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Characterization of matrix transformation of complex uncertain sequences via expected value operator

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The aim of this paper is to study the concept of matrix transformation between complex uncertain sequences in mean. The characterization of the matrix transformation has been made by applying the concept of convergence of complex uncertain series. Moreover, in this context, some well-known theorems of real sequence spaces have been established by considering complex uncertain sequence via expected value operator.

Key words and phrases: uncertainty space, complex uncertain series, complex uncertain sequence, matrix transformation.

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Introduction

To deal with indeterminacy B. Liu [12] defined the concept of uncertainty theory, which is based on uncertain measure. The uncertain measure follows the axioms of normality, duality, subadditivity and product. In the year 2007, the notion of uncertain sequences and their convergences was introduced by B. Liu [12] and then the same was extended by C. You [20]. Thereafter, to describe the complex uncertain quantities, the notions of complex uncertain variable and complex uncertain distribution are presented by Z. Peng [16]. X. Chen et al. [1] explored the work considering the sequence of complex uncertain variables due to Z. Peng [16]. They reported (see [1]) five convergence concepts of sequence of complex uncertain variables, namely convergence in almost surely, convergence in measure, convergence in mean, convergence in distribution and convergence with respect to uniformly almost surely by establishing interrelationships among them. Since its initiation, the study of complex uncertain sequences got the full attention of the researchers. These convergence concept of complex uncertain sequence has also been generalised by D. Datta and B.C. Tripathy [10], B. Das et al. [2–9], S. Saha et al. [18]. In this context, we focus to study matrix transformation of complex uncertain sequence by using the notion of convergent complex uncertain series.

The study of sequence space through matrices are very much relevant in the current research flow [17, 19]. Interest in general matrix transformation was first stimulated to some extent by establishing results on special type of matrices in summability theory, which were obtained by E. Cesàro, É. Borel and others. However, it was O. Toeplitz, who first made a

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detailed study on matrix transformation on sequence spaces and then mathematicians made progress enormously in this particular direction [11, 14, 15]. As part of the frame of reference, we state the book [13].

Let $A = (a_{nk})$, $n, k = 1, 2, 3, \ldots$, is an infinite matrix and $x = \{x_k\} \in c_0$ (c_0 being the family of all null sequences). Then

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \ldots \\ a_{21} & a_{22} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \ldots \\ a_{21}x_1 + a_{22}x_2 + \ldots \\ \ldots \quad \ldots \quad \ldots \end{pmatrix}.$$

Thus A is said to be an operator, which maps the sequence x into Ax, where $(Ax)_n \equiv A_n(x) = \sum\limits_{k=1}^{\infty} a_{nk}x_k$, provided that each of the series converges.

This motivates us to define the convergence of complex uncertain series to study the matrix transformation of such sequences. We also study the famous Silverman-Toeplitz theorem and Kojima-Schur theorem via complex uncertain sequences.

1 Preliminaries

Before going to the main section we need some basic and preliminary ideas about the existing definitions and results, which will play a major role in this study.

Definition 1 ([12])**.** *Let $\mathcal{L}$ be σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following three axioms.*

Normality axiom. $\mathcal{M}\{\Gamma\} = 1$.

Duality axiom. $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any* $\Lambda \in L$.

Subadditivity axiom. *For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$, we have*

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and each element Λ in $\mathcal{L}$ is called an event.

In order to obtain an uncertain measure of compound events, a product uncertain measure is defined as follows.

Product axiom. *Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, 3, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{j=1}^{\infty} \Lambda_j\right\} = \bigwedge_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\},$$

where Λ_j are arbitrarily chosen events from Γ_j for $j = 1, 2, 3, \ldots$, respectively.

Also, the monotonicity axiom of uncertain measure is given as follows.

Monotonicity axiom. *For any two events Λ_1 and Λ_2 with $\Lambda_1 \subseteq \Lambda_2$ we have*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}.$$

Definition 2 ([16]). *A complex uncertain variable is a measurable function ζ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e. for any Borel set B of complex numbers, the set $\{\zeta \in B\} = \{\gamma \in \Gamma : \zeta(\gamma) \in B\}$ is an event.*

Definition 3 ([12]). *The expected value operator of an uncertain variable ζ is defined by*

$$E[\zeta] = \int_0^{+\infty} \mathcal{M}\{\zeta \geq r\}\, dr - \int_{-\infty}^{0} \mathcal{M}\{\zeta \leq r\}\, dr,$$

provided that at least one of the two integrals is finite.

Definition 4 ([1]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent almost surely (a.s.) to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{n\to\infty} \|\zeta_n(\gamma) - \zeta(\gamma)\| = 0 \ \text{ for every } \ \gamma \in \Lambda.$$

Definition 5 ([1]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent in measure to ζ if for any $\varepsilon \geq 0$*

$$\lim_{n\to\infty} \mathcal{M}\{\|\zeta_n - \zeta)\| \geq \varepsilon\} = 0.$$

Definition 6 ([1]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent in mean to ζ if*

$$\lim_{n\to\infty} E\big[\|\zeta_n - \zeta\|\big] = 0.$$

Definition 7 ([1]). *Let $\Phi_1, \Phi_2, \Phi_3, \ldots$ be the complex uncertainty distributions of complex uncertain variables $\zeta_1, \zeta_2, \zeta_3, \ldots$, respectively. Then the complex uncertain sequence $\{\zeta_n\}$ is convergent in distribution to ζ if*

$$\lim_{n\to\infty} \Phi_n(c) = \Phi(c)$$

for all $c \in \mathbb{C}$, at which $\Phi(c)$ is continuous.

Definition 8 ([1]). *The complex uncertain sequence $\{\zeta_n\}$ is said to be convergent uniformly almost surely (u.a.s.) to ζ if there exists a sequence of events $\{E'_k\}$, $\mathcal{M}\{E'_k\} \to 0$, such that $\{\zeta_n\}$ converges uniformly to ζ in $\Gamma - E'_k$ for any fixed $k \in \mathbb{N}$.*

Definition 9 ([5]). *Suppose that $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An infinite complex uncertain series $\sum_{k=1}^{\infty} \zeta_k(\gamma)$ is said to be convergent in mean if the sequence of its partial sums $\{S_n(\gamma)\}$ is convergent in mean to some finite limit S for all $\gamma \in \Gamma$, where $S_n(\gamma) = \sum_{k=1}^{n} \zeta_k(\gamma)$ for any event $\gamma \in \Gamma$. That is,*

$$\lim_{n\to\infty} E\big[\|S_n(\gamma) - S(\gamma)\|\big] = 0.$$

If $\zeta_n = \xi_n + i\eta_n, \zeta = \xi + i\eta$ are complex uncertain variables in an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, then $\zeta_n(\gamma), \zeta(\gamma)$ are complex numbers, where ξ_n, η_n, ξ and η are real uncertain variables. So, the norm $\|\zeta_n - \zeta\|$ is the usual norm of complex numbers. The following is due to X. Chen et al. [1].

Remark 1 ([1]). *If $\zeta_n = \xi_n + i\eta_n, \zeta = \xi + i\eta$ are complex uncertain variables, where ξ_n, η_n, ξ and η are real uncertain variables, then the norm $\|\zeta_n - \zeta\|$ is given by*

$$\|\zeta_n - \zeta\| = \sqrt{(\xi_n - \xi)^2 - (\eta_n - \eta)^2}.$$

Also, $\|\zeta_n - \zeta\|$ is a real uncertain variable.

Theorem 1 ([12]). *If ζ and η are independent uncertain variables with finite expected values, then $E[a\zeta + b\eta] = aE[\zeta] + bE[\eta]$, where a and b are scalars.*

Throughout the article, the family of all convergent complex uncertain sequences in mean, in measure, in distribution, in almost surely and with respect to uniformly almost surely is denoted by $c(\Gamma_E)$, $c(\Gamma_{\mathcal{M}})$, $c(\Gamma_{\mathcal{D}})$, $c(\Gamma_{a.s})$, $c(\Gamma_{u.a.s})$, respectively. Similarly, the collection of all null sequences in mean, in measure, in distribution, in almost surely and with respect to uniformly almost surely is denoted by $c_0(\Gamma_E)$, $c_0(\Gamma_{\mathcal{M}})$, $c_0(\Gamma_{\mathcal{D}})$, $c_0(\Gamma_{a.s})$ and $c_0(\Gamma_{u.a.s})$, respectively.

2 Matrix transformation of complex uncertain sequences

Consider an infinite matrix $A = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix}$ and a complex uncertain sequence $\zeta = \{\zeta_n\}$. We apply A to ζ as follows: $A\zeta(\gamma) = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} \zeta_1(\gamma) \\ \zeta_2(\gamma) \\ \dots \end{pmatrix}$, where $\gamma \in \Gamma$. Then, by usual matrix multiplication, we have $A\zeta = \begin{pmatrix} a_{11}\zeta_1(\gamma) + a_{12}\zeta_2(\gamma) + \dots \\ a_{21}\zeta_1(\gamma) + a_{22}\zeta_2(\gamma) + \dots \\ \dots \quad \dots \quad \dots \end{pmatrix}$. Thus, we can write $(A\zeta)_n = A_n(\zeta)$ and it is given by

$$A_n(\zeta) = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma),$$

provided that the infinite series converges in mean for each n.

In this paper, we are dealing with expectation with respect to matrix maps. Since the terms of the matrix in the transformation are considered to be weights, so the terms of the matrix, i.e. a_{nk} should be non-negative. Hence throughout the paper we consider the matrices $A = (a_{nk})$ of non-negative terms.

Also to prove the following Theorems 2, 4, 5, 6, 9, the linearity property of uncertain variables are used. To hold the linearity property the variables needs to be independent and have finite expected values, due to B. Liu [12]. So for the above mentioned theorems, the complex uncertain variables $\zeta_1, \zeta_2, \dots$ are assumed to be independent in nature and all the variables has finite expected values.

Theorem 2. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $\{\zeta_n\}$ is a complex uncertain sequence. Suppose $a_{nk} \to 0$ as $n \to \infty$ uniformly for all $k \in \mathbb{N}$ and let $M = \sup_n \sum_k |a_{nk}| < \infty$. Then one may call A a bounded linear operator on $c_0(\Gamma_E)$ into itself and $\|A\| = M$.*

Proof. Consider an uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$ and $\{\zeta_n(\gamma)\} \in c_0(\Gamma_E)$. We first show that $A\zeta(\gamma) \in c_0(\Gamma_E)$, that is $A_n(\zeta(\gamma)) \to 0$ as $n \to \infty$.

This is true when the complex uncertain series $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ is absolutely convergent in mean for each n.

Now, for any $m \geq 1$ we have

$$\begin{aligned} E\big[\|A_n(\zeta(\gamma))\|\big] &\leq E\bigg[\sum_{k=1}^{\infty}\|a_{nk}\zeta_k(\gamma)\|\bigg] = E\bigg[\sum_{k=1}^{m}\|a_{nk}\zeta_k(\gamma)\|\bigg] + E\bigg[\sum_{k=m+1}^{\infty}\|a_{nk}\zeta_k(\gamma)\|\bigg] \\ &\leq E\bigg[\sup_{k\leq m}\|\zeta_k(\gamma)\|\sum_{k=1}^{m}|a_{nk}|\bigg] + E\bigg[\max_{k\geq m+1}\|\zeta_k(\gamma)\|M\bigg] \\ \text{(since } E \text{ is monotonic)} \quad &= E\bigg[\sup_{k\leq m}\|\zeta_k(\gamma)\|\bigg]\sum_{k=1}^{m}|a_{nk}| + M\max_{k\geq m+1}E\big[\|\zeta_k(\gamma)\|\big]. \end{aligned}$$

Take m and n so large that for any arbitrary small $\varepsilon > 0$ we have

$$\max\big\{E\big[\|\zeta_k(\gamma)\| : k \geq m+1\big]\big\} < \varepsilon \quad \text{and} \quad \sum_{k=1}^{m}|a_{nk}| < \varepsilon,$$

since $a_{nk} \to 0$ as $n \to \infty$ (k is fixed). Therefore, $A(\zeta(\gamma)) \in c_0(\Gamma_E)$ and hence A defines an operator from $c_0(\Gamma_E)$ into $c_0(\Gamma_E)$.

Now, for any scalar λ and complex uncertain sequences $\zeta = \{\zeta_n\}, \eta = \{\eta_n\}$, we have

$$\begin{aligned} A(\lambda\zeta(\gamma) + \eta(\gamma)) &= \sum_{k=1}^{\infty}(a_{nk}\lambda\zeta_k(\gamma) + a_{nk}\eta_k(\gamma)) \\ &= \lambda\sum_{k=1}^{\infty}(a_{nk}\zeta_k(\gamma)) + \sum_{k=1}^{\infty}(a_{nk}\eta_k(\gamma)) = \lambda A(\zeta(\gamma)) + A(\eta(\gamma)), \quad n \in \mathbb{N}. \end{aligned}$$

Therefore, A is linear. Again,

$$\|A(\zeta(\gamma))\| = \sup_n\Big\|\sum_k a_{nk}\zeta_k(\gamma)\Big\| \leq \|\zeta(\gamma)\|\sup_n\sum_k|a_{nk}| = M\|\zeta(\gamma)\|$$

for every $\zeta \in c_0(\Gamma_E)$. Hence, $\|A\| \leq M$ for all $\zeta \in c_0(\Gamma_E)$ and so A is bounded.

For the reverse inequality, let $M = \sup_n\sum_k|a_{nk}| < \infty$. Then there exists a positive integer n_0 such that $\sum_k|a_{nk}| > M - \frac{\varepsilon}{2}$ for all $n > n_0$ and since $\sum_k|a_{nk}|$ is finite, there exists $p_0 \in \mathbb{N}$ such that $\sum_{k>p}|a_{nk}| < \frac{\varepsilon}{2}$ for all $p > p_0$. For all $\gamma \in \Gamma$ we define $\zeta = \{\zeta_k\} \in c_0(\Gamma_E)$ by

$$\zeta_k(\gamma) = \begin{cases} sgn\ a_{n_0k}, & 1 \leq k \leq p; \\ 0, & k > p. \end{cases}$$

Then $\|\zeta(\gamma)\| = 1$ and $\frac{\|A(\zeta(\gamma))\|}{\|\zeta(\gamma)\|} = \sup_n\|A_n(\zeta(\gamma))\| \geq \|A_n(\zeta(\gamma))\| > M - \varepsilon$ uniformly for all n, i.e. for each row element of the transformed matrix $A\zeta$. This implies

$$M = \sup\left\{\frac{\|A(\zeta(\gamma))\|}{\|\zeta(\gamma)\|} : \zeta(\gamma) \neq 0\right\} = \|A\|.$$

□

Theorem 3. *Let A be any bounded linear operator defined on $c_0(\Gamma_E)$ into itself. Then A determines a matrix (a_{mn}) such that $(A\zeta(\gamma))_n = \sum_k a_{nk}\zeta_k(\gamma)$ for every $\gamma \in c_0(\Gamma_E)$ and*

$$\|A\| = \sup_n\sum_k|a_{nk}| < \infty.$$

Also, $a_{nk} \to 0$ as $n \to \infty$ (keeping k fixed).

Proof. Let $\zeta \in c_0(\Gamma_E)$. Then $\zeta(\gamma) = \sum_k \zeta_k(\gamma) e_k$, where $\{e_k\}$ is a basis in $c_0(\Gamma_E)$, which is given by $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots), \dots$. Now, $A\zeta(\gamma) = \sum_{k=1}^{\infty} \zeta_k(\gamma) A e_k = \sum_{k=1}^{\infty} \zeta_k(\gamma)(a_k^n)_{n\in\mathbb{N}}$, where Ae_k is a sequence $\{a_k^1, a_k^2, \dots\} \in c_0(\Gamma_E)$, $k = 1, 2, 3, \dots$. Then, we obtain, $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_k^n \zeta_k(\gamma)$, $n = 1, 2, \dots$. Since $e_k \in c_0(\Gamma_E)$, then $Ae_k \in c_0(\Gamma_E)$ also for $k = 1, 2, 3, \dots$. Thus, we can say that the sequence Ae_k converges to 0 in mean for each k. That implies $a_{nk} \to 0$ as $n \to \infty$, keeping k fixed. Thus, $\lim_{n\to\infty} A_n\zeta(\gamma) = \lim_{n\to\infty} \sum_k a_{nk}\zeta_k(\gamma) = 0$, since $\lim_{n\to\infty} a_{nk} = 0$. We are to prove that $\|A\| = \sup_n \sum_k |a_{nk}|$. Now $\|A_n\zeta(\gamma)\| \le \|A\zeta(\gamma)\| \le \|A\|\|\zeta\|$ for each n. Since A is a bounded linear operator and $\zeta \in c_0(\Gamma_E)$, then A_n is a bounded linear functional on $c_0(\Gamma_E)$. Thus we have the sequence $\{A_n\} \in c_0^*(\Gamma_E)$ such that $\lim_{n\to\infty} A_n(\zeta(\gamma)) = 0$. Then, by Banach-Steinhaus theorem $\|A_n\| \le H$ for some constant H for all n.

From the table of dual spaces in [13, page 110], $\|A_n\| = \sum_k |a_{nk}|$. Then $M = \sup_n \sum_k |a_{nk}| < \infty$ and by the above theorem

$$\|A\| = M.$$

□

Definition 10. *A complex uncertain sequence $\zeta = \{\zeta_k\}$ is said to be bounded in mean if $\sup_k E[\|\zeta_k\|]$ is finite. The collection of all bounded complex uncertain sequences in mean is denoted by $\ell_\infty(\Gamma_E)$.*

Theorem 4. *The infinite matrix operator A is a bounded linear operator from $\ell_\infty(\Gamma_E)$ into itself (we write this as $A \in (\ell_\infty(\Gamma_E), \ell_\infty(\Gamma_E))$) if and only if*

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty.$$

Proof. Let $\zeta = \{\zeta_k\}$ be a complex uncertain sequence, which is bounded in mean. Then for any given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $E[\|\zeta_k(\gamma)\|] < \varepsilon$ for all $\gamma \in \Gamma$ and $k \ge n_0$.

Let us consider an infinite bounded real matrix $A = (a_{nk})$ in such a way that $\sup_n \sum_{k=1}^{\infty} |a_{nk}|$ is finite. Since $A = (a_{nk})$ is itself bounded uniformly for each n, therefore $(A\zeta(\gamma))_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ exists for all $\gamma \in \Gamma$. Then

$$E[\|(A\zeta)_n\|] = E\Big[\Big\| \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) \Big\|\Big] \le E[\|\zeta(\gamma)\|] \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$$

uniformly for all n, since $\zeta \in \ell_\infty(\Gamma_E)$. Therefore, $A\zeta \in \ell_\infty(\Gamma_E)$. Hence, A defines a bounded linear operator from $\ell_\infty(\Gamma_E)$ into $\ell_\infty(\Gamma_E)$.

Conversely, let $A \in (\ell_\infty(\Gamma_E), \ell_\infty(\Gamma_E))$. That is A transforms a complex uncertain sequence $\zeta \in \ell_\infty(\Gamma_E)$ to another sequence $A\zeta \in \ell_\infty(\Gamma_E)$. This implies

$$\sup_n E[\|(A\zeta)_n\|] = \sup_n E\Big[\Big\| \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) \Big\|\Big] < \infty.$$

Then, by an application of Banach-Steinhaus theorem, we have $\|A\| = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$. □

Theorem 5. *$A : c_0(\Gamma_E) \to \ell_\infty(\Gamma_E)$ is a bounded linear operator if $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.*

Proof. Let $\zeta = \{\zeta_k\}$ be a complex uncertain null sequence. Then by the definition we have that for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $E\big[\|\zeta_n(\gamma) - 0\|\big] < \varepsilon$ for all $n > n_0$, that is $E\big[\|\zeta_n(\gamma)\|\big] < \varepsilon$ for all $\gamma \in \Gamma$. Let

$$H = \max\{E\big[\|\zeta_1(\gamma)\|\big], E\big[\|\zeta_2(\gamma)\|\big], \dots, E\big[\|\zeta_{n_0}(\gamma)\|\big], \varepsilon\}$$

for all $\gamma \in \Gamma$. Thus, $E\big[\|\zeta_n(\gamma)\|\big] < H$, whenever $\gamma \in \Gamma$. Therefore, $\zeta = \{\zeta_n\} \in \ell_\infty(\Gamma_E)$. Hence, we have $A\zeta \in \ell_\infty(\Gamma_E)$, by Theorem 4. □

Theorem 6. *$A \in (c(\Gamma_E), \ell_\infty(\Gamma_E))$ is a bounded linear operator if $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.*

Proof. Let $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$. Since every convergent complex uncertain sequence in mean is bounded in mean, therefore

$$\zeta \in c(\Gamma_E) \implies \zeta \in \ell_\infty(\Gamma_E)$$

and so $A\zeta \in \ell_\infty(\Gamma_E)$, by Theorem 4. Hence, $A \in (c(\Gamma_E), \ell_\infty(\Gamma_E))$. □

Theorem 7. *If $A \in (c(\Gamma_E), c(\Gamma_E))$, then also $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.*

Proof. Let $\zeta = \{\zeta_n\} \in c(\Gamma_E)$ be a complex uncertain sequence, which convergent in mean and $(A\zeta)_n = \sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ exists. Then by the assumption $A\zeta \in c(\Gamma_E)$. Let it converges to some finite limit. The existence of $A\zeta$ for each n and $\zeta \in c(\Gamma_E)$ proves the boundedness of $\sum_{k=1}^{\infty} |a_{nk}|$ uniformly for all n in mean and hence the result follows.

Also, by Banach-Steinhaus theorem, $\|A_n\| = \sum_{k=1}^{\infty} |a_{nk}|$. □

We now study the famous Silverman-Toeplitz theorem via expected value operator considering a complex uncertain sequence, which is convergent in mean.

Theorem 8. *A bounded linear operator A, which transforms a complex uncertain sequence $\{\zeta_n\} \in c(\Gamma_E)$ into $\{A\zeta_n\} \in c(\Gamma_E)$, preserves the limit if and only if the following conditions are satisfied:*

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}|$ *is finite;*

(ii) $\lim_{n\to\infty} a_{nk} = 0$, *while k is fixed;*

(iii) $\sum_{k=1}^{\infty} a_{nk} = 1$ *for $n \to \infty$.*

Proof. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space and $A : c(\Gamma_E) \to c(\Gamma_E)$ be a bounded linear operator, which preserves limit. Define the complex uncertain variables ζ_n for uniform values of k (k being a fixed natural at any instance) by

$$\zeta_n(\gamma) = \begin{cases} 1, & n = k; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\zeta(\gamma) = 0$ for all $\gamma \in \Gamma$. Then in each cases we get $\lim_{n\to\infty} E\left[\|\zeta_n - \zeta\|\right] = 0$. Hence, the complex uncertain sequence $\{\zeta_n\}$ is convergent in mean and it converges to zero.

Thus, $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = 0$ (by our hypothesis), which implies $\lim_{n\to\infty} a_{nk} = 0$ for fixed k. Thus, the condition *(ii)* is proved.

For the necessity of *(iii)*, let us consider the complex uncertain sequence $\{\zeta_k\}$ such that $\zeta_k = 1$ for all k and let $\zeta = 1$. Then $E\left[\|\zeta_k - \zeta\|\right] \to 0$ as $k \to \infty$. Thus the sequence $\{\zeta_k\}$ converges to $\zeta = 1$ in mean. Therefore, the transformed complex uncertain sequence is also converges to $\zeta = 1$ in mean. Consequently, $E\left[\left\|\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) - 1\right\|\right] \to 0$ as $n \to \infty$. This implies that $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} = 1$. Now, $\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)$ exists for each n and tends to ζ, whenever $\{\zeta_k\}$ converges to ζ in mean. Then by Theorem 7, we can say that $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$.

For sufficiency, let the three conditions hold true and the complex uncertain sequence $\{\zeta_k\}$ converges in mean to ζ. Now,

$$E\left[\left\|\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma)\right\|\right] \leq \sum_{k=1}^{\infty} a_{nk} E\left[\|\zeta_k(\gamma) - \zeta(\gamma)\|\right] + E\left[\left\|\zeta(\gamma)\sum_{k=1}^{\infty} a_{nk}\right\|\right].$$

Using condition *(i)* and the fact that $\zeta_k \to \zeta$ in mean, we have the first term of the right hand side of the above equation is zero. Again by condition *(iii)*, the second term of the right hand side tends to $E\left[\|\zeta(\gamma)\|\right]$ as $n \to \infty$. Therefore, we can write $\lim_{n\to\infty} E\left[\left\|\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) - \zeta(\gamma)\right\|\right] = 0$. Hence, the transformed complex uncertain sequence converges to ζ in mean. Hence, $A \in (c_0(\Gamma_E), c_0(\Gamma_E))$ and it keeps the limit preserved. □

Finally, we establish the Kojima-Schur theorem related to matrix transformation of sequences in uncertain environment.

Theorem 9. $A : c(\Gamma_E) \to c(\Gamma_E)$ *is a bounded linear operator if and only if the following conditions are satisfied:*

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$;

(ii) for each $p \in \mathbb{N}$ *there exists* $a_p = \lim_{n\to\infty}\sum_{k=p}^{\infty} a_{nk}$.

Proof. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertain space and $A \in (c(\Gamma_E), c(\Gamma_E))$. Suppose the complex uncertain sequence $\{\zeta_n\} \in c(\Gamma_E)$ converges to ζ in mean. Then from Theorem 7 it follows $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and thus *(i)* is proved. Consider the complex uncertain sequence $\{\zeta_n\}$ in such a way that

$$\zeta_n(\gamma) = \begin{cases} 0, & n < p; \\ 1, & \text{otherwise} \end{cases}$$

for some finite p and $\zeta(\gamma) = 1$ for all $\gamma \in \Gamma$. Then $\lim_{n\to\infty} E\left[\|\zeta_n - \zeta\|\right] = 0$ and so $\{\zeta_n\} \in c(\Gamma_E)$, which converges to $\zeta = 1$ in mean. Thus

$$\lim_{n\to\infty}(A\zeta(\gamma))_n = \lim_{n\to\infty} A\zeta_n(\gamma) = \lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}\zeta_n(\gamma) = \lim_{n\to\infty}\sum_{k=p}^{\infty} a_{nk} = a_p,$$

since $\zeta_n(\gamma) = 0$ for $n < p$.

Conversely, let conditions *(i)* and *(ii)* holds true and $\{\zeta_n\} \in c(\Gamma_E)$ converges to ζ in mean. Then,

$$\sum_{k=1}^{\infty} a_{nk}\zeta_k(\gamma) = \sum_{k=1}^{\infty} a_{nk}\{\zeta_k(\gamma) - \zeta(\gamma)\} + \zeta(\gamma)\sum_{k=1}^{\infty} a_{nk} = S_{\Gamma_n} + \zeta(\gamma)\sum_{k=1}^{\infty} a_{nk},$$

where $S_{\Gamma_n} = \sum_{k=1}^{\infty} a_{nk}\{\zeta_k(\gamma) - \zeta(\gamma)\}$. Now, by condition *(i)*, the term $\zeta(\gamma)\sum_{k=1}^{\infty} a_{nk}$ tends to $a_1\zeta(\gamma)$. Suppose

$$b_k = \lim_{n\to\infty} a_{nk} = \lim_{n\to\infty}\Big(\sum_{j=k}^{\infty} a_{nj} - \sum_{j=k+1}^{\infty} a_{nj}\Big) = a_k - a_{k+1}$$

for each k. So, $\sum_k |b_k| = \sum_k |\lim_{n\to\infty} a_{nk}| \le \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$, by condition *(i)*. Again,

$$E\Big[\sum_{k=1}^{\infty}(a_{nk} - b_k)\|\zeta_k(\gamma) - \zeta(\gamma)\|\Big] = \sum_{k=1}^{\infty}(a_{nk} - b_k)E\big[\|\zeta_k(\gamma) - \zeta(\gamma)\|\big].$$

Since $\{\zeta_n(\gamma)\}$ converges to ζ in mean, so S_{Γ_n} tends to $\sum_{k=1}^{\infty} b_k(\zeta_k(\gamma) - \zeta(\gamma))$. Hence, we obtain $A \in (c(\Gamma_E), c(\Gamma_E))$. □

3 Conclusion

In an uncertainty space, the convergence of a complex uncertain sequence and complex uncertain series are defined via five concepts viz. convergence in mean, in measure, in distribution, in almost surely and with respect to uniformly almost surely. In this article, we have studied matrix transformation of complex uncertain sequences in mean via expected value operator. These results can be proved by considering the other concepts on convergences too. These results can be generalized for further studies in this direction.

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Метою статті є вивчити поняття матричного перетворення між комплексними невизначеними послідовностями в середньому. Характеризацію матричного перетворення зроблено шляхом застосування поняття збіжності комплексних невизначених рядів. Більше того, у цьому контексті деякі добре відомі теореми для просторів дійсних послідовностей були встановлені шляхом розгляду комплексної невизначеної послідовності через оператор математичного сподівання.

Ключові слова і фрази: простір невизначеності, комплексний невизначений ряд, комплексна невизначена послідовність, матричне перетворення.