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A study on conformal Ricci solitons and conformal Ricci almost solitons within the framework of almost contact geometry

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The goal of this paper is to find some important Einstein manifolds using conformal Ricci solitons and conformal Ricci almost solitons. We prove that a Kenmotsu metric as a conformal Ricci soliton is Einstein if it is an η-Einstein or the potential vector field V is infinitesimal contact transformation or collinear with the Reeb vector field ξ. Next, we prove that a Kenmotsu metric as gradient conformal Ricci almost soliton is Einstein if the Reeb vector field leaves the scalar curvature invariant. Finally, we have embellished an example to illustrate the existence of conformal Ricci soliton and gradient almost conformal Ricci soliton on Kenmotsu manifold.

Key words and phrases: conformal Ricci soliton, Kenmotsu manifold, Einstein manifold, infinitesimal contact transformation.

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1 Introduction

In recent years, geometric flows, in particular, the Ricci flow have been an interesting research topic in differential geometry. The concept of Ricci flow was first introduced by R.S. Hamilton and developed to answer Thurston's geometric conjecture. A Ricci soliton can be considered as a fixed point of Hamilton's Ricci flow (see details in [17]) and a natural generalization of the Einstein metric (i.e., the Ricci tensor Ric is a constant multiple of the pseudo-Riemannian metric g), defined on a pseudo-Riemannian manifold (M, g) by

$$\frac{1}{2}\pounds_V g + Ric = \lambda g,$$

where $\pounds_V$ denotes the Lie-derivative in the direction of $V \in \chi(M)$, Ric is the Ricci tensor of g and λ is a constant. The Ricci soliton is said to be shrinking, steady, and expanding accordingly if λ is negative, zero, and positive respectively. Otherwise, it will be called indefinite. A Ricci soliton is trivial if V is either zero or Killing on M. First, S. Pigola et al. [22] assume the soliton constant λ to be a smooth function on M and named as Ricci almost soliton. After that, A. Barros et al. studied Ricci almost soliton detailed in [1,2]. Recently, J.T. Cho and M. Kimura [4] generalized the notion of Ricci soliton to η-Ricci soliton, C. Călin and M. Crasmareanu [5] studied this in Hopf hypersurfaces of complex space forms.

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In 2005, A.E. Fischer [11] has introduced conformal Ricci flow which is a mere generalisation of the classical Ricci flow equation that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation was given by

$$\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg,$$
$$r(g) = -1,$$

where $r(g)$ is the scalar curvature of the manifold, p is a scalar non-dynamical field and n is the dimension of the manifold. Corresponding to the conformal Ricci flow equation in 2015, N. Basu and A. Bhattacharyya [3] introduced the notion of conformal Ricci soliton equation as a generalization of Ricci soliton equation given by

$$\frac{1}{2}\pounds_V g + Ric + \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big)\Big]g = 0. \tag{1}$$

If the potential vector field V is a gradient of a smooth function f on M then the manifold is called a gradient conformal Ricci almost soliton. In this case the equation (1) can be exibited as

$$Hess\ f + Ric + \Big[\lambda - \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big)\Big]g = 0, \tag{2}$$

where $Hess\ f$ denotes the Hessian of f. The function f is known as the potential function.

It is worthy to mention that R. Sharma [28] first initiated the study of Ricci solitons in contact geometry. However, A. Ghosh [15] is the first to consider 3-dimensional Kenmotsu metric as a Ricci soliton. After that, Kenmotsu manifold is studied on many context of Ricci soliton by many authors like A. Ghosh [14], W. Wang [30] etc. In [29], authors have considered $*$-Ricci solitons and gradient almost $*$-Ricci solitons on Kenmotsu manifolds and obtained some beautiful results. Many authors studied conformal Ricci solitons and their generalizations in the framework of almost contact and paracontact geometries, e.g., Kenmotsu manifold [3,7,24], Sasakian manifold [6,23], f-Kenmotsu manifold [18,21], para-Kähler manifold [7] and (κ, μ)-paracontact metric manifolds [27]. Ricci solitons and their generalizations have been enormously studied by many authors within the framework of contact and paracontact metric manifolds (see in details [6, 8–10, 12, 13, 19, 20, 25, 26]).

This paper is organized as follows. After collecting some basic definitions and formulas on Kenmotsu manifold in Section 2, we prove in Section 3 that a Kenmotsu metric as a conformal Ricci soliton is Einstein if it is an η-Einstein or the potential vector field V is infinitesimal contact transformation or V is collinear with the Reeb vector field ξ. We also have constructed an example of 5-dimensional Kenmotsu manifold admitting conformal Ricci soliton. In Section 4, we consider conformal Ricci almost solitons on Kenmotsu manifold and find some η-Einstein and Einstein manifolds using conformal Ricci almost solitons. Next, we construct an example of almost conformal Ricci soliton on Kenmotsu manifold to prove our findings.

2 Notes on almost contact metric manifolds

In this section, we will present some preliminaries which will be used during the paper. A smooth manifold M^{2n+1} of dimension $(2n+1)$ is said to be contact if it has a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. The 1-form η is known as a contact form.

Corresponding to this 1-form one can find a unit vector field ξ, called the Reeb vector field, such that $d\eta(\xi, .) = 0$ and $\eta(\xi) = 1$. Polarization of $d\eta$ on the contact subbundle D (defined by $\eta = 0$), yields a Riemannian metric g and a (1, 1) tensor field φ, that satisfy $\eta(X) = g(X, \xi)$

$$\varphi^2 = -I + \eta \otimes \xi, \tag{3}$$

and $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ for all vector fields X, Y on M. The metric g is called associated metric of η and (φ, ξ, η, g) is a contact metric structure. It is well known on an almost contact metric manifold that $\varphi(\xi) = 0$, $\eta o \varphi = 0$. An almost contact metric structure is said to be contact metric if it satisfies $d\eta(X, Y) = g(X, \varphi Y)$ for all vector fields X, Y on M. If ξ is Killing, then M is said to be K-contact manifold and a normal almost contact metric manifold is said to be Sasakian. An almost contact metric manifold is said to be Kenmotsu manifold if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{4}$$

for any X, $Y \in \chi(M)$, and

$$\begin{aligned} \nabla_X \xi &= X - \eta(X)\xi, & (5)\\ R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, & (6)\\ R(X, \xi)Y &= g(X, Y)\xi - \eta(Y)X, & (7)\\ Q\xi &= -2n\xi & (8) \end{aligned}$$

for all X, $Y \in \chi(M)$, where ∇, R and Q denote respectively, the Riemannian connection, the curvature tensor and the Ricci operator of g associated with the Ricci tensor given by $Ric(X, Y) = g(QX, Y)$ for all X, $Y \in \chi(M)$. Now, we recall the following lemma on Kenmotsu manifold.

Lemma 1 ([29]). *On Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ the following formulas hold for any X, $Y \in \chi(M)$*

$$\begin{aligned} (\nabla_X Q)\xi &= -QX - 2nX, & (9)\\ (\nabla_\xi Q)X &= -2QX - 4nX. & (10) \end{aligned}$$

3 On conformal Ricci soliton

In this section, we study the conformal Ricci solitons on Kenmotsu manifold and find some important conditions so that a Kenmotsu metric as a conformal Ricci soliton is Einstein.

First, we recall a defination. A contact metric manifold M^{2n+1} is said to be η-Einstein, if the Ricci tensor Ric can be written as

$$Ric(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y), \tag{11}$$

where α, β are smooth functions on M. For an η-Einstein K-contact manifold [32] and para-Sasakian manifold [33] of dimension > 3, it is well known that the functions α, β are constants, but for an η-Einstein para-Kenmotsu manifold this is not true. So, we continue α, β as functions. In [15], A. Ghosh studied a 3-dimensional Kenmotsu metric as a Ricci soliton and for higher dimension in [16]. Before formulate our main results first we derive the following lemma.

Lemma 2. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a Kenmotsu manifold. If g represents a conformal Ricci soliton with potential vector field V then we have*

$$(\pounds_V R)(X,\xi)\xi = 0 \tag{12}$$

for any $X \in \chi(M)$.

Proof. Taking the covariant derivative of (1) along an arbitrary vector field Z on M and using (5) we get

$$(\nabla_Z \pounds_V g)(X,Y) = -2(\nabla_Z Ric)(X,Y) \tag{13}$$

for any X, $Y \in \chi(M)$. Next, recalling the following commutation formula (see [31, p.23]) we obtain

$$(\pounds_V \nabla_Z g - \nabla_Z \pounds_V g - \nabla_{[V,Z]} g)(X,Y) = -g((\pounds_V \nabla)(Z,X),Y) - g((\pounds_V \nabla)(Z,Y),X) \tag{14}$$

for all X, Y, $Z \in \chi(M)$. In view of the parallel Riemannian metric g, it follows from (8) that $(\nabla_Z \pounds_V g)(X,Y) = g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X)$ for all X, Y, $Z \in \chi(M)$. Plugging it into (13) we obtain

$$g((\pounds_V \nabla)(Z,X),Y) + g((\pounds_V \nabla)(Z,Y),X) = -2(\nabla_Z Ric)(X,Y) \tag{15}$$

for any X, Y, $Z \in \chi(M)$. Interchanging cyclicly the roles of X, Y, Z in (15) we can get

$$g((\pounds_V \nabla)(X,Y),Z) = (\nabla_Z Ric)(X,Y) - (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(Z,X) \tag{16}$$

for all Y, $Z \in \chi(M)$. Now, substituting ξ for Y in (16) and using (10), the formula (9) yields

$$(\pounds_V \nabla)(X,\xi) = 2QX + 4nX \tag{17}$$

for all $X \in \chi(M)$. Next, using (5), (17) in the covariant derivative of (17) along Y, we obtain

$$(\nabla_Y \pounds_V \nabla)(X,\xi) = (\pounds_V \nabla)(X,Y) - 2(\nabla_Y Q)X + 2\eta(Y)(QX + 2nX)$$

for any $X \in \chi(M)$. Making use of this in the following commutation formula (see [31, p.23]) $(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z)$, we can derive

$$(\pounds_V R)(X,Y)\xi = 2\{(\nabla_Y Q)X - (\nabla_X Q)Y\} + 2\{\eta(X)QY - \eta(Y)QX\} + 4n\{\eta(X)Y - \eta(Y)X\} \tag{18}$$

for all vector fields X, $Y \in \chi(M)$. Substituting Y by ξ in (18) and using (8), (9) and (10), we get the required result. □

Theorem 1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$, $n > 1$, be an η-Einstein Kenmotsu manifold. If g represents a conformal Ricci soliton with potential vector field V, then g is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Proof. First, tracing the equation (11) gives $r = (2n+1)\alpha + \beta$ and putting $X = Y = \xi$ in (11) and using (8) we get $\alpha + \beta = -2n$. Therefore, by computation, (11) transforms into

$$Ric(X,Y) = \left(1 + \frac{r}{2n}\right)g(X,Y) - \left\{(2n+1) + \frac{r}{2n}\right\}\eta(X)\eta(Y) \tag{19}$$

for all X, Y on M. This gives

$$(\nabla_Y Q)X = \frac{(Yr)}{2n}\Big\{X - \eta(X)\xi\Big\} + \Big\{(2n+1) + \frac{r}{2n}\Big\}\Big\{g(X,Y))\xi + \eta(X)(Y - 2\eta(Y)\xi)\Big\}$$

for all $X, Y \in \chi(M)$. By virtue of this, (18) provides

$$(\pounds_V R)(X,Y)\xi = \frac{1}{n}\Big\{(Xr)(Y - \eta(Y)\xi) - (Yr)(X - \eta(X)\xi)\Big\}$$

for all $X, Y \in \chi(M)$. Setting $Y = \xi$ in the above formula and using the Lemma 2, we get $(\xi r)\varphi^2 X = 0$ for any $X \in \chi(M)$. Using this in the trace of (10), we get $r = -2n(2n+1)$. It follows from (19) that M is Einstein. Hence the proof. □

Now, taking the Lie-derivative of $g(\xi,\xi) = 1$ along the potential vector field V and applying (1) one can obtain

$$\eta(\pounds_V \xi) = \lambda - 2n - \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big). \tag{20}$$

Further, from (5) we get $R(X,\xi)\xi = -X + \eta(X)\xi$ and the Lie derivative of this along V yields

$$(\pounds_V R)(X,\xi)\xi + R(X,\pounds_V\xi)\xi + R(X,\xi)\pounds_V\xi = \{(\pounds_V\eta)X\}\xi + \eta(X)\pounds_V\xi \tag{21}$$

for any $X \in \chi(M)$. If g represents a conformal Ricci soliton with potential vector field V then the Lemma 2 holds, i.e. $(\pounds_V R)(X,\xi)\xi = 0$. Plugging it into (21) and using (6) provides

$$(\pounds_V g)(X,\xi) + 2\eta(\pounds_V\xi)X = 0 \tag{22}$$

for any $X \in \chi(M)$. Again, applying (1) and (20) in (22) yields $(2n - \lambda)\varphi^2 X = 0$ for any $X \in \chi(M)$. Next, using (3) and then tracing yields $2n\big(2n - \lambda + \frac{1}{2}(p + \frac{2}{2n+1})\big) = 0$. This implies

$$\lambda = 2n + \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big). \tag{23}$$

So, from the previous identity we can say that conformal Ricci soliton is expanding if $p \geq 0$.

Theorem 2. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a Kenmotsu manifold. If g represents a conformal Ricci soliton with non-zero potential vector field V, that is collinear with ξ, then g is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Proof. Since the potential vector field V is collinear with ξ, i.e. $V = \nu\xi$ for some smooth function ν on M. Making use of (5) in the covariant derivative of $V = \nu\xi$ along X yields

$$\nabla_X V = (X\nu)\xi + \nu\{(X - \eta(X)\xi)\}$$

for any $X \in \chi(M)$. By virtue of this, the soliton equation (1) reduces to

$$2Ric(X,Y) + (X\nu)\eta(Y) + (Y\nu)\eta(X) + 2\Big(\lambda - \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big) + \nu\Big)g(X,Y) - 2\nu\eta(X)\eta(Y) = 0 \tag{24}$$

for all $X, Y \in \chi(M)$. Setting $X = Y = \xi$ in (24) and using (8), (23) we get $\xi\nu = 0$. It follows from (24) that $X\nu = 0$. Putting it into (24) provides

$$Ric(X,Y) = -\Big(\nu + \lambda - \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big)\Big)g(X,Y) + \nu\eta(X)\eta(Y) \tag{25}$$

for all $X, Y \in \chi(M)$. This shows that M is η-Einstein and therefore from Theorem 1 we conclude that M is Einstein. Thus, from (24) we have $\nu = 0$ and so $\nu + \lambda = 2n + \frac{1}{2}(p + \frac{2}{2n+1})$ (it follows from (23)). Hence we have from (25) that $Ric = -2ng$ and therefore $r = -2n(2n+1)$, as required. □

Now, we discuss an example of Kenmotsu manifold that admits a conformal Ricci soliton.

Example 1. *Let $M^5 = \{(x,y,z,u,v) \in \mathbb{R}^5\}$ be a 5-dimensional manifold, where (x,y,z,u,v) be the standard coordinates in $\mathbb{R}^5$. Now, consider an orthonormal basis $\{e_1,e_2,e_3,e_4,e_5\}$ of vector fields on M, where $e_1 = e^{-v}\frac{\partial}{\partial x}$, $e_2 = e^{-v}\frac{\partial}{\partial y}$, $e_3 = e^{-v}\frac{\partial}{\partial z}$, $e_4 = e^{-v}\frac{\partial}{\partial u}$, $e_5 = \frac{\partial}{\partial v}$. Define $(1,1)$ tensor field φ as follows:*

$$\varphi(e_1) = e_3, \ \ \varphi(e_2) = e_4, \ \ \varphi(e_3) = -e_1, \ \ \varphi(e_4) = -e_2, \ \ \varphi(e_5) = 0.$$

The Riemannian metric is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\eta(X) = g(X,e_5)$ for any $X \in \chi(M^5)$.

Then $\eta(e_5) = 1$, $\varphi^2 X = -X + \eta(X)\xi$, and $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$ for all $X, Y \in \chi(M^5)$. Thus, for $\xi = e_5$, (φ,ξ,η,g) is an almost contact structure. The non-zero components of the Levi-Civita connection ∇ (using Koszul's formula) are

$$\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = \nabla_{e_3}e_3 = \nabla_{e_4}e_4 = -e_5, \ \nabla_{e_1}e_5 = e_1, \ \nabla_{e_2}e_5 = e_2, \ \nabla_{e_3}e_5 = e_3, \ \nabla_{e_4}e_5 = e_4. \quad (26)$$

By virtue of this we can verify (4) *and therefore $M^5(\varphi,\xi,\eta,g)$ is a Kenmotsu manifold. Using the well knownn expression of curvatute tensor $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we now compute the following non-zero components*

$$\begin{array}{llll}
R(e_1,e_2)e_2 = -e_1, & R(e_1,e_3)e_3 = -e_1, & R(e_1,e_4)e_4 = -e_1, & R(e_1,e_5)e_5 = -e_1, \\
R(e_1,e_2)e_1 = e_2, & R(e_1,e_3)e_1 = e_3, & R(e_1,e_4)e_1 = e_4, & R(e_1,e_5)e_1 = e_5, \\
R(e_2,e_3)e_2 = e_3, & R(e_2,e_4)e_2 = e_4, & R(e_2,e_5)e_2 = -e_5, & R(e_2,e_3)e_3 = -e_2, \\
R(e_2,e_4)e_4 = -e_2, & R(e_2,e_5)e_5 = -e_2, & R(e_3,e_4)e_3 = e_4, & R(e_3,e_5)e_3 = e_5, \\
R(e_3,e_4)e_4 = -e_3, & R(e_4,e_5)e_4 = e_5, & R(e_5,e_3)e_5 = e_3, & R(e_5,e_4)e_5 = e_4.
\end{array}$$

Using this, we compute the components of the Ricci tensor as $Ric(e_i,e_i) = -4$ for $i = 1,2,3,4,5$, and therefore

$$Ric(X,Y) = -4g(X,Y) \quad (27)$$

for all $X, Y \in \chi(M^5)$. Let us consider the potential vector field $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$. Then with the help of (26) *we can show that*

$$(\pounds_V g)(X,Y) = 4\{g(X,Y) - \eta(X)\eta(Y)\} \quad (28)$$

for all $X, Y \in \chi(M^5)$. So, combining (27) *and* (28)*, we observe that soliton equation* (1) *holds for $\lambda = \frac{21}{5} + \frac{p}{2}$, i.e. the metric g is a conformal Ricci soliton with this potential vector field V and the constant $\lambda = \frac{21}{5} + \frac{p}{2}$, which also satisfies the $\lambda = 2n + \frac{1}{2}(p + \frac{2}{2n+1})$, here $n = 2$.*

4 On conformal Ricci almost soliton

In this section, we consider conformal Ricci almost soliton on Kenmotsu manifold. It follows from (1) that conformal Ricci almost soliton is the generalization of Ricci almost soliton because it involve a smooth function λ. So, first we study gradiant conformal Ricci almost soliton on Kenmotsu manifold in order to extend the result for gradiant Ricci almost soliton by A. Ghosh [14]. Thus, equations (1) and (2) hold for smooth function λ. Now, we prove the following result for later use.

Lemma 3. *If $Xf = (\xi f)\eta(X)$ for any vector field X and a smooth function f on a contact metric manifold M, then f is constant on M.*

Proof. From the hypothesis we have $df = \xi(f)\eta$. Operating it by d and applying Poincare lemma, namely $d^2 = 0$, we obtain $fd\eta + df \wedge \eta = 0$. Now if we take wedge product with η and using $\eta \wedge \eta = 0$ and $d\eta \wedge \eta$ is non-vanishing everywhere on M, we find $\xi f = 0$ and so we get $df = 0$. Hence f is constant on M. □

Theorem 3. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold. If M admits a gradient conformal Ricci almost soliton and the Reeb vector field ξ leaves the scalar curvature r invariant, then it is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof. The equation (2) can be exhibited as

$$\nabla_X Df + QX + \Big\{\lambda - \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big)\Big\}X = 0$$

for any $X \in \chi(M)$. Using this in $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we can easily obtain the curvature tensor expression in the follwing form

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + (Y\lambda)X - (X\lambda)Y \tag{29}$$

for all X, $Y \in \chi(M)$. Taking contraction of (29) over X with respect to an orthonormal basis $\{e_i : i = 1, 2, \dots, 2n+1\}$, we obtain

$$Ric(Y, Df) = -\sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)Y, e_i) + (Yr) + 2n(Y\lambda)$$

for any $Y \in \chi(M)$. Now, contracting Bianchi's second identity we have

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)Y, e_i) = \frac{1}{2}(Yr).$$

Plugging it into the previous equation gives

$$Ric(Y, Df) = \frac{1}{2}(Yr) + 2n(Y\lambda) \tag{30}$$

for any $Y \in \chi(M)$.

Substituting Y by Df in (7) and contracting we get $Ric(\xi, Df) = -2n(\xi f)$. Using this in (30) yields

$$(\xi r) + 4n\{\xi(\lambda + f)\} = 0. \tag{31}$$

Now, substituting ξ for Y in (29) and using Lemma 1 provides

$$R(X,\xi)Df = -QX - 2nX + \xi(\lambda)X - X(\lambda)\xi \tag{32}$$

for any $X \in \chi(M)$. Next, taking inner product of (32) with ξ and using (7) we obtain the equation $g(R(X,\xi)Df,\xi) = \xi(\lambda)\eta(X) - X(\lambda)$ for any $X \in \chi(M)$. By virtue of (7), the preceeding equation reduces to $X(f+\lambda) = \xi(f+\lambda)\eta(X)$ for any $X \in \chi(M)$. We can conclude from Lemma 3 that $\lambda + f = c$ is a constant on M. Contracting (10) provides

$$\xi r = -2\{r + 2n(2n+1)\}. \tag{33}$$

As $\lambda + f$ is constant, so from (31) and (33) we get $\{\frac{r}{2n} + (2n+1)\} = 0$, which implies $r = -2n(2n+1)$. Further, using (8) in (32), we obtain

$$X(f+\lambda)\xi = -QX + \{\xi(f+\lambda) - 2n\}X$$

for any $X \in \chi(M)$. By virtue of this, the preceeding equation transforms into $QX = -2nX$ for any X on $\in \chi(M)$. This shows that M is Einstein. So, we complete the proof. □

Next, considering a Kenmotsu metric as a conformal Ricci almost soliton with the potential vector field V, that is pointwise collinear with the Reeb vector field ξ, we extend the Theorem 3 from gradient conformal Ricci almost soliton to conformal Ricci almost soliton and prove the following assertion.

Theorem 4. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a Kenmotsu manifold. If M admits a conformal Ricci almost soliton with non-zero potential vector field V, that is collinear with ξ, then g is η-Einstein. Moreover, if the Reeb vector field ξ leaves the scalar curvature r invariant, then g is Einstein with constant scalar curvature $-2n(2n+1)$.*

Proof. By hypothesis we have that $V = \sigma\xi$ for some smooth function σ on M. It follows that

$$(\pounds_V g)(X,Y) = (X\sigma)\eta(Y) + (Y\sigma)\eta(X) + 2\sigma\{g(X,Y) - \eta(X)\eta(Y)\}$$

for all X, $Y \in \chi(M)$. By virtue of this, the soliton equation (1) transforms into

$$\begin{aligned} 2Ric(X,Y) &+ (X\sigma)\eta(Y) + (Y\sigma)\eta(X) \\ &+ 2\Big(\sigma + \lambda - \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big)\Big)g(X,Y) = 2\sigma\eta(X)\eta(Y) \end{aligned} \tag{34}$$

for all X, $Y \in \chi(M)$.

Now, putting $X = Y = \xi$ in (34) and using (14) yields $\xi\sigma = 2n - \lambda + \frac{1}{2}(p + \frac{2}{2n+1})$. Thus, the equation (34) gives $X\sigma = \big(2n - \lambda + \frac{1}{2}(p + \frac{2}{2n+1}) - \sigma\big)\eta(X)$. Making use of this in (34) entails that

$$\begin{aligned} Ric(X,Y) = &-\Big(\sigma + \lambda - \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big)\Big)g(X,Y) \\ &- \Big(2n - \lambda + \frac{1}{2}\Big(p + \frac{2}{2n+1}\Big) - \sigma\Big)\eta(X)\eta(Y) \end{aligned} \tag{35}$$

for all X, $Y \in \chi(M)$. Hence M is η-Einstein. Moreover, if the Reeb vector field ξ leaves the scalar curvature r invariant, i.e. $\xi r = 0$, then tracing (12) yields $(\xi r) = -2\{r + 2n(2n+1)\}$ and therefore $r = -2n(2n+1)$. Using this in the trace of (35) gives $\lambda + \sigma = 2n + \frac{1}{2}(p + \frac{2}{2n+1})$. Thus, from (35) we have $QX = -2nX$ and therefore M is Einstein. □

If we consider $V = \sigma\xi$ for some constant σ instead of a function, then (34) holds and therefore inserting $X = Y = \xi$ into (34) and using (7) gives $\xi\sigma = 2n - \lambda$. Using this in (34) yields $\lambda + \sigma = 2n + \frac{1}{2}(p + \frac{2}{2n+1})$, where we have used that σ is a constant. Thus, from (35) we can conclude the following result.

Corollary 1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold. If M admits a non-trivial conformal Ricci almost soliton with $V = \sigma\xi$ for some constant σ, then it is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Now, we discuss an example of Kenmotsu manifold that admits a gradient conformal Ricci soliton.

Example 2. *Let $M^5 = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ be a 5-dimensional manifold, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. Now, consider a orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ of vector fields on M, where $e_1 = v\frac{\partial}{\partial x}$, $e_2 = v\frac{\partial}{\partial y}$, $e_3 = v\frac{\partial}{\partial z}$, $e_4 = v\frac{\partial}{\partial u}$, $e_5 = -v\frac{\partial}{\partial v}$. Define $(1,1)$ tensor field φ as follows:*

$$\varphi(e_1) = e_2, \;\; \varphi(e_2) = -e_1, \;\; \varphi(e_3) = e_4, \;\; \varphi(e_4) = -e_3, \;\; \varphi(e_5) = 0.$$

The Riemannian metric is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\eta(X) = g(X, e_5)$ for any $X \in \chi(M^5)$.

Then $\eta(e_5) = 1$, $\varphi^2 X = -X + \eta(X)\xi$, and $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ for all $X, Y \in \chi(M^5)$. Thus, for $\xi = e_5$, (φ, ξ, η, g) is an almost contact structure. The non-zero components of the Levi-Civita connection ∇ (using Koszul's formula) are

$$\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = \nabla_{e_3}e_3 = \nabla_{e_4}e_4 = -e_5, \; \nabla_{e_1}e_5 = e_1, \; \nabla_{e_2}e_5 = e_2, \; \nabla_{e_3}e_5 = e_3, \; \nabla_{e_4}e_5 = e_4. \quad (36)$$

By virtue of this we can verify (4) and therefore $M^5(\varphi, \xi, \eta, g)$ is a Kenmotsu manifold. Using the well knownn expression of curvatute tensor $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we now compute the following non-zero components

$$\begin{array}{llll} R(e_1, e_2)e_2 = -e_1, & R(e_1, e_3)e_3 = -e_1, & R(e_1, e_4)e_4 = -e_1, & R(e_1, e_5)e_5 = -e_1, \\ R(e_1, e_2)e_1 = e_2, & R(e_1, e_3)e_1 = e_3, & R(e_1, e_4)e_1 = e_4, & R(e_1, e_5)e_1 = e_5, \\ R(e_2, e_3)e_2 = e_3, & R(e_2, e_4)e_2 = e_4, & R(e_2, e_5)e_2 = e_5, & R(e_2, e_3)e_3 = -e_2, \\ R(e_2, e_4)e_4 = -e_2, & R(e_2, e_5)e_5 = -e_2, & R(e_3, e_4)e_3 = e_4, & R(e_3, e_5)e_3 = e_5, \\ R(e_3, e_4)e_4 = -e_3, & R(e_4, e_5)e_4 = e_5, & R(e_5, e_3)e_5 = e_3, & R(e_5, e_4)e_5 = e_4. \end{array}$$

Using this, we compute the components of the Ricci tensor as $Ric(e_i, e_i) = -4$ for $i = 1, 2, 3, 4, 5$ and therefore

$$Ric(X, Y) = -4g(X, Y) \quad (37)$$

for all $X, Y \in \chi(M^5)$.

Let $f : M \to \mathbb{R}$ be a smooth function defined by $f(x,y,z,u,v) = x^2 + y^2 + z^2 + u^2 + \frac{v^2}{2}$. The gradient Df of f is given by

$$Df = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.$$

Then with the help of (36) *we can show that*

$$(\pounds_{Df}g)(X,Y) = 2\{g(X,Y) - \eta(X)\eta(Y)\} \tag{38}$$

for all $X, Y \in \chi(M^5)$. So, combining (37) *and* (38)*, we observe that soliton equation* (1) *holds for $\lambda = \frac{17}{5} + \frac{p}{2}$, i.e. the metric g is a gradient conformal Ricci almost soliton with this potential vector field $V = Df$, $\lambda = \frac{17}{5} + \frac{p}{2}$.*

5 Geometrical and physical motivations

The notion of conformal Ricci solitons can be charecterized as a kinematic solution of conformal Ricci flow, whose profile develop a characterization of spaces of constant sectional curvature along with the locally symmetric spaces. Also, geometric phenomenon of conformal Ricci solitons can evolve an aqueduct between a sectional curvature inheritance symmetry of space time and class of Ricci solitons. The mathematical notion of a conformal Ricci soliton should not be confused with the notion of soliton solutions, which arise in several areas of methematical or theoretical physics and its applications. Conformal Ricci soliton is important as it can help in understanding the concepts of energy or entropy in general realtivity. This property is the same as that of heat equation due to which an isolated system loses the heat for a thermal equilibrium. It deals a geometric and physical applications with relativistic viscous fluid spacetime admitting heat flux and stress, dark and dust fluid general relativistic spacetime, radiation era in general relativistic spacetime. Also there is further scope of research in this direction of various types of complex manifolds like Kaehler manifolds, para-Kaehler manifolds, Hopf manifolds etc.

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Дей С. *Дослідження конформних солітонів Річчі та конформних майже солітонів Річчі в рамках майже контактної геометрії* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 31–42.

Метою цієї статті є побудова деяких важливих многовидів Айнштайна за допомогою конформних солітонів Річчі та конформних майже солітонів Річчі. Ми доводимо, що метрика Кенмоцу як конформний солітон Річчі є айнштайнівською, якщо вона є η-айнштайнівською, або потенційне векторне поле V є інфінітезимальним контактним перетворенням або колінеарним з векторним полем Ріба ξ. Також ми доводимо, що метрика Кенмоцу як градієнтний конформний майже солітон Річчі є айнштайнівською, якщо векторне поле Ріба залишає незмінною скалярну кривизну. Насамкінець, ми побудували приклад, щоб проілюструвати існування конформного солітону Річчі та градієнтного майже конформного солітону Річчі на многовиді Кенмоцу.

Ключові слова і фрази: конформний солітон Річчі, многовид Кенмоцу, многовид Айнштайна, інфінітезимальне контактне перетворення.