ is a regular part of the asymptotics, $\Pi x(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \Pi_s x(\tau)$, $\tau = t/\varepsilon$, and $Qx(\xi,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s Q_s x(\xi)$, $\xi = (t-T)/\varepsilon$, is a singular part of the asymptotics.

Substituting representation (7) into the system (5), we get
$$\varepsilon^2 A(t,\varepsilon)\frac{d^2\overline{x}(t,\varepsilon)}{dt^2} + A(\varepsilon\tau,\varepsilon)\frac{d^2\Pi x(\tau,\varepsilon)}{d\tau^2} + A(\xi\varepsilon + T,\varepsilon)\frac{d^2 Qx(\xi,\varepsilon)}{d\xi^2}$$
$$= f(\overline{x}(t,\varepsilon) + \Pi x(\tau,\varepsilon) + Qx(\xi,\varepsilon), t, \varepsilon).$$

Then we find the functions $\overline{x}(t,\varepsilon)$, $\Pi x(\tau,\varepsilon)$, $Qx(\xi,\varepsilon)$, solving the following systems
$$\varepsilon^2 A(t,\varepsilon)\frac{d^2\overline{x}}{dt^2} = \overline{f}(t,\varepsilon), \tag{8}$$
$$A(\varepsilon\tau,\varepsilon)\frac{d^2\Pi x}{d\tau^2} = \Pi f(\tau,\varepsilon), \tag{9}$$
$$A(\xi\varepsilon + T,\varepsilon)\frac{d^2 Qx}{d\xi^2} = Qf(\xi,\varepsilon), \tag{10}$$

where

$$\begin{aligned}\overline{f}(t,\varepsilon) &= f(\overline{x}(t,\varepsilon),t,\varepsilon),\\ \Pi f(\tau,\varepsilon) &= f(\overline{x}(\varepsilon\tau,\varepsilon)+\Pi x(\tau,\varepsilon),\varepsilon\tau,\varepsilon) - f(\overline{x}(\varepsilon\tau,\varepsilon),\varepsilon\tau,\varepsilon),\\ Qf(\xi,\varepsilon) &= f(\overline{x}(\xi\varepsilon+T,\varepsilon)+Qx(\xi,\varepsilon),\xi\varepsilon+T,\varepsilon) - f(\overline{x}(\xi\varepsilon+T,\varepsilon),\xi\varepsilon+T,\varepsilon).\end{aligned}$$

Let

$$\overline{f}(t,\varepsilon) = \sum_{s=0}^{\infty}\varepsilon^s\overline{f}_s(t), \quad \Pi f(\tau,\varepsilon) = \sum_{s=0}^{\infty}\varepsilon^s\Pi_s f(\tau), \quad Qf(\xi,\varepsilon) = \sum_{s=0}^{\infty}\varepsilon^s Q_s f(\xi).$$

Here, in particular

$$\begin{aligned}\overline{f}_0(t) &= f(\overline{x}_0(t),t,0),\ \Pi_0 f(\tau) = f(\overline{x}_0(0)+\Pi_0 x(\tau),0,0) - f(\overline{x}_0(0),0,0),\\ Q_0 f(\xi) &= f(\overline{x}_0(T)+Q_0x(\xi),T,0) - f(\overline{x}_0(T),T,0),\\ \overline{f}_s(t) &= f'_x(\overline{x}_0(t),t,0)\overline{x}_s(t) + \overline{g}_s(t),\\ \Pi_s f(\tau) &= f'_x(\overline{x}_0(0)+\Pi_0x(\tau),0,0)\Pi_s x(\tau) + g_s(\tau),\\ Q_s f(\xi) &= f'_x(\overline{x}_0(T)+Q_0x(\xi),T,0)Q_s x(\xi) + h_s(\xi), \quad s\in N,\end{aligned}$$

the functions $\overline{g}_s(t)$, $g_s(\tau)$ and $h_s(\xi)$ are expressed recursively through $\overline{x}_k(t)$, $\Pi_k x(\tau)$ and $Q_k(\xi)$, $k < s$.

Note that, generally speaking,

$$f(\overline{x}(t,\varepsilon)+\Pi x(\tau,\varepsilon)+Qx(\xi,\varepsilon),t,\varepsilon) \neq \overline{f}(t,\varepsilon)+\Pi f(\tau,\varepsilon)+Qf(\xi,\varepsilon). \quad (11)$$

But since in a neighborhood of $t = 0$ function $Qx(\xi,\varepsilon)$ should be as small as, like a function $\Pi x(\tau,\varepsilon)$ in a neighborhood of $t = T$ [24], then (11) can be considered as an approximate equality.

Substituting (7) in boundary conditions (6), we can write

$$\overline{x}(0,\varepsilon)+\Pi x(0,\varepsilon) = x_0, \overline{x}(T,\varepsilon)+Qx(0,\varepsilon) = x_T. \quad (12)$$

Assume that the following conditions are satisfied:

$$\begin{aligned}A(t,\varepsilon) &= \sum_{s=0}^{\infty}\varepsilon^s A_s(t) \equiv \sum_{s=0}^{\infty}\varepsilon^s\frac{1}{s!}\frac{\partial^s A(t,0)}{\partial\varepsilon^s},\\ A(\varepsilon\tau,\varepsilon) &= \sum_{s=0}^{\infty}\varepsilon^s\Pi_s A(\tau) \equiv \sum_{s=0}^{\infty}\varepsilon^s\sum_{i=0}^{s}\frac{\tau^{s-i}}{i!(s-i)!}\frac{\partial^s A(0,0)}{\partial t^{s-i}\partial\varepsilon^i},\\ A(\xi\varepsilon+T,\varepsilon) &= \sum_{s=0}^{\infty}\varepsilon^s Q_s A(\xi) \equiv \sum_{s=0}^{\infty}\varepsilon^s\sum_{i=0}^{s}\frac{\xi^{s-i}}{i!(s-i)!}\frac{\partial^s A(T,0)}{\partial t^{s-i}\partial\varepsilon^i}.\end{aligned}$$

Let us equate coefficients at the similar powers of ε in (8)–(10). For the leading terms of the asymptotics ($\overline{x}_0(t)$, $\Pi_0x(\tau)$ and $Q_0x(\xi)$), we obtain

$$f(\overline{x}_0(t),t,0) = 0,$$

$$A(0,0)\frac{d^2\Pi_0 x}{d\tau^2} = f(\overline{x}_0(0)+\Pi_0 x,0,0) - f(\overline{x}_0(0),0,0), \quad (13)$$

$$A(T,0)\frac{d^2 Q_0 x}{d\xi^2} = f(\overline{x}_0(T)+Q_0 x,T,0) - f(\overline{x}_0(T),T,0). \quad (14)$$

In view of condition 3 systems (13)–(14) will have the form

$$A(0,0)\frac{d^2\Pi_0 x}{d\tau^2} = f(\overline{x}_0(0) + \Pi_0 x, 0, 0),$$
$$A(T,0)\frac{d^2 Q_0 x}{d\xi^2} = f(\overline{x}_0(T) + Q_0 x, T, 0).$$

Let us denote through $\Pi_{01}x$, $f_1(\overline{x}_0(0) + \Pi_0 x, 0, 0)$, $Q_{01}x$, $\overline{x}_{01}(t)$, x_{01} and x_{T1} the first components of the vectors $\Pi_0 x$, $f(\overline{x}_0(0) + \Pi_0 x, 0, 0)$, $Q_0 x$, $\overline{x}_0(t)$, x_0 and x_T respectively, and through $\Pi_{02}x$, $f_2(\overline{x}_0(0) + \Pi_0 x, 0, 0)$, $Q_{02}x$, $\overline{x}_{02}(t)$, x_{02} and x_{T2} we denote the vectors, containing other components of vectors $\Pi_0 x$, $f(\overline{x}_0(0) + \Pi_0 x, 0, 0)$, $Q_0 x$, $\overline{x}_0(t)$, x_0 and x_T.

Further we make the following assumptions.

6. The equations $f_1(\overline{x}_0(0) + \Pi_0 x, 0, 0) = 0$ and $f_1(\overline{x}_0(T) + Q_0 x, T, 0) = 0$ have the solutions $\Pi_{01}x = \Pi_{01}x(\Pi_{02}x)$ and $Q_{01}x = Q_{01}x(Q_{02}x)$, which are continuous in the field of parameters change $\Pi_{02}x$, $Q_{02}x$ respectively, and

$$\Pi_{01}x(x_{02} - \overline{x}_{02}(0)) = x_{01} - \overline{x}_{01}(0); \quad \Pi_{01}x(\Pi_{02}x) \to 0, \quad \Pi_{02}x \to 0,$$
$$Q_{01}x(x_{T2} - \overline{x}_{02}(T)) = x_{T1} - \overline{x}_{01}(T); \quad Q_{01}x(Q_{02}x) \to 0, \quad Q_{02}x \to 0.$$

7. The problems

$$\frac{d^2\Pi_{02}x}{d\tau^2} = f_2(\overline{x}_0(0) + \Pi_0 x, 0, 0), \tag{15}$$
$$\Pi_{02}x(0) = x_{02} - \overline{x}_{02}(0); \quad \Pi_{02}x(\tau) \to 0, \quad \tau \to \infty,$$

and

$$\frac{d^2 Q_{02}x}{d\xi^2} = f_2(\overline{x}_0(T) + Q_0 x, T, 0),$$
$$Q_{02}x(0) = x_{T2} - \overline{x}_{02}(T); \quad Q_{02}x(\xi) \to 0, \quad \xi \to -\infty,$$

have such solutions $\Pi_{02}x = \Pi_{02}x(\tau)$, $Q_{02}x = Q_{02}x(\xi)$, that

$$\|\overline{x}_0(t) + \Pi_0 x(t/\varepsilon)\| < c, \quad t \in [0;T],$$

and

$$\|\overline{x}_0(t) + Q_0 x((t-T)/\varepsilon)\| < c, \quad t \in [0;T].$$

8. $\{f'_x(\overline{x}_0(0) + \Pi_0 x(\tau), 0, 0)\}_{11} \neq 0$, $\tau \geq 0$; $\{f'_x(\overline{x}_0(T) + Q_0 x(\xi), T, 0)\}_{11} \neq 0$, $\xi \leq 0$, where, for example, $\{f'_x(\overline{x}_0(0) + \Pi_0 x(\tau), 0, 0)\}_{11}$ is a corresponding element of the matrix $f'_x(\overline{x}_0(0) + \Pi_0 x(\tau), 0, 0)$.

9. $\operatorname{Re}\lambda_i(t,0) > 0$, $t \in [0;T]$, $i = \overline{1, n-1}$.

Then we can assume that $\operatorname{Re}\sqrt{\lambda_i(t,0)} > 0$, $t \in [0;T]$, $i = \overline{1, n-1}$. From the Conditions 6–8 it follows, that $\Pi_0 x(\tau) \in C^\infty[0;\infty)$ and $Q_0 x(\xi) \in C^\infty(-\infty;0]$ [9].

Let us show that there are constants α_0, c_0, for which

$$\|\Pi_0 x(\tau)\| \leq c_0 \exp(-\alpha_0 \tau), \quad \tau \geq 0.$$

For this purpose we will represent the system (15) in the following form

$$\frac{d^2\Pi_{02}x}{d\tau^2} = W_{n-1}(0,0)\Pi_{02}x + G_2(\Pi_0 x),$$

where $G_2(\Pi_0 x) = f_2(\overline{x}_0(0) + \Pi_0 x, 0, 0) - f'_{2x}(\overline{x}_0(0), 0, 0)\Pi_{02}x$, $f'_{2x} = (\partial f_{2i}/\partial x_j)_{i,j=\overline{2,n}}$. Note that $G_2(0) = 0$.

Suppose that $\tau \geq \tau_0$, where τ_0 will be defined below. The solution of the problem

$$\begin{aligned} &\frac{d^2 y_i}{d\tau^2} - \lambda_i(0,0) y_i = \{G_2(y)\}_i, \\ &y_i(\tau_0) = \{\Pi_{02}x(\tau_0)\}_i; \quad y_i(\tau) \to 0, \quad \tau \to \infty, \end{aligned} \tag{16}$$

$i = \overline{2,n}$, satisfies the integral equation

$$\begin{aligned} y_i(\tau) = e^{-\sqrt{\lambda_i(0,0)}\,(\tau-\tau_0)} &\left(\{\Pi_{02}x(\tau_0)\}_i - e^{\sqrt{\lambda_i(0,0)}\,\tau_0} \int_{\infty}^{\tau_0} \frac{\{G_2(y)\}_i}{2\sqrt{\lambda_i(0,0)}} e^{-\sqrt{\lambda_i(0,0)}\,s} ds \right) \\ - e^{-\sqrt{\lambda_i(0,0)}\,\tau} \int_{\tau_0}^{\tau} \frac{\{G_2(y)\}_i}{2\sqrt{\lambda_i(0,0)}} e^{\sqrt{\lambda_i(0,0)}\,s} ds &+ e^{\sqrt{\lambda_i(0,0)}\,\tau} \int_{\infty}^{\tau} \frac{\{G_2(y)\}_i}{2\sqrt{\lambda_i(0,0)}} e^{-\sqrt{\lambda_i(0,0)}\,s} ds. \end{aligned} \tag{17}$$

According to the Lagrange finite-increments formula we get $\|G_2(y) - G_2(0)\| \leq \delta\|y\|$, where $\delta = \delta(y(\tau))$, $\delta(y(\tau)) \to 0$, $\tau \to \infty$.

Consider an equation

$$z_i = \Phi_i y_i, \quad i = \overline{2,n}, \tag{18}$$

where the operator Φ_i is determined by the right-hand side of the formula (17) on the set

$$T = \{y_i(\tau) \in C[\tau_0; +\infty) : \ |y_i(\tau)| \leq c_1 \exp(-\alpha_0(\tau - \tau_0))\}, \quad 0 < \alpha_0 < \operatorname{Re}\sqrt{\lambda_i(0,0)},\ i = \overline{2,n}.$$

For a sufficiently large τ_0 the mapping $z_i = \Phi_i y_i$ is a contraction mapping of the set T into itself. That is why the equation (18) has a unique solution on the set T [11]. Therefore, $|y_i(\tau)| \leq c_1 \exp(-\alpha_0(\tau - \tau_0))$, $\tau \geq \tau_0$.

For $0 \leq \tau \leq \tau_0$ the solution of the equation (16) satisfying the condition $y_i(0) = \{x_{02}\}_i - \{\overline{x}_{02}(0)\}_i$ is bounded by some constant c_2, $|y_i(\tau)| \leq c_2$, $0 \leq \tau \leq \tau_0$.

We set $c_0 = \max\{c_1 \exp(\alpha_0\tau_0), c_2 \exp(\alpha_0\tau_0)\}$. Then according to the construction we get $|y_i(\tau)| \leq c_0 \exp(-\alpha_0\tau)$, $\tau \geq 0$, $i = \overline{2,n}$. Thus, $\|\Pi_0 x(\tau)\| \leq c_0 \exp(-\alpha_0\tau)$, $\tau \geq 0$.

Similarly, we prove the existence of such a constant β_0, $0 < \beta_0 < \operatorname{Re}\sqrt{\lambda_i(0,0)}$, $i = \overline{2,n}$, that $\|Q_0 x(\xi)\| \leq c_0 \exp(\beta_0\xi)$, $\xi \leq 0$.

Equating the coefficients of like powers of ε in the equations (8)–(10), we obtain

$$f'_x(\overline{x}_0(t), t, 0)\overline{x}_s = \sum_{i=0}^{s-2} A_i(t) \frac{d^2\overline{x}_{s-i-2}(t)}{dt^2} - \overline{g}_s(t),$$

$$A(0,0)\frac{d^2\Pi_s x}{d\tau^2} = f'_x(\overline{x}_0(0) + \Pi_0 x(\tau), 0, 0)\Pi_s x(\tau) + r_s(\tau), \tag{19}$$

$$A(T,0)\frac{d^2 Q_s x}{d\xi^2} = f'_x(\overline{x}_0(T) + Q_0 x(\xi), T, 0)Q_s x(\xi) + q_s(\xi), \tag{20}$$

where $\overline{x}_i(t) \equiv 0$, $t \in [0;T]$, $i < 0$,

$$r_s(\tau) = g_s(\tau) - \sum_{i=1}^{s} \Pi_i A(\tau) \frac{d^2 \Pi_{s-i} x(\tau)}{d\tau^2},$$
$$q_s(\xi) = h_s(\xi) - \sum_{i=1}^{s} Q_i A(\xi) \frac{d^2 Q_{s-i} x(\xi)}{d\xi^2}.$$

From the Conditions 5 and 9 it follows, that $\det f'_x(\overline{x}_0(t), t, 0) \neq 0$, $t \in [0;T]$. That is why

$$\overline{x}_s(t) = (f'_x(\overline{x}_0(t), t, 0))^{-1} \Big(\sum_{i=0}^{s-2} A_i(t) \frac{d^2 \overline{x}_{s-i-2}(t)}{dt^2} - \overline{g}_s(t) \Big), \quad s \in N.$$

We set

$$f'_x(\overline{x}_0(0) + \Pi_0 x(\tau), 0, 0) = \begin{pmatrix} C_1(\tau) & C_2(\tau) \\ C_3(\tau) & C_4(\tau) \end{pmatrix},$$

where $C_4(\tau)$ is the square matrix of the $(n-1)$th order. Note that

$$f'_x(\overline{x}_0(0) + \Pi_0 x(\tau), 0, 0) \to f'_x(\overline{x}_0(0), 0, 0), \quad \tau \to \infty.$$

Let us consider the system (19):

$$\Pi_{s1} x = -\frac{1}{C_1(\tau)} (C_2(\tau) \Pi_{s2} x + r_{s1}(\tau)),$$
$$\frac{d^2 \Pi_{s2} x}{d\tau^2} = \Big(C_4(\tau) - \frac{C_3(\tau) C_2(\tau)}{C_1(\tau)} \Big) \Pi_{s2} x(\tau) + \frac{C_3(\tau) r_{s1}(\tau)}{C_1(\tau)} + r_{s2}(\tau), \tag{21}$$

where $\Pi_{s1} x$, $r_{s1}(\tau)$ are the first components of the vectors $\Pi_s x$, $r_s(\tau)$, and $\Pi_{s2} x$, $r_{s2}(\tau)$ are the rest components of the vectors $\Pi_s x$, $r_s(\tau)$.

Equating the coefficients at the similar powers of ε in the first identity (12), we get initial conditions for the solution of the system (21)

$$\Pi_{s2} x(0) = -\overline{x}_{s2}(0). \tag{22}$$

Moreover, the functions $\Pi_s x(\tau)$ are boundary functions. That is $\Pi_s x(\tau) \to 0$, $\tau \to \infty$.

Thinking as in the case of proving an exponential estimate for $\Pi_0 x(\tau)$, using the method of mathematical induction, we show that the problem (21)–(22) has a solution $\Pi_{s2} x = \Pi_{s2} x(\tau)$ for which $\|\Pi_{s2} x(\tau)\| \leq c_0 \exp(-\alpha_0 \tau)$, $\tau \geq 0$. According to the construction of the solution we get the estimate

$$\|\Pi_s x(\tau)\| \leq c_0 \exp(-\alpha_0 \tau), \quad \tau \geq 0, \quad s \in N,$$

where constants c_0, α_0 are different than in previous estimates.

10. Suppose the following $\overline{x}_{s1}(0) + \Pi_{s1} x(0) = 0$, $s \in N$.

Then $\overline{x}_s(0) + \Pi_s x(0) = 0$, $s \in N$.

Remark. *For example, the conditions*

$$\frac{\partial^s a(0,0)}{\partial \varepsilon^s} = 0, \quad \frac{\partial^s f_1(x_0, 0, 0)}{\partial \varepsilon^s} = 0, \quad s \in N.$$

are sufficient for the fulfilling the condition 10.

Let us consider the system (20) with the initial conditions

$$Q_{s2}x(0) = -\overline{x}_{s2}(T), \quad Q_s x(\xi) \to 0, \quad \xi \to -\infty.$$

11. Suppose, that $\overline{x}_{s1}(T) + Q_{s1}x(0) = 0$, $s \in N$.

We prove that the system (20) has the solution $Q_s x = Q_s x(\xi)$ such that $\overline{x}_s(T) + Q_s x(0) = 0$, $s \in N$. Besides,

$$\|Q_s x(\xi)\| \leq c_0 \exp(\beta_0 \xi), \quad \xi \leq 0, \quad s \in N.$$

2 Investigation of the asymptotic behaviour of the constructed formal solution

Let us prove that the constructed formal solution of the problem (5)–(6) has the asymptotic properties. For this purpose we make the substitution $x(t,\varepsilon) = x_m(t,\varepsilon) + y(t,\varepsilon)$ in system (5), where $x_m(t,\varepsilon) = \sum_{s=0}^{m} \varepsilon^s (\overline{x}_s(t) + \Pi_s x(\tau) + Q_s(\xi))$, and $y(t,\varepsilon)$ is a new unknown function. Then system (5) can be written in the form

$$\varepsilon^2 A(t,\varepsilon)\frac{d^2 y}{dt^2} = f(x_m(t,\varepsilon) + y, t, \varepsilon) - \varepsilon^2 A(t,\varepsilon)\frac{d^2 x_m(t,\varepsilon)}{dt^2}. \tag{23}$$

Then boundary conditions for the system (23) take the form

$$y(0,\varepsilon) = -\sum_{s=0}^{m} \varepsilon^s Q_s x(-T/\varepsilon), \quad y(T,\varepsilon) = -\sum_{s=0}^{m} \varepsilon^s \Pi_s x(T/\varepsilon).$$

Thus, we can state that

$$y(0,\varepsilon) = O(\varepsilon^{m+1}), \quad y(T,\varepsilon) = O(\varepsilon^{m+1}), \quad \varepsilon \to 0+. \tag{24}$$

We put

$$\overline{x}_m(t,\varepsilon) = \sum_{s=0}^{m} \varepsilon^s \overline{x}_s(t), \quad \Pi_m x(\tau,\varepsilon) = \sum_{s=0}^{m} \varepsilon^s \Pi_s(\tau), \quad Q_m(\xi,\varepsilon) = \sum_{s=0}^{m} \varepsilon^s Q_s(\xi).$$

Then

$$\begin{aligned} f(\overline{x}_m(t,\varepsilon) + \Pi_m x(\tau,\varepsilon) + Q_m x(\xi,\varepsilon), t, \varepsilon) &= f(\overline{x}_m(t,\varepsilon), t, \varepsilon) + (f(\overline{x}_m(t,\varepsilon) + \Pi_m x(\tau,\varepsilon), t, \varepsilon) \\ &- f(\overline{x}_m(t,\varepsilon), t, \varepsilon)) + (f(\overline{x}_m(t,\varepsilon) + Q_m x(\xi,\varepsilon), t, \varepsilon) - f(\overline{x}_m(t,\varepsilon), t, \varepsilon)) \\ &+ f(\overline{x}_m(t,\varepsilon) + \Pi_m x(\tau,\varepsilon) + Q_m x(\xi,\varepsilon), t, \varepsilon) - f(\overline{x}_m(t,\varepsilon) + \Pi_m x(\tau,\varepsilon), t, \varepsilon) \\ &- f(\overline{x}_m(t,\varepsilon) + Q_m x(\xi,\varepsilon), t, \varepsilon) + f(\overline{x}_m(t,\varepsilon), t, \varepsilon). \end{aligned}$$

Considering this expression on the segments $[0; T/2]$ and $[T/2; T]$, we obtain

$$\begin{aligned} f(\overline{x}_m(t,\varepsilon) + \Pi_m x(\tau,\varepsilon) + Q_m x(\xi,\varepsilon), t, \varepsilon) &= f(\overline{x}_m(t,\varepsilon), t, \varepsilon) + (f(\overline{x}_m(t,\varepsilon) + \Pi_m x(\tau,\varepsilon), t, \varepsilon) \\ &- f(\overline{x}_m(t,\varepsilon), t, \varepsilon)) + +(f(\overline{x}_m(t,\varepsilon) + Q_m x(\xi,\varepsilon), t, \varepsilon) - f(\overline{x}_m(t,\varepsilon), t, \varepsilon)) + O(\varepsilon^{m+1}) \\ &= \sum_{s=0}^{m} \varepsilon^s (\overline{f}_s(t) + \Pi_s f(\tau) + Q_s f(\xi)) + O(\varepsilon^{m+1}). \end{aligned}$$

Thus, the system (23) can be written as

$$\varepsilon^2 A(t,\varepsilon)\frac{d^2y}{dt^2} = f'_x(\overline{x}_0(t),t,0)y + g(y,t,\varepsilon),$$

where $g(y,t,\varepsilon) = f(x_m(t,\varepsilon)+y,t,\varepsilon) - f(x_m(t,\varepsilon),t,\varepsilon) - f'_x(\overline{x}_0(t),t,0)y + O(\varepsilon^{m+1})$. Note, that

$$\|g(y^{(1)},t,\varepsilon) - g(y^{(2)},t,\varepsilon)\| \le d_1(\varepsilon + \exp(-\alpha_0 t/\varepsilon) + \exp(\beta_0(t-T)/\varepsilon))\|y^{(1)} - y^{(2)}\| \tag{25}$$

and

$$\|g(0,t,\varepsilon)\| \le d_2\varepsilon^{m+1}, \quad t \in [0;T], \tag{26}$$

for all $y^{(1)}, y^{(2)} \in D_{m+1}$, $D_{m+1} = \{y(t,\varepsilon) \in C[0;T] : \ \|y(t,\varepsilon)\| \le k\varepsilon^{m+1}\}$.

Suppose the following

$$y(t,\varepsilon) = z(t,\varepsilon) + \frac{\varphi(\varepsilon)}{T}(T-t) + \frac{\psi(\varepsilon)}{T}t,$$

where the functions $\varphi(\varepsilon) = y(0,\varepsilon)$, $\psi(\varepsilon) = y(T,\varepsilon)$ are defined from the conditions (24). Then we have

$$\varepsilon^2 A(t,\varepsilon)\frac{d^2z}{dt^2} = f'_x(\overline{x}_0(t),t,0)z + q(z,t,\varepsilon), \tag{27}$$

$$z(0,\varepsilon) = 0, \quad z(T,\varepsilon) = 0. \tag{28}$$

It should be noted, that function $q(z,t,\varepsilon)$ has asymptotic estimates (25), (26).

Setting $z(t,\varepsilon) = Q(t,\varepsilon)u(t,\varepsilon)$, the problem (27)–(28) can be represented in the form

$$\varepsilon^2 H(t,\varepsilon)\frac{d^2u}{dt^2} = \Omega(t,\varepsilon)u + r(u,t,\varepsilon), \tag{29}$$

$$u(0,\varepsilon) = 0, \quad u(T,\varepsilon) = 0, \tag{30}$$

where $r(u,t,\varepsilon) = P(t,\varepsilon)q(Q(t,\varepsilon)u,t,\varepsilon) - \varepsilon^2 H(t,\varepsilon)Q^{-1}(t,\varepsilon)(Q''(t,\varepsilon)u + 2Q'(t,\varepsilon)u')$. Note, that $P(t,0) = E_n$, $Q(t,0) = E_n$ and $Q'(t,0) = 0$ [21,23].

12. Suppose that $a(t,\varepsilon) = \varepsilon^s a_1(t,\varepsilon)$, $s \in N$, where $\operatorname{Re} a_1(t,0) > 0$, $t \in [0;T]$. We also suppose, that $u_i^{(1)}(t,\varepsilon)$ and $u_i^{(2)}(t,\varepsilon)$ are the linearly independent solutions of the equation

$$\varepsilon^2 h_i(t,\varepsilon)\frac{d^2u_i}{dt^2} = \omega_i(t,\varepsilon)u_i, \tag{31}$$

where $h_i(t,\varepsilon)$, $\omega_i(t,\varepsilon)$ are diagonal matrix elements $H(t,\varepsilon)$ and $\Omega(t,\varepsilon)$.

According to the construction of the solutions we have [7]

$$u_i^{(1)}(t,\varepsilon) = \theta_i^{-1/4}(t,\varepsilon)\exp\Big(-\frac{1}{\varepsilon}\int_0^t \sqrt{\theta_i(t,\varepsilon)}\,dt\Big)e_i^{(1)}(t,\varepsilon),$$

$$u_i^{(2)}(t,\varepsilon) = \theta_i^{-1/4}(t,\varepsilon)\exp\Big(\frac{1}{\varepsilon}\int_0^t \sqrt{\theta_i(t,\varepsilon)}\,dt\Big)e_i^{(2)}(t,\varepsilon),$$

where $\theta_i(t,\varepsilon) = \omega_i(t,\varepsilon)/h_i(t,\varepsilon)$, $i = \overline{1,n}$; $e_i^{(j)}(t,\varepsilon) = 1 + O(\varepsilon)$, $\varepsilon \to 0+$, $j = 1,2$.

The functions

$$v_i^{(1)}(t,\varepsilon) = u_i^{(1)}(t,\varepsilon)e_i^{(2)}(0,\varepsilon) - u_i^{(2)}(t,\varepsilon)e_i^{(1)}(0,\varepsilon),$$
$$v_i^{(2)}(t,\varepsilon) = u_i^{(1)}(t,\varepsilon)e_i^{(2)}(T,\varepsilon) - u_i^{(2)}(t,\varepsilon)\exp\Big(-\frac{2}{\varepsilon}\int_0^T \sqrt{\theta_i(t,\varepsilon)}\,dt\Big)e_i^{(1)}(T,\varepsilon)$$

are the solutions of the equation (43) and

$$v_i^{(1)}(0,\varepsilon) = 0, \quad v_i^{(2)}(T,\varepsilon) = 0. \tag{32}$$

The Wronskian of the functions $v_i^{(1)}(t,\varepsilon), v_i^{(2)}(t,\varepsilon)$ is $\Delta(t,\varepsilon) = (2/\varepsilon)(1+O(\varepsilon)), \varepsilon \to 0+$.

Then the solution of the problem (29)–(30) satisfies the system of integral equations

$$u_1(t,\varepsilon) = \frac{1}{\varepsilon^{2+s}}\int_0^T G_1(t,s,\varepsilon)r_1(u,s,\varepsilon)\,ds, \tag{33}$$
$$u_i(t,\varepsilon) = \frac{1}{\varepsilon^{2}}\int_0^T G_i(t,s,\varepsilon)r_i(u,s,\varepsilon)\,ds, \quad i=\overline{2,n}, \tag{34}$$

where

$$G_i(t,s,\varepsilon) = \frac{1}{\Delta(t,\varepsilon)}\begin{cases} v_i^{(1)}(t,\varepsilon)v_i^{(2)}(s,\varepsilon), & 0\le t\le s,\\ v_i^{(2)}(t,\varepsilon)v_i^{(1)}(s,\varepsilon), & s\le t\le T,\end{cases}$$

is the Green's function for the boundary-value problem (31)–(32). According to the construction we get

$$|G_i(t,s,\varepsilon)| \le \frac{\varepsilon}{2}\,\theta_i^{-1/4}(t,\varepsilon)\theta_i^{-1/4}(s,\varepsilon)\exp\Big(\frac{1}{\varepsilon}\int_s^t \sqrt{\theta_i(t,\varepsilon)}\,dt\Big)(1+O(\varepsilon)),\ 0\le t\le s,$$

and

$$|G_i(t,s,\varepsilon)| \le \frac{\varepsilon}{2}\,\theta_i^{-1/4}(t,\varepsilon)\theta_i^{-1/4}(s,\varepsilon)\exp\Big(\frac{1}{\varepsilon}\int_t^s \sqrt{\theta_i(t,\varepsilon)}\,dt\Big)(1+O(\varepsilon)), \quad s\le t\le T.$$

Let d_3 and d_4 be constants such that

$$\varepsilon^{s/2}\,\mathrm{Re}\sqrt{\theta_1(t,\varepsilon)} \ge d_3 > 0, \quad \varepsilon^{-s/2}|\theta_1^{-1/4}(t,\varepsilon)\theta_1^{-1/4}(s,\varepsilon)| \le d_4$$

and

$$\mathrm{Re}\sqrt{\theta_i(t,\varepsilon)} \ge d_3 > 0, \quad |\theta_i^{-1/4}(t,\varepsilon)\theta_i^{-1/4}(s,\varepsilon)| \le d_4, \quad i=\overline{2,n}; \quad t,s\in[0;T].$$

13. Suppose that $d_1d_4 < 2d_3$.

For sufficiently large k the operator

$$T\varphi = \frac{1}{\varepsilon^2}\int_0^T \widetilde{G}(t,s,\varepsilon)r(\varphi,s,\varepsilon)ds,$$

where $\widetilde{G}(t,s,\varepsilon) = \mathrm{diag}\{(1/\varepsilon^s)G_1(t,s,\varepsilon), G_2(t,s,\varepsilon),\dots,G_n(t,s,\varepsilon)\}$, maps the set D_{m+1} into itself. This mapping is a contraction mapping. Consequently, the systems (33)–(34) has one and only one solution on the set D_{m+1}. That is why the problem (29)–(30) has unique solution $u = u(t,\varepsilon)$ as well. Besides, $\|u(t,\varepsilon)\| \le k\varepsilon^{m+1}$, $t\in[0;T]$.

Thus, the main result of the paper can be formulated as follows.

Theorem 1. *If $A(t,\varepsilon) \in C^{m+1}(G)$, $f(x,t,\varepsilon) \in C^{m+1}(K)$ and the assumptions 3–13 are satisfied, then there exists a unique solution $x = x(t,\varepsilon)$ of the boundary-value problem* (5)–(6) *for sufficiently small ε, $0<\varepsilon\le\varepsilon_1\le\varepsilon_0$, such that*

$$\|x(t,\varepsilon) - x_m(t,\varepsilon)\| = O(\varepsilon^{m+1}), \quad t\in[0;T], \quad \varepsilon\to 0+. \tag{35}$$

3 The special case

Suppose, that

$$f(x,t,\varepsilon) = B(t)(x-\psi(t)) + \varepsilon\widetilde{f}(x,t,\varepsilon), \tag{36}$$

where $B(t)$ is an $(n\times n)$-matrix [2,25]. Thus, $f'_x(x,t,0) = B(t)$ and $\overline{x}_0(t) = \psi(t)$.

14. Assume, that the pencil $B(t) - \mu A(t,0)$, $t \in [0;T]$, is regular and it has $n-1$ distinct eigenvalues.

Then, without loss of generality, we can state that $B(t) = \Omega(t,0)$, $A(t,0) = H(t,0)$. According to (36) the equations $f_1(\overline{x}_0(0) + \Pi_0 x(\tau), 0, 0) = 0$ and $f_1(\overline{x}_0(T) + Q_0 x(\xi), T, 0) = 0$ are solvable for $\Pi_{01}x(\tau)$ and $Q_{01}x(\xi)$ respectively, and $\Pi_{01}x = 0$, $Q_{01}x = 0$.

15. Suppose, that $x_{01} = \overline{x}_{01}(0)$ and $x_{T1} = \overline{x}_{01}(T)$.

Note, that in this case constant d_1 is determined from inequality

$$\|g(y^{(1)},t,\varepsilon) - g(y^{(2)},t,\varepsilon)\| \le \varepsilon d_1 \|y^{(1)} - y^{(2)}\|.$$

Theorem 2. *If $A(t,\varepsilon) \in C^{m+1}(G)$, $f(x,t,\varepsilon) \in C^{m+1}(K)$,*

$$f(x,t,\varepsilon) = B(t)(x-\psi(t)) + \varepsilon\widetilde{f}(x,t,\varepsilon)$$

and conditions 4, 9–12, 14, and 15 are satisfied, then, for sufficiently small ε, $0 < \varepsilon \le \varepsilon_1 \le \varepsilon_0$, there exists a unique solution $x = x(t,\varepsilon)$ of the boundary-value problem (5)–(6)*, for which the asymptotic estimate* (35) *is valid.*

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Самусенко П.Ф., Віра М.Б. *Асимптотичні розв'язки крайової задачі для сингулярно збуреної системи диференціально-алгебраїчних рівнянь* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 49–60.

У роботі розглядається крайова задача для сингулярно збуреної диференціально-алгебраїчної системи рівнянь другого порядку. Розглянуто випадок простих коренів характеристичного рівняння. Отримано достатні умови існування та єдиності розв'язку крайової задачі для диференціально-алгебраїчної системи рівнянь. Розроблено метод побудови асимптотичних розв'язків поставленої задачі.

Ключові слова і фрази: крайова задача, асимптотичний розв'язок, диференціально-алгебраїчна система, сингулярно збурена система.