ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2022, **14** (1), 6–19
doi:10.15330/cmp.14.1.6-19

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2022, Т.14, №1, С.6–19

Identities relating six members of the Fibonacci family of sequences

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In this paper, we prove several identities each relating a sum of products of three terms coming from different members of the Fibonacci family of sequences with a comparable sum whose terms come from three other sequences. These identities are obtained as special cases of formulas relating two linear combinations of products of three generalized Fibonacci or Lucas sequences. The latter formulas are in turn obtained from a more general generating function result for the product of three terms coming from second-order linearly recurrent sequences with arbitrary initial values. We employ algebraic arguments to establish our results, making use of the Binet-like formulas of the underlying sequences. Among the sequences for which the aforementioned identities are found include the Fibonacci, Pell, Jacobsthal and Mersenne numbers, along with their associated Lucas companion sequences.

Key words and phrases: Horadam sequence, Fibonacci sequence, Lucas sequence, Pell sequence, Jacobsthal sequence, gibonacci sequence, generating function.

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1 Introduction

Let $U_n = U_n(p,q)$ denote the *generalized Fibonacci sequence* defined recursively by

$$U_0 = 0,\ U_1 = 1,\ U_n = pU_{n-1} + qU_{n-2}, \quad n \geq 2,$$

and $V_n = V_n(p,q)$ the *generalized Lucas sequence* defined by

$$V_0 = 2,\ V_1 = p,\ V_n = pV_{n-1} + qV_{n-2}, \quad n \geq 2.$$

Recall the special cases of $U_n(p,q)$ given by $F_n = U_n(1,1)$, $P_n = U_n(2,1)$, $J_n = U_n(1,2)$, $B_n = U_n(6,-1)$ and $M_n = U_n(3,-2)$ corresponding respectively to the Fibonacci, Pell, Jacobsthal, balancing and Mersenne number sequences. The respective companion sequences are given by $L_n = V_n(1,1)$, $Q_n = V_n(2,1)$, $j_n = V_n(1,2)$, $C_n = V_n(6,-1)$ and $K_n = V_n(3,-2)$, the first three of which are referred to as the Lucas, Pell-Lucas and Jacobsthal-Lucas numbers. Note that C_n is twice of what is referred to as the n-th Lucas-balancing number. As special

УДК 511.176
2020 *Mathematics Subject Classification:* 11B37, 11B39.

Statements and conclusions made in this paper by R. Frontczak are entirely those of the author. They do not necessarily reflect the views of LBBW.

cases of the results in the next section, we obtain several identities each involving exactly six of the aforementioned sequences.

This paper is a continuation of our study of relations for the Fibonacci family of sequences. The principal tool that will be used are generating functions of second-order sequences, which have been discussed in detail recently by I. Mező [5]. In [3], we derived the ordinary generating function for products of two arbitrary second-order sequences. Some special cases of this formula also appear in [1]. From the formula found in [3], several identities are derived involving linear combinations of convolutions of the generalized Fibonacci and Lucas sequences. As a consequence, various identities are obtained as special cases which relate exactly four sequences amongst the Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas number sequences.

In this paper, we proceed one step further and prove relations involving six members of the Fibonacci sequence family. A related formula has been found by D.L. Russell [6] for the sum of products of three different Fibonacci sequences. D.L. Russell proves his formula by induction using an identity relating three members of the Fibonacci family. Here, we employ a generating function approach in extending the results from [3] and our formulas relate certain linear combinations of products of terms coming from three sequences.

The organization of this paper is as follows. In the next section, we first establish a general generating function formula for products of three linearly recurrent second-order sequences with constant coefficients where there is no restriction on the constants or initial values. We then work out several cases of this formula where the terms come from the generalized Fibonacci or Lucas sequences. Comparing the generating functions derived from these cases leads to identities relating sums of products of three members of the Fibonacci family of sequences with a comparable sum involving three other sequences. See Theorems 2 and 3 below and the examples that follow. Further related formulas can be obtained that involve sums of products of three gibonacci numbers. In the third section, some possible generalizations of the foregoing results are discussed.

2 Main results

Let a, b, p, q denote indeterminates or arbitrary (possibly complex) numbers with $p^2 + 4q \neq 0$. Let $T_n = T_n(a, b, p, q)$ be defined recursively by

$$T_0 = a, \ T_1 = b, \ T_n = pT_{n-1} + qT_{n-2}, \quad n \geq 2.$$

Note that T_n reduces to U_n when $a = 0, b = 1$ and to V_n when $a = 2$, $b = p$. Let $T_n^{(i)} = T_n(a_i, b_i, p_i, q_i)$ for $1 \leq i \leq 3$, where (a_i, b_i, p_i, q_i) is arbitrary for each i, with $U_n^{(i)} = U_n^{(i)}(p_i, q_i)$ and $V_n^{(i)} = V_n^{(i)}(p_i, q_i)$ corresponding to the cases of $T_n^{(i)}$ when (a_i, b_i) is given by $(0, 1)$ and $(2, p_i)$, respectively.

Let $\Delta_i = \sqrt{p_i^2 + 4q_i}$. Suppose $f(x)$ is a polynomial in x whose coefficients are rational functions in the variables a_i, b_i, q_i, p_i and the quantities Δ_i for $1 \leq i \leq 3$. Then let $\overline{f}(x)$ be the polynomial obtained by replacing each occurrence of Δ_3 in $f(x)$ with $-\Delta_3$ (i.e., $\overline{f}(x)$ is the conjugate of $f(x)$ with respect to the radical quantity Δ_3).

We will need the following factorization result in proofs of subsequent formulas.

Lemma 1. *We have* $M(x) = G(x)\overline{G}(x)$*, where*

$$G(x) = 1 - \frac{p_1(p_2p_3 + \Delta_2\Delta_3)}{2}x + \frac{2p_1^2q_2q_3 - q_1(p_2^2 + 2q_2)(p_3^2 + 2q_3) - p_2p_3q_1\Delta_2\Delta_3}{2}x^2 + \frac{p_1q_1q_2q_3(p_2p_3 + \Delta_2\Delta_3)}{2}x^3 + (q_1q_2q_3)^2x^4$$

and

$$M(x) = 1 + \sum_{i=1}^{8} m_i x^i,$$

with

$$\begin{aligned}
m_1 &= -p_1p_2p_3,\\
m_2 &= -\big(p_1^2p_2^2q_3 + p_1^2q_2p_3^2 + q_1p_2^2p_3^2 + 2(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2) + 4q_1q_2q_3\big),\\
m_3 &= -p_1p_2p_3\big(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2 + 5q_1q_2q_3\big),\\
m_4 &= p_1^4q_2^2q_3^2 + q_1^2p_2^4q_3^2 + q_1^2q_2^2p_3^4 + 6(q_1q_2q_3)^2\\
&\qquad + 4q_1q_2q_3(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2) - q_1q_2q_3(p_1p_2p_3)^2,\\
m_5 &= p_1p_2p_3q_1q_2q_3\big(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2 + 5q_1q_2q_3\big),\\
m_6 &= -(q_1q_2q_3)^2\big(p_1^2p_2^2q_3 + p_1^2q_2p_3^2 + q_1p_2^2p_3^2 + 2(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2) + 4q_1q_2q_3\big),\\
m_7 &= p_1p_2p_3(q_1q_2q_3)^3,\\
m_8 &= (q_1q_2q_3)^4.
\end{aligned}$$

Proof. This can be shown by expanding the product $G(x)\overline{G}(x)$ and computing the coefficient of x^i for each $0 \le i \le 8$, making use of the definitions. We illustrate the cases $i = 3$ and $i = 4$, with similar arguments applying to the others. When $i = 3$, the coefficient of x^3 in $G(x)\overline{G}(x)$ is given by

$$\begin{aligned}
p_1p_2p_3q_1q_2q_3 &- \frac{p_1}{4}(p_2p_3 + \Delta_2\Delta_3)\big(2p_1^2q_2q_3 - q_1(p_2^2 + 2q_2)(p_3^2 + 2q_3) + p_2p_3q_1\Delta_2\Delta_2\big)\\
&- \frac{p_1}{4}(p_2p_3 - \Delta_2\Delta_3)\big(2p_1^2q_2q_3 - q_1(p_2^2 + 2q_2)(p_3^2 + 2q_3) - p_2p_3q_1\Delta_2\Delta_2\big)\\
&= \frac{p_1p_2p_3}{2}\big(2q_1q_2q_3 - 2p_1^2q_2q_3 + q_1(p_2^2 + 2q_2)(p_3^2 + 2q_3) - q_1\Delta_2^2\Delta_3^2\big)\\
&= -\frac{p_1p_2p_3}{2}\big(2p_1^2q_2q_3 - 2q_1q_2q_3 + q_1\big((p_2^2 + 4q_2)(p_3^2 + 4q_3) - (p_2^2 + 2q_2)(p_3^2 + 2q_3)\big)\big),
\end{aligned}$$

which simplifies to the formula above for m_3. When $i = 4$, the coefficient of x^4 in $G(x)\overline{G}(x)$ is given by

$$2(q_1q_2q_3)^2 - \frac{p_1^2q_1q_2q_3(p_2^2p_3^2 - \Delta_2^2\Delta_3^2)}{2} + \big(p_1^2q_2q_3 - \frac{q_1}{2}(p_2^2 + 2q_2)(p_3^2 + 2q_3)\big)^2 - \frac{p_2^2p_3^2q_1^2}{4}\Delta_2^2\Delta_3^2,$$

upon considering a difference of squares when accounting for the contribution coming from the product of the x and the x^3 terms in G and $\overline{G}$ as well as from the two x^2 terms. This expression may be rewritten as

$$\begin{aligned}
2(q_1q_2q_3)^2 &+ 2p_1^2q_1q_2q_3(p_2^2q_3 + p_3^2q_2 + 4q_2q_3) - \frac{p_2^2p_3^2q_1^2}{4}(p_2^2p_3^2 + 4p_2^2q_3 + 4p_3^2q_2 + 16q_2q_3)\\
&+ p_1^4q_2^2q_3^2 - p_1^2q_1q_2q_3(p_2^2p_3^2 + 2p_2^2q_3 + 2p_3^2q_2 + 4q_2q_3)\\
&+ \frac{q_1^2}{4}(p_2^4 + 4p_2^2q_2 + 4q_2^2)(p_3^4 + 4p_3^2q_3 + 4q_3^2),
\end{aligned}$$

which simplifies to the formula for m_4. $\square$

For a fixed i, let

$$r_1^{(i)} = \frac{p_i + \Delta_i}{2}, \quad r_2^{(i)} = \frac{p_i - \Delta_i}{2}, \quad \alpha_i = \frac{2b_i - a_i p_i + a_i \Delta_i}{2\Delta_i}, \quad \beta_i = \frac{a_i p_i - 2b_i + a_i \Delta_i}{2\Delta_i}.$$

Note that $T_n^{(i)}$ is given explicitly by the Binet-like formula $T_n^{(i)} = \alpha_i \big(r_1^{(i)}\big)^n + \beta_i \big(r_2^{(i)}\big)^n$ for $n \geq 0$.

We have the following general formula for the generating function of the product of three arbitrary T_n sequences.

Theorem 1. *We have*

$$\sum_{n\geq 0} T_n^{(1)} T_n^{(2)} T_n^{(3)} x^n = \frac{G(x)\overline{H}(x) + \overline{G}(x)H(x)}{M(x)}, \tag{1}$$

where $G(x)$ and $M(x)$ are given in Lemma 1 and

$$\begin{aligned} H(x) = {} & \alpha_2\alpha_3\big(a_1 - r_1^{(2)} r_1^{(3)}(a_1p_1 - b_1)x\big)\big(1 - p_1 r_2^{(2)} r_2^{(3)} x - q_1\big(r_2^{(2)} r_2^{(3)} x\big)^2\big) \\ & + \beta_2\beta_3\big(a_1 - r_2^{(2)} r_2^{(3)}(a_1p_1 - b_1)x\big)\big(1 - p_1 r_1^{(2)} r_1^{(3)} x - q_1\big(r_1^{(2)} r_1^{(3)} x\big)^2\big). \end{aligned}$$

Proof. First note that

$$\begin{aligned} \sum_{n\geq 0} T_n^{(1)} T_n^{(2)} T_n^{(3)} x^n &= \sum_{n\geq 0} \big(\alpha_1\big(r_1^{(1)}\big)^n + \beta_1\big(r_2^{(1)}\big)^n\big)\big(\alpha_2\big(r_1^{(2)}\big)^n + \beta_2\big(r_2^{(2)}\big)^n\big)\big(\alpha_3\big(r_1^{(3)}\big)^n + \beta_3\big(r_2^{(3)}\big)^n\big)x^n \\
&= \alpha_2\alpha_3\left(\frac{\alpha_1}{1 - r_1^{(1)} r_1^{(2)} r_1^{(3)} x} + \frac{\beta_1}{1 - r_2^{(1)} r_1^{(2)} r_1^{(3)} x}\right) + \beta_2\beta_3\left(\frac{\alpha_1}{1 - r_1^{(1)} r_2^{(2)} r_2^{(3)} x} + \frac{\beta_1}{1 - r_2^{(1)} r_2^{(2)} r_2^{(3)} x}\right) \\
&\quad + \alpha_2\beta_3\left(\frac{\alpha_1}{1 - r_1^{(1)} r_1^{(2)} r_2^{(3)} x} + \frac{\beta_1}{1 - r_2^{(1)} r_1^{(2)} r_2^{(3)} x}\right) + \beta_2\alpha_3\left(\frac{\alpha_1}{1 - r_1^{(1)} r_2^{(2)} r_1^{(3)} x} + \frac{\beta_1}{1 - r_2^{(1)} r_2^{(2)} r_1^{(3)} x}\right) \\
&= \alpha_2\alpha_3 \cdot \frac{a_1 - r_1^{(2)} r_1^{(3)}\big(\alpha_1 r_2^{(1)} + \beta_1 r_1^{(1)}\big)x}{\big(1 - r_1^{(1)}\big(r_1^{(2)} r_1^{(3)} x\big)\big)\big(1 - r_2^{(1)}\big(r_1^{(2)} r_1^{(3)} x\big)\big)} + \beta_2\beta_3 \cdot \frac{a_1 - r_2^{(2)} r_2^{(3)}\big(\alpha_1 r_2^{(1)} + \beta_1 r_1^{(1)}\big)x}{\big(1 - r_1^{(1)}\big(r_2^{(2)} r_2^{(3)} x\big)\big)\big(1 - r_2^{(1)}\big(r_2^{(2)} r_2^{(3)} x\big)\big)} \\
&\quad + \alpha_2\beta_3 \cdot \frac{a_1 - r_1^{(2)} r_2^{(3)}\big(\alpha_1 r_2^{(1)} + \beta_1 r_1^{(1)}\big)x}{\big(1 - r_1^{(1)}\big(r_1^{(2)} r_2^{(3)} x\big)\big)\big(1 - r_2^{(1)}\big(r_1^{(2)} r_2^{(3)} x\big)\big)} + \beta_2\alpha_3 \cdot \frac{a_1 - r_2^{(2)} r_1^{(3)}\big(\alpha_1 r_2^{(1)} + \beta_1 r_1^{(1)}\big)x}{\big(1 - r_1^{(1)}\big(r_2^{(2)} r_1^{(3)} x\big)\big)\big(1 - r_2^{(1)}\big(r_2^{(2)} r_1^{(3)} x\big)\big)} \\
&= \alpha_2\alpha_3 \cdot \frac{a_1 - r_1^{(2)} r_1^{(3)}(a_1p_1 - b_1)x}{1 - p_1 r_1^{(2)} r_1^{(3)} x - q_1\big(r_1^{(2)} r_1^{(3)} x\big)^2} + \beta_2\beta_3 \cdot \frac{a_1 - r_2^{(2)} r_2^{(3)}(a_1p_1 - b_1)x}{1 - p_1 r_2^{(2)} r_2^{(3)} x - q_1\big(r_2^{(2)} r_2^{(3)} x\big)^2} \\
&\quad + \alpha_2\beta_3 \cdot \frac{a_1 - r_1^{(2)} r_2^{(3)}(a_1p_1 - b_1)x}{1 - p_1 r_1^{(2)} r_2^{(3)} x - q_1\big(r_1^{(2)} r_2^{(3)} x\big)^2} + \beta_2\alpha_3 \cdot \frac{a_1 - r_2^{(2)} r_1^{(3)}(a_1p_1 - b_1)x}{1 - p_1 r_2^{(2)} r_1^{(3)} x - q_1\big(r_2^{(2)} r_1^{(3)} x\big)^2}, \end{aligned} \tag{2}$$

where we have used the facts $\alpha_1 + \beta_1 = a_1$, $r_1^{(1)} + r_2^{(1)} = p_1$, $\alpha_1 r_2^{(1)} + \beta_1 r_1^{(1)} = a_1 p_1 - b_1$, and $r_1^{(1)} r_2^{(1)} = -q_1$.

Observe that the sum of the first two expressions in (2) is given by

$$\frac{H(x)}{\Big(1 - p_1 r_1^{(2)} r_1^{(3)} x - q_1\big(r_1^{(2)} r_1^{(3)} x\big)^2\Big)\Big(1 - p_1 r_2^{(2)} r_2^{(3)} x - q_1\big(r_2^{(2)} r_2^{(3)} x\big)^2\Big)},$$

where $H(x)$ is as defined above. Expanding the denominator in the last expression, we obtain

$$1 - p_1\big(r_1^{(2)}r_1^{(3)} + r_2^{(2)}r_2^{(3)}\big)x + \Big(p_1^2 r_1^{(2)}r_2^{(2)}r_1^{(3)}r_2^{(3)} - q_1\Big(\big(r_1^{(2)}r_1^{(3)}\big)^2 + \big(r_2^{(2)}r_2^{(3)}\big)^2\Big)\Big)x^2 \\ + p_1q_1r_1^{(2)}r_2^{(2)}r_1^{(3)}r_2^{(3)}\big(r_1^{(2)}r_1^{(3)} + r_2^{(2)}r_2^{(3)}\big)x^3 + q_1^2\big(r_1^{(2)}r_2^{(2)}r_1^{(3)}r_2^{(3)}\big)^2x^4.$$

From the definitions, we have

$$r_1^{(2)}r_1^{(3)} + r_2^{(2)}r_2^{(3)} = \frac{(p_2+\Delta_2)(p_3+\Delta_3)+(p_2-\Delta_2)(p_3-\Delta_3)}{4} = \frac{p_2p_3+\Delta_2\Delta_3}{2}$$

and

$$\begin{aligned}\big(r_1^{(2)}r_1^{(3)}\big)^2 + \big(r_2^{(2)}r_2^{(3)}\big)^2 &= \frac{(p_2+\Delta_2)^2(p_3+\Delta_3)^2+(p_2-\Delta_2)^2(p_3-\Delta_3)^2}{16} \\ &= \frac{(p_2^2+2q_2+p_2\Delta_2)(p_3^2+2q_3+p_3\Delta_3)+(p_2^2+2q_2-p_2\Delta_2)(p_3^2+2q_3-p_3\Delta_3)}{4} \\ &= \frac{(p_2^2+2q_2)(p_3^2+2q_3)+p_2p_3\Delta_2\Delta_3}{2}.\end{aligned}$$

Since $r_1^{(i)}r_2^{(i)} = -q_i$ for each i, we thus get for the denominator

$$1 - \frac{p_1(p_2p_3+\Delta_2\Delta_3)}{2}x + \frac{2p_1^2q_2q_3 - q_1(p_2^2+2q_2)(p_3^2+2q_3) - p_2p_3q_1\Delta_2\Delta_3}{2}x^2 \\ + \frac{p_1q_1q_2q_3(p_2p_3+\Delta_2\Delta_3)}{2}x^3 + (q_1q_2q_3)^2x^4,$$

which coincides with $G(x)$ as defined above. Hence, the sum of the first two expressions in (2) equals $H(x)/G(x)$. Since the last two expressions in (2) can be obtained from the first two by replacing Δ_3 with $-\Delta_3$, their sum is given by $\overline{H}(x)/\overline{G}(x)$. Note that in terms of the notation above $\overline{H}(x)$ is given explicitly as

$$\overline{H}(x) = \alpha_2\beta_3\big(a_1 - r_1^{(2)}r_2^{(3)}(a_1p_1-b_1)x\big)\big(1 - p_1r_2^{(2)}r_1^{(3)}x - q_1\big(r_2^{(2)}r_1^{(3)}x\big)^2\big) \\ + \beta_2\alpha_3\big(a_1 - r_2^{(2)}r_1^{(3)}(a_1p_1-b_1)x\big)\big(1 - p_1r_1^{(2)}r_2^{(3)}x - q_1\big(r_1^{(2)}r_2^{(3)}x\big)^2\big).$$

By Lemma 1, we thus get

$$\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}T_n^{(3)}x^n = \frac{H(x)}{G(x)} + \frac{\overline{H}(x)}{\overline{G}(x)} = \frac{G(x)\overline{H}(x)+\overline{G}(x)H(x)}{M(x)},$$

as desired. □

In general, the formula for the numerator of $\sum_{n\geq 0} T_n^{(1)}T_n^{(2)}T_n^{(3)}x^n$ in (1) is somewhat complicated. However, for several particular choices of initial values a_i and b_i, there are significant simplifications. For example, when $T_n^{(i)} = U_n^{(i)}$ for each i, we get the following result.

Corollary 1. *We have*

$$\sum_{n\geq 0} U_n^{(1)}U_n^{(2)}U_n^{(3)}x^n = \frac{\sum_{i=1}^{7} s_ix^i}{M(x)}, \tag{3}$$

where $M(x)$ is as in Lemma 1, $s_1 = 1$, $s_2 = s_6 = 0$ and

$$\begin{aligned} s_3 &= -\big(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2 + 3q_1q_2q_3\big), \\ s_4 &= -2p_1p_2p_3q_1q_2q_3, \\ s_5 &= q_1q_2q_3\big(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2 + 3q_1q_2q_3\big), \\ s_7 &= -(q_1q_2q_3)^3. \end{aligned}$$

Proof. Note first that $\alpha_i = -\beta_i = \frac{1}{\Delta_i}$ when $a_i = 0$ and $b_i = 1$. Thus, in this case, $H(x)$ reduces to

$$\begin{aligned}
&\frac{x}{\Delta_2\Delta_3}\left(r_1^{(2)}r_1^{(3)}\left(1 - p_1 r_2^{(2)}r_2^{(3)}x - q_1\left(r_2^{(2)}r_2^{(3)}x\right)^2\right) + r_2^{(2)}r_2^{(3)}\left(1 - p_1 r_1^{(2)}r_1^{(3)}x - q_1\left(r_1^{(2)}r_1^{(3)}x\right)^2\right)\right)\\
&\quad = \frac{x}{\Delta_2\Delta_3}\left(r_1^{(2)}r_1^{(3)} + r_2^{(2)}r_2^{(3)} - 2p_1q_2q_3x - q_1q_2q_3\left(r_1^{(3)}r_1^{(3)} + r_2^{(2)}r_2^{(3)}\right)x^2\right)\\
&\quad = \frac{x}{\Delta_2\Delta_3}\left(\frac{p_2p_3 + \Delta_2\Delta_3}{2} - 2p_1q_2q_3x - \frac{q_1q_2q_3(p_2p_3 + \Delta_2\Delta_3)}{2}x^2\right).
\end{aligned}$$

By Theorem 1, the numerator of $\sum_{n\geq 0} U_n^{(1)}U_n^{(2)}U_n^{(3)}x^n$ equals $G(x)\overline{H}(x) + \overline{G}(x)H(x)$, where $H(x)$ is as given. Note that the denominator is unchanged from Theorem 1 since no restrictions have been imposed concerning the p_i and q_i. In computing the numerator, it is a simpler matter to consider only the product $\overline{G}(x)H(x)$ and then double all terms in which Δ_3 appears with an even exponent (including zero or negative even powers) and to ignore all terms containing an odd power of Δ_3. We illustrate by finding the coefficients of x^3 and x^5 below. Note that $\overline{G}(x)H(x)$ is given explicitly as

$$\begin{aligned}
&\frac{x}{4\Delta_2\Delta_3}\left(p_2p_3 + \Delta_2\Delta_3 - 4p_1q_2q_3x - q_1q_2q_3(p_2p_3 + \Delta_2\Delta_3)x^2\right)\\
&\qquad \times \left(2 - p_1(p_2p_3 - \Delta_2\Delta_3)x + \left(2p_1^2q_2q_3 - q_1(p_2^2 + 2q_2)(p_3^2 + 2q_3) + p_2p_3q_1\Delta_2\Delta_2\right)x^2\right.\\
&\qquad \left. + p_1q_1q_2q_3(p_2p_3 - \Delta_2\Delta_3)x^3 + 2(q_1q_2q_3)^2x^4\right).
\end{aligned}$$

For the x^3 term in $G(x)\overline{H}(x) + \overline{G}(x)H(x)$, we thus get

$$\frac{1}{2}\left(p_2^2p_3^2q_1 + 2p_1^2q_2q_3 - q_1(p_2^2 + 2q_2)(p_3^2 + 2q_3) - 4p_1^2q_2q_3 - 2q_1q_2q_3\right),$$

which reduces to the formula above for s_3. For x^5, we have

$$\begin{aligned}
&\frac{1}{2}\left(2(q_1q_2q_3)^2 + 4p_1^2q_1q_2^2q_3^2 - p_2^2p_3^2q_1^2q_2q_3 - q_1q_2q_3(2p_1^2q_2q_3 - q_1(p_2^2 + 2q_2)(p_3^2 + 2q_3))\right)\\
&\quad = \frac{q_1q_2q_3}{2}\left(2q_1q_2q_3 + 2p_1^2q_2q_3 - q_1p_2^2p_3^2 + q_1(p_2^2 + 2q_2)(p_3^2 + 2q_3)\right),
\end{aligned}$$

which reduces to s_5. The remaining s_i can be found similarly; note that only terms in which Δ_3 appears with odd exponent occur in the expressions for the coefficients of x^2 and x^6 in $\overline{G}(x)H(x)$ and hence the coefficient for these powers in the numerator in (3) is zero. □

Taking particular values of the parameters p_i and q_i in Corollary 1 (for instance, letting $p_1 = q_1 = 1$, $p_2 = 2$, $q_2 = 1$ and $p_3 = 1$, $q_3 = 2$ for the first formula) yields the following.

Example 1.

$$\begin{aligned}
\sum_{n\geq 0} F_nP_nJ_nx^n &= \frac{x - 17x^3 - 8x^4 + 34x^5 - 8x^7}{1 - 2x - 43x^2 - 42x^3 + 173x^4 + 84x^5 - 172x^6 + 16x^7 + 16x^8},\\
\sum_{n\geq 0} F_nP_nB_nx^n &= \frac{x - 28x^3 + 24x^4 - 28x^5 + x^7}{1 - 12x - 234x^2 - 312x^3 + 1339x^4 - 312x^5 - 234x^6 - 12x^7 + x^8},\\
\sum_{n\geq 0} J_nM_nB_nx^n &= \frac{x + 148x^3 - 144x^4 - 592x^5 - 64x^7}{1 - 18x - 263x^2 + 2520x^3 + 17304x^4 - 10080x^5 - 4208x^6 + 1152x^7 + 256x^8}.
\end{aligned}$$

We next consider the case when $T_n^{(i)} = V_n^{(i)}$ for all i in Theorem 1.

Corollary 2. *We have*

$$\sum_{n\geq 0} V_n^{(1)} V_n^{(2)} V_n^{(3)} x^n = \frac{8+\sum_{i=1}^{7} t_i x^i}{M(x)},$$

where $M(x)$ is as in Lemma 1 and

$$\begin{aligned}
t_1 &= -7p_1p_2p_3,\\
t_2 &= -6\big(p_1^2p_2^2q_3 + p_1^2q_2p_3^2 + q_1p_2^2p_3^2 + 2(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2) + 4q_1q_2q_3\big),\\
t_3 &= -5p_1p_2p_3\big(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2 + 5q_1q_2q_3\big),\\
t_4 &= 16q_1q_2q_3(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2) + 24(q_1q_2q_3)^2\\
&\qquad - 4q_1q_2q_3(p_1p_2p_3)^2 + 4(p_1^4q_2^2q_3^2 + q_1^2p_2^4q_3^2 + q_1^2q_2^2p_3^4),\\
t_5 &= 3p_1p_2p_3q_1q_2q_3(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2 + 5q_1q_2q_3),\\
t_6 &= -2(q_1q_2q_3)^2\big(p_1^2p_2^2q_3 + p_1^2q_2p_3^2 + q_1p_2^2p_3^2 + 2(p_1^2q_2q_3 + q_1p_2^2q_3 + q_1q_2p_3^2) + 4q_1q_2q_3\big),\\
t_7 &= p_1p_2p_3(q_1q_2q_3)^3.
\end{aligned}$$

Proof. First note $a_i = 2$, $b_i = p_i$ implies $\alpha_i = \beta_i = 1$. Thus, in this case, we have that $H(x)$ reduces to

$$\begin{aligned}
&\big(2 - p_1r_1^{(2)}r_1^{(3)}x\big)\big(1 - p_1r_2^{(2)}r_2^{(3)}x - q_1\big(r_2^{(2)}r_2^{(3)}x\big)^2\big)\\
&\quad + \big(2 - p_1r_2^{(2)}r_2^{(3)}x\big)\big(1 - p_1r_1^{(2)}r_1^{(3)}x - q_1\big(r_1^{(2)}r_1^{(3)}x\big)^2\big)\\
&= 4 - 3p_1\big(r_1^{(2)}r_1^{(3)} + r_2^{(2)}r_2^{(3)}\big)x + 2\left(p_1^2q_2q_3 - q_1\big(r_1^{(2)}r_1^{(3)}\big)^2 - q_1\big(r_2^{(2)}r_2^{(3)}\big)^2\right)x^2\\
&\quad + p_1q_1q_2q_3\big(r_1^{(2)}r_1^{(3)} + r_2^{(2)}r_2^{(3)}\big)x^3\\
&= 4 - 3p_1\frac{p_2p_3+\Delta_2\Delta_3}{2}x + \left(2p_1^2q_2q_3 - q_1(p_2^2+2q_2)(p_3^2+2q_3) - p_2p_3q_1\Delta_2\Delta_3\right)x^2\\
&\quad + p_1q_1q_2q_3\frac{p_2p_3+\Delta_2\Delta_3}{2}x^3.
\end{aligned}$$

Therefore, the numerator in (1) is given by $G(x)\overline{H}(x) + \overline{G}(x)H(x)$, where $\overline{G}(x)H(x)$ equals

$$\begin{aligned}
&\frac{1}{4}\Big(8 - 3p_1(p_2p_3+\Delta_2\Delta_3)x + 2\big(2p_1^2q_2q_3 - q_1(p_2^2+2q_2)(p_3^2+2q_3) - p_2p_3q_1\Delta_2\Delta_3\big)x^2\\
&\quad + p_1q_1q_2q_3(p_2p_3+\Delta_2\Delta_3)x^3\Big)\Big(2 - p_1(p_2p_3-\Delta_2\Delta_3)x + \big(2p_1^2q_2q_3\\
&\quad - q_1(p_2^2+2q_2)(p_3^2+2q_3) + p_2p_3q_1\Delta_2\Delta_3\big)x^2 + p_1q_1q_2q_3(p_2p_3-\Delta_2\Delta_3)x^3 + 2(q_1q_2q_3)^2x^4\Big).
\end{aligned}$$

Computing (and doubling) only the terms in this product where Δ_3 occurs with an even exponent leads to the formulas above for the t_i. □

From Corollary 2, we get results such as the following.

Example 2.

$$\sum_{n\geq 0} L_nQ_nj_nx^n = \frac{8 - 14x - 258x^2 - 210x^3 + 692x^4 + 252x^5 - 344x^6 + 16x^7}{1 - 2x - 43x^2 - 42x^3 + 173x^4 + 84x^5 - 172x^6 + 16x^7 + 16x^8},$$

$$\sum_{n\geq 0} L_nQ_nC_nx^n = \frac{8 - 84x - 1404x^2 - 1560x^3 + 5356x^4 - 936x^5 - 468x^6 - 12x^7}{1 - 12x - 234x^2 - 312x^3 + 1339x^4 - 312x^5 - 234x^6 - 12x^7 + x^8},$$

$$\sum_{n\geq 0} j_nK_nC_nx^n = \frac{8 - 126x - 1578x^2 + 12600x^3 + 69216x^4 - 30240x^5 - 8416x^6 + 1152x^7}{1 - 18x - 263x^2 + 2520x^3 + 17304x^4 - 10080x^5 - 4208x^6 + 1152x^7 + 256x^8}.$$

Letting $T_n^{(i)} = U_n^{(i)}$ for $i = 1, 2$ and $T_n^{(3)} = V_n^{(3)}$ gives the following formula.

Corollary 3. *We have*

$$\sum_{n\geq 0} U_n^{(1)} U_n^{(2)} V_n^{(3)} x^n = \frac{\sum_{i=1}^{7} r_i x^i}{M(x)},$$

where $M(x)$ *is as in Lemma 1 and*

$$r_1 = p_3, \qquad r_2 = 2p_1p_2q_3, \qquad r_3 = p_3\left(p_1^2q_2q_3 + q_1p_2^2q_3 - q_1q_2p_3^2 - q_1q_2q_3\right),$$
$$r_4 = 0, \qquad r_5 = p_3q_1q_2q_3(p_1^2q_2q_3 + q_1p_2^2q_3 - q_1q_2p_3^2 - q_1q_2q_3),$$
$$r_6 = -2p_1p_2q_1^2q_2^2q_3^3, \qquad r_7 = p_3(q_1q_2q_3)^3.$$

Proof. Note in this case that $\alpha_i = -\beta_i = \frac{1}{\Delta_i}$ for $i = 1, 2$ with $\alpha_3 = \beta_3 = 1$ and thus $H(x)$ reduces to

$$\frac{x}{\Delta_2}\left(r_1^{(2)}r_1^{(3)}\left(1 - p_1r_2^{(2)}r_2^{(3)}x - q_1\left(r_2^{(2)}r_2^{(3)}x\right)^2\right) - r_2^{(2)}r_2^{(3)}\left(1 - p_1r_1^{(2)}r_1^{(3)}x - q_1\left(r_1^{(2)}r_1^{(3)}x\right)^2\right)\right)$$
$$= \frac{x}{\Delta_2}\left(r_1^{(2)}r_1^{(3)} - r_2^{(2)}r_2^{(3)} - q_1q_2q_3\left(r_2^{(2)}r_2^{(3)} - r_1^{(2)}r_1^{(3)}\right)x^2\right) = \frac{x(1+q_1q_2q_3x^2)\left(r_1^{(2)}r_1^{(3)} - r_2^{(2)}r_2^{(3)}\right)}{\Delta_2}$$
$$= \frac{x(1+q_1q_2q_3x^2)((p_2+\Delta_2)(p_3+\Delta_3)-(p_2-\Delta_2)(p_3-\Delta_3))}{4\Delta_2} = \frac{x(1+q_1q_2q_3x^2)(p_2\Delta_3 + p_3\Delta_2)}{2\Delta_2}.$$

Applying (1), and considering $\overline{G}(x)H(x)$ as before, yields the desired formulas for the r_i. □

Example 3.

$$\sum_{n\geq 0} F_nP_nj_nx^n = \frac{x + 8x^2 + 7x^3 + 14x^5 - 32x^6 + 8x^7}{1 - 2x - 43x^2 - 42x^3 + 173x^4 + 84x^5 - 172x^6 + 16x^7 + 16x^8},$$
$$\sum_{n\geq 0} F_nJ_nC_nx^n = \frac{6x - 2x^2 - 438x^3 + 876x^5 + 8x^6 - 48x^7}{1 - 6x - 237x^2 - 354x^3 + 4733x^4 - 708x^5 - 948x^6 - 48x^7 + 16x^8},$$
$$\sum_{n\geq 0} P_nM_nj_nx^n = \frac{x + 24x^2 + 8x^3 - 32x^5 - 384x^6 - 64x^7}{1 - 6x - 57x^2 + 120x^3 + 824x^4 + 480x^5 - 912x^6 - 384x^7 + 256x^8}.$$

Finally, when $T_n^{(i)} = V_n^{(i)}$ for $i = 1, 2$ and $T_n^{(3)} = U_n^{(3)}$ in Theorem 1, one gets the following result.

Corollary 4. *We have*

$$\sum_{n\geq 0} V_n^{(1)} V_n^{(2)} U_n^{(3)} x^n = \frac{\sum_{i=1}^{7} l_i x^i}{M(x)},$$

where $M(x)$ *is as in Lemma 1 and*

$$\begin{aligned}
l_1 &= p_1p_2,\\
l_2 &= 2p_3(p_1^2q_2 + q_1p_2^2 + 2q_1q_2),\\
l_3 &= p_1p_2(p_1^2q_2q_3 + q_1p_2^2q_3 + 3q_1q_2p_3^2 + 5q_1q_2q_3),\\
l_4 &= 2q_1q_2p_3(p_1^2p_2^2q_3 - 2q_1q_2p_3^2 - 4q_1q_2q_3),\\
l_5 &= -p_1p_2q_1q_2q_3(p_1^2q_2q_3 + q_1p_2^2q_3 + 3q_1q_2p_3^2 + 5q_1q_2q_3),\\
l_6 &= 2p_3(q_1q_2q_3)^2(p_1^2q_2 + q_1p_2^2 + 2q_1q_2),\\
l_7 &= -p_1p_2(q_1q_2q_3)^3.
\end{aligned}$$

Proof. Note that $\alpha_i = \beta_i = 1$ for $i = 1, 2$ and $\alpha_3 = -\beta_3 = \frac{1}{\Delta_3}$ so that $H(x)$ is given by

$$\begin{aligned}
&\frac{1}{\Delta_3}\Big(\big(2 - p_1 r_1^{(2)} r_1^{(3)} x\big)\big(1 - p_1 r_2^{(2)} r_2^{(3)} x - q_1 \big(r_2^{(2)} r_2^{(3)} x\big)^2\big) \\
&\qquad - \big(2 - p_1 r_2^{(2)} r_2^{(3)} x\big)\big(1 - p_1 r_1^{(2)} r_1^{(3)} x - q_1 \big(r_1^{(2)} r_1^{(3)} x\big)^2\big)\Big) \\
&\quad = \frac{1}{\Delta_3}\Big(p_1\big(r_1^{(2)} r_1^{(3)} - r_2^{(2)} r_2^{(3)}\big)x + 2q_1\big(r_1^{(2)} r_1^{(3)} + r_2^{(2)} r_2^{(3)}\big)\big(r_1^{(2)} r_1^{(3)} - r_2^{(2)} r_2^{(3)}\big)x^2 \\
&\qquad + p_1 q_1 q_2 q_3 \big(r_2^{(2)} r_2^{(3)} - r_1^{(2)} r_1^{(3)}\big)x^3\Big) \\
&\quad = \frac{x(p_2\Delta_3 + p_3\Delta_2)}{2\Delta_3}\big(p_1 + q_1(p_2 p_3 + \Delta_2\Delta_3)x - p_1 q_1 q_2 q_3 x^2\big).
\end{aligned}$$

The proof is then completed in a similar manner as before. □

Example 4.

$$\begin{aligned}
\sum_{n\geq 0} Q_n j_n F_n x^n &= \frac{2x + 26x^2 + 50x^3 - 32x^4 - 100x^5 + 104x^6 - 16x^7}{1 - 2x - 43x^2 - 42x^3 + 173x^4 + 84x^5 - 172x^6 + 16x^7 + 16x^8}, \\
\sum_{n\geq 0} Q_n j_n M_n x^n &= \frac{2x + 78x^2 + 32x^3 - 336x^4 + 128x^5 + 1248x^6 + 128x^7}{1 - 6x - 57x^2 + 120x^3 + 824x^4 + 480x^5 - 912x^6 - 384x^7 + 256x^8}, \\
\sum_{n\geq 0} L_n C_n J_n x^n &= \frac{6x + 66x^2 + 342x^3 - 164x^4 + 684x^5 + 264x^6 + 48x^7}{1 - 6x - 237x^2 - 354x^3 + 4733x^4 - 708x^5 - 948x^6 - 48x^7 + 16x^8}.
\end{aligned}$$

Let $f(x) = a(x)/M(x)$ and $g(x) = b(x)/M(x)$ denote the generating functions in Corollaries 1 and 2, respectively. Then comparing coefficients of x^n on both sides of the equality $b(x)f(x) = a(x)g(x)$ leads to the following general identity.

Theorem 2 (sequence triples $U_n^{(1)} U_n^{(2)} U_n^{(3)}$ and $V_n^{(1)} V_n^{(2)} V_n^{(3)}$)**.** *For $n \geq 7$, we have*

$$8U_n^{(1)} U_n^{(2)} U_n^{(3)} + \sum_{i=1}^{7} t_i U_{n-i}^{(1)} U_{n-i}^{(2)} U_{n-i}^{(3)} = \sum_{i=1}^{7} s_i V_{n-i}^{(1)} V_{n-i}^{(2)} V_{n-i}^{(3)}, \tag{4}$$

where s_i and t_i are defined in Corollaries 1 and 2.

Taking particular values of the parameters in Theorem 2 leads to formulas such as the following each of which relates six sequences from the Fibonacci family of sequences.

Example 5.

$$\begin{aligned}
&8F_n P_n J_n - 14F_{n-1}P_{n-1}J_{n-1} - 258F_{n-2}P_{n-2}J_{n-2} - 210F_{n-3}P_{n-3}J_{n-3} \\
&\qquad + 692F_{n-4}P_{n-4}J_{n-4} + 252F_{n-5}P_{n-5}J_{n-5} - 344F_{n-6}P_{n-6}J_{n-6} + 16F_{n-7}P_{n-7}J_{n-7} \\
&\quad = L_{n-1}Q_{n-1}j_{n-1} - 17L_{n-3}Q_{n-3}j_{n-3} - 8L_{n-4}Q_{n-4}j_{n-4} \\
&\qquad + 34L_{n-5}Q_{n-5}j_{n-5} - 8L_{n-7}Q_{n-7}j_{n-7},
\end{aligned} \tag{5}$$

$$\begin{aligned}
&8F_n J_n B_n - 42F_{n-1}J_{n-1}B_{n-1} - 1422F_{n-2}J_{n-2}B_{n-2} - 1770F_{n-3}J_{n-3}B_{n-3} \\
&\qquad + 18932F_{n-4}J_{n-4}B_{n-4} - 2124F_{n-5}J_{n-5}B_{n-5} - 1896F_{n-6}J_{n-6}B_{n-6} - 48F_{n-7}J_{n-7}B_{n-7} \\
&\quad = L_{n-1}j_{n-1}C_{n-1} - 63L_{n-3}j_{n-3}C_{n-3} + 24L_{n-4}j_{n-4}C_{n-4} \\
&\qquad - 126L_{n-5}j_{n-5}C_{n-5} + 8L_{n-7}j_{n-7}C_{n-7},
\end{aligned}$$

$$
\begin{aligned}
&8P_nB_nM_n - 252P_{n-1}B_{n-1}M_{n-1} + 828P_{n-2}B_{n-2}M_{n-2} + 11340P_{n-3}B_{n-3}M_{n-3} \\
&\qquad + 8708P_{n-4}B_{n-4}M_{n-4} - 13608P_{n-5}B_{n-5}M_{n-5} + 1104P_{n-6}B_{n-6}M_{n-6} + 288P_{n-7}B_{n-7}M_{n-7} \\
&= Q_{n-1}C_{n-1}K_{n-1} + 67Q_{n-3}C_{n-3}K_{n-3} - 144Q_{n-4}C_{n-4}K_{n-4} \\
&\qquad - 134Q_{n-5}C_{n-5}K_{n-5} - 8Q_{n-7}C_{n-7}K_{n-7}.
\end{aligned}
$$

A similar comparison as before this time using Corollaries 3 and 4 leads to the following further result.

Theorem 3 (sequence triples $U_n^{(1)}U_n^{(2)}V_n^{(3)}$ and $V_n^{(1)}V_n^{(2)}U_n^{(3)}$)**.** *For $n \geq 6$, we have*

$$
\sum_{i=1}^{7} l_i U_{n-i+1}^{(1)} U_{n-i+1}^{(2)} V_{n-i+1}^{(3)} = \sum_{i=1}^{7} r_i V_{n-i+1}^{(1)} V_{n-i+1}^{(2)} U_{n-i+1}^{(3)},
$$

where r_i and l_i are defined in Corollaries 3 and 4.

From Theorem 3, one gets further identities relating six sequences such as the following.

Example 6.

$$
\begin{aligned}
&2F_nP_nj_n + 14F_{n-1}P_{n-1}j_{n-1} + 46F_{n-2}P_{n-2}j_{n-2} - 4F_{n-3}P_{n-3}j_{n-3} \\
&\qquad - 92F_{n-4}P_{n-4}j_{n-4} + 56F_{n-5}P_{n-5}j_{n-5} - 16F_{n-6}P_{n-6}j_{n-6} \\
&= L_nQ_nJ_n + 8L_{n-1}Q_{n-1}J_{n-1} + 7L_{n-2}Q_{n-2}J_{n-2} + 14L_{n-4}Q_{n-4}J_{n-4} \\
&\qquad - 32L_{n-5}Q_{n-5}J_{n-5} + 8L_{n-6}Q_{n-6}J_{n-6}, \\
&F_nJ_nC_n + 84F_{n-1}J_{n-1}C_{n-1} + 203F_{n-2}J_{n-2}C_{n-2} - 3288F_{n-3}J_{n-3}C_{n-3} \\
&\qquad + 406F_{n-4}J_{n-4}C_{n-4} + 336F_{n-5}J_{n-5}C_{n-5} + 8F_{n-6}J_{n-6}C_{n-6} \\
&= 6L_nj_nB_n - 2L_{n-1}j_{n-1}B_{n-1} - 438L_{n-2}j_{n-2}B_{n-2} + 876L_{n-4}j_{n-4}B_{n-4} \\
&\qquad + 8L_{n-5}j_{n-5}B_{n-5} - 48L_{n-6}j_{n-6}B_{n-6}, \\
&P_nM_nC_n - 6P_{n-1}M_{n-1}C_{n-1} - 207P_{n-2}M_{n-2}C_{n-2} - 400P_{n-3}M_{n-3}C_{n-3} \\
&\qquad + 414P_{n-4}M_{n-4}C_{n-4} - 24P_{n-5}M_{n-5}C_{n-5} - 8P_{n-6}M_{n-6}C_{n-6} \\
&= Q_nK_nB_n - 2Q_{n-1}K_{n-1}B_{n-1} + 69Q_{n-2}K_{n-2}B_{n-2} \\
&\qquad + 138Q_{n-4}K_{n-4}B_{n-4} + 8Q_{n-5}K_{n-5}B_{n-5} + 8Q_{n-6}K_{n-6}B_{n-6}.
\end{aligned}
$$

We conclude this section with analogous results for a different generalization of the Fibonacci numbers wherein the initial values are arbitrary instead of the coefficients of the recurrence. Recall that the *gibonacci sequence* $G_n = G_n(a,b)$ is defined by

$$
G_0 = a, \ G_1 = b, \ G_n = G_{n-1} + G_{n-2}, \quad n \geq 2.
$$

See, for example, [2, Chapter 2] or [4, Chapter 7] and references therein. Then there is the following explicit formula for $\sum_{n\geq 0} G_n^{(1)} G_n^{(2)} G_n^{(3)} x^n$, where $G_n^{(i)} = G_n(a_i, b_i)$.

Corollary 5. *We have*

$$
\sum_{n\geq 0} G_n^{(1)} G_n^{(2)} G_n^{(3)} x^n = \frac{\sum_{i=0}^{7} u_i x^i}{1 - x - 13x^2 - 8x^3 + 20x^4 + 8x^5 - 13x^6 + x^7 + x^8},
$$

where

$$
\begin{aligned}
u_0 &= a_1a_2a_3,\\
u_1 &= -a_1a_2a_3 + b_1b_2b_3,\\
u_2 &= a_1(-12a_2a_3 + a_2b_3 + b_2a_3 + b_2b_3) + b_1(a_2a_3 + a_2b_3 + b_2a_3),\\
u_3 &= a_1(-8a_2a_3 + a_2b_3 + b_2a_3 + 3b_2b_3) + b_1(a_2a_3 + 3a_2b_3 + 3b_2a_3 - 6b_2b_3),\\
u_4 &= a_1(14a_2a_3 - 3a_2b_3 - 3b_2a_3 + b_2b_3) + b_1(-3a_2a_3 + a_2b_3 + b_2a_3 - 2b_2b_3),\\
u_5 &= a_1(6a_2a_3 - a_2b_3 - b_2a_3 - 3b_2b_3) + b_1(-a_2a_3 - 3a_2b_3 - 3b_2a_3 + 6b_2b_3),\\
u_6 &= a_1(-7a_2a_3 + 3a_2b_3 + 3b_2a_3 - b_2b_3) + b_1(3a_2a_3 - a_2b_3 - b_2a_3),\\
u_7 &= (a_1 - b_1)(a_2 - b_2)(a_3 - b_3).
\end{aligned}
$$

Proof. Note that $\alpha_i = \frac{a_i}{2} + \frac{(2b_i - a_i)\sqrt{5}}{10}$ and $\beta_i = \frac{a_i}{2} - \frac{(2b_i - a_i)\sqrt{5}}{10}$ when $p_i = q_i = 1$. In this case, we have

$$
\begin{aligned}
H(x) = \alpha_2\alpha_3 &\left(a_1 - \Big(\frac{1+\sqrt{5}}{2}\Big)^2(a_1 - b_1)x\right)\left(1 - \Big(\frac{1-\sqrt{5}}{2}\Big)^2 x - \Big(\frac{1-\sqrt{5}}{2}\Big)^4 x^2\right)\\
&+ \beta_2\beta_3\left(a_1 - \Big(\frac{1-\sqrt{5}}{2}\Big)^2(a_1 - b_1)x\right)\left(1 - \Big(\frac{1+\sqrt{5}}{2}\Big)^2 x - \Big(\frac{1+\sqrt{5}}{2}\Big)^4 x^2\right).
\end{aligned}
$$

Note that the second expression in the above formula is seen to be the conjugate (with respect to $\sqrt{5}$) of the first. Furthermore,

$$
\begin{aligned}
&\left(a_1 - \Big(\frac{1+\sqrt{5}}{2}\Big)^2(a_1 - b_1)x\right)\left(1 - \Big(\frac{1-\sqrt{5}}{2}\Big)^2 x - \Big(\frac{1-\sqrt{5}}{2}\Big)^4 x^2\right)\\
&\quad = a_1 - \left(3a_1 - \frac{3+\sqrt{5}}{2}b_1\right)x - \left(\frac{5-3\sqrt{5}}{2}a_1 + b_1\right)x^2 + \frac{3-\sqrt{5}}{2}(a_1 - b_1)x^3
\end{aligned}
$$

and

$$
\alpha_2\alpha_2 = \frac{3a_2a_3 - a_2b_3 - b_2a_3 + 2b_2b_3}{10} + \frac{(-a_2a_3 + a_2b_3 + b_2a_3)\sqrt{5}}{10}.
$$

This implies

$$
\begin{aligned}
H(x) = &\frac{(3a_2a_3 - a_2b_3 - b_2a_3 + 2b_2b_3)(2a_1 - 3(2a_1 - b_1)x - (5a_1 + 2b_1)x^2 + 3(a_1 - b_1)x^3)}{10}\\
&+ \frac{(-a_2a_3 + a_2b_3 + b_2a_3)(b_1x + 3a_1x^2 - (a_1 - b_1)x^3)}{2}.
\end{aligned}
$$

By similar reasoning, we have

$$
\overline{H}(x) = \frac{1}{5}(2a_2a_3 + a_2b_3 + b_2a_3 - 2b_2b_3)(a_1 + (2a_1 - b_1)x - b_1x^2 - (a_1 - b_1)x^3).
$$

Moreover, $p_i = q_i = 1$ for all i implies

$$
G(x) = 1 - 3x - 6x^2 + 3x^3 + x^4, \qquad \overline{G}(x) = 1 + 2x - x^2 - 2x^3 + x^4,
$$

where $G(x)$ is as in Lemma 1. An explicit computation of the quotient on the right side of (1) now completes the proof. $\square$

Comparing the formulas for $\sum_{n\geq 0} G_n^{(1)} G_n^{(2)} G_n^{(3)} x^n$ and $\sum_{n\geq 0} G_n^{(4)} G_n^{(5)} G_n^{(6)} x^n$ gives the following general result.

Theorem 4 (sequence triples $G_n^{(1)} G_n^{(2)} G_n^{(3)}$ and $G_n^{(4)} G_n^{(5)} G_n^{(6)}$). *For $n \geq 7$, we have*

$$\sum_{i=0}^{7} u_i' G_{n-i}^{(1)} G_{n-i}^{(2)} G_{n-i}^{(3)} = \sum_{i=0}^{7} u_i G_{n-i}^{(4)} G_{n-i}^{(5)} G_{n-i}^{(6)},$$

where the u_i are as in Corollary 5 and u_i' is obtained from u_i by replacing (a_j, b_j) with (a_{j+3}, b_{j+3}) for $1 \leq j \leq 3$.

3 Concluding remarks

Note that Theorem 1 and subsequent results are seen to apply even when the $T_n^{(i)}$ sequences are not all distinct. Further, it is possible to derive formulas analogous to Theorems 2 and 3 above relating terms from the sequence $T_n^{(1)} T_n^{(2)} T_n^{(3)}$ to terms from $T_n^{(4)} T_n^{(5)} T_n^{(6)}$ using Theorem 1 directly. However, in general, such formulas are going to be rather complicated and involve multiple convolution sums (which we leave for the reader to explore) since the denominators in (1) for the generating functions $\sum_{n\geq 0} T_n^{(1)} T_n^{(2)} T_n^{(3)} x^n$ and $\sum_{n\geq 0} T_n^{(4)} T_n^{(5)} T_n^{(6)} x^n$ would not be the same in that case. This is due to the fact that the three parameter pairs (p_i, q_i) for $1 \leq i \leq 3$ need not be a permutation of the pairs (p_{i+3}, q_{i+3}), which allows for the corresponding denominators M to be different. Indeed, when these two sets of parameter pairs can be obtained from another by such a permutation, the denominators are seen to be the same in the two corresponding generating functions. Further, from Corollaries 1–4, one may derive analogues of Theorems 2 and 3 in which terms from a specific sequence appear on both sides of the resulting identity. For example, comparing $\sum_{n\geq 0} U_n^{(1)} U_n^{(2)} V_n^{(3)} x^n$ and $\sum_{n\geq 0} V_n^{(1)} V_n^{(3)} U_n^{(2)} x^n$ using Corollaries 3 and 4 leads to an analogue of Theorem 3 relating $U_n^{(1)} U_n^{(2)} V_n^{(3)}$ and $V_n^{(1)} U_n^{(2)} V_n^{(3)}$.

One may apply all of the results above, starting with the generating function formulas, to polynomial analogues of the preceding numerical sequences. Indeed, Theorems 2 and 3 may be viewed as relating certain linear combinations of products of Fibonacci and Lucas polynomials in the quantities p_i and q_i for $i = 1, 2, 3$. Recall that the Chebyshev polynomials $A_n = A_n(x)$ and $B_n = B_n(x)$ of the first and second kind satisfy the second-order recurrence $w_n = 2xw_{n-1} - w_{n-2}$ for $n \geq 2$, with initial conditions $A_0 = 1$, $A_1 = x$ and $B_0 = 1$, $B_1 = 2x$, where x is an indeterminate. Note that $2A_n$ and B_{n-1} for $n \geq 0$ correspond respectively to special cases of V_n and U_n, assuming $B_{-1} = 0$, and thus the theorems above may be applied to the sequences $2A_n$ and B_{n-1}. Furthermore, the preceding results may also be extended to Fibonacci-type sequences with negative indices. To do so, consider $U_n' = U_{-n}$ and $V_n' = V_{-n}$ for $n \geq 0$. Note that U_n' and V_n' both satisfy the general recurrence $r_n = -\frac{p}{q} r_{n-1} + \frac{1}{q} r_{n-2}$ for $n \geq 2$, but with respective initial values $U_0' = 0$, $U_1' = \frac{1}{q}$ and $V_0' = 2$, $V_1' = -\frac{p}{q}$. Thus, one may apply the results above to the sequences qU_n' and V_n'.

Finally, it is possible to generalize the foregoing identities to subsequences of the original sequences in which the indices are multiples of a given integer. Let $k \geq 1$ be fixed. Then for the sequence $U_{nk} = U_{nk}(p, q)$, we have the recurrence $U_{nk} = V_k U_{(n-1)k} - (-q)^k U_{(n-2)k}$ for $n \geq 2$, which can be shown using the Binet formulas for U_n and V_n. The same recurrence also holds for V_{nk}. Therefore, the sequences U_{nk}/U_k and V_{nk} for $n \geq 0$ may be viewed as generalized Fibonacci and Lucas sequences per the definition given in the introduction where p and q are replaced by V_k and $-(-q)^k$, respectively. Thus, Theorem 2 may be generalized to

$$\begin{aligned}
8U_{nk}^{(1)}U_{nk}^{(2)}U_{nk}^{(3)} &+ t_1'U_{(n-1)k}^{(1)}U_{(n-1)k}^{(2)}U_{(n-1)k}^{(3)} + t_2'U_{(n-2)k}^{(1)}U_{(n-2)k}^{(2)}U_{(n-2)k}^{(3)} \\
&+ t_3'U_{(n-3)k}^{(1)}U_{(n-3)k}^{(2)}U_{(n-3)k}^{(3)} + t_4'U_{(n-4)k}^{(1)}U_{(n-4)k}^{(2)}U_{(n-4)k}^{(3)} + t_5'U_{(n-5)k}^{(1)}U_{(n-5)k}^{(2)}U_{(n-5)k}^{(3)} \\
&+ t_6'U_{(n-6)k}^{(1)}U_{(n-6)k}^{(2)}U_{(n-6)k}^{(3)} + t_7'U_{(n-7)k}^{(1)}U_{(n-7)k}^{(2)}U_{(n-7)k}^{(3)} \\
&= U_k^{(1)}U_k^{(2)}U_k^{(3)}\big(V_{(n-1)k}^{(1)}V_{(n-1)k}^{(2)}V_{(n-1)k}^{(3)} + s_3'V_{(n-3)k}^{(1)}V_{(n-3)k}^{(2)}V_{(n-3)k}^{(3)} \\
&+ s_4'V_{(n-4)k}^{(1)}V_{(n-4)k}^{(2)}V_{(n-4)k}^{(3)} + s_5'V_{(n-5)k}^{(1)}V_{(n-5)k}^{(2)}V_{(n-5)k}^{(3)} + s_7'V_{(n-7)k}^{(1)}V_{(n-7)k}^{(2)}V_{(n-7)k}^{(3)}\big),
\end{aligned} \tag{6}$$

where s_i' and t_i' are obtained from s_i and t_i respectively by replacing (p_i, q_i) with $(V_k^{(i)}, -(-q_i)^k)$ for $1 \leq i \leq 3$. A comparable generalization may be given for Theorem 3. Note that (6) reduces to (4) when $k = 1$.

Taking, for example, (p_1, q_1), (p_2, q_2) and (p_3, q_3) to be $(1, 1)$, $(2, 1)$ and $(1, 2)$ in (6) gives

$$\begin{aligned}
8F_{nk}P_{nk}J_{nk} &+ b_1F_{(n-1)k}P_{(n-1)k}J_{(n-1)k} + b_2F_{(n-2)k}P_{(n-2)k}J_{(n-2)k} + b_3F_{(n-3)k}P_{(n-3)k}J_{(n-3)k} \\
&+ b_4F_{(n-4)k}P_{(n-4)k}J_{(n-4)k} + b_5F_{(n-5)k}P_{(n-5)k}J_{(n-5)k} \\
&+ b_6F_{(n-6)k}P_{(n-6)k}J_{(n-6)k} + b_7F_{(n-7)k}P_{(n-7)k}J_{(n-7)k} \\
= F_kP_kJ_k\big(&L_{(n-1)k}Q_{(n-1)k}j_{(n-1)k} + a_3L_{(n-3)k}Q_{(n-3)k}j_{(n-3)k} + a_4L_{(n-4)k}Q_{(n-4)k}j_{(n-4)k} \\
&+ a_5L_{(n-5)k}Q_{(n-5)k}j_{(n-5)k} + a_7L_{(n-7)k}Q_{(n-7)k}j_{(n-7)k}\big),
\end{aligned} \tag{7}$$

where

$$a_3 = 3(-2)^k - j_k^2 - 2^k(L_k^2 + Q_k^2), \qquad a_4 = -(-2)^{k+1}L_kQ_kj_k,$$
$$a_5 = 3\cdot 4^k - (-2)^kj_k^2 - (-4)^k(L_k^2 + Q_k^2), \qquad a_7 = (-8)^k$$

and

$$\begin{aligned}
b_1 &= -7L_kQ_kj_k, \\
b_2 &= 6\big((-2)^kL_k^2Q_k^2 + (-2)^{k+2} - (2^{k+1} - (-1)^kj_k^2)(L_k^2 + Q_k^2) - 2j_k^2\big), \\
b_3 &= -5L_kQ_kj_k\big(2^k(L_k^2 + Q_k^2) + j_k^2 - 5(-2)^k\big), \\
b_4 &= -(-2)^{k+4}\big(2^k(L_k^2 + Q_k^2) + j_k^2\big) + 6\cdot 4^{k+1} + (-2)^{k+2}(L_kQ_kj_k)^2 + 4^{k+1}(L_k^4 + Q_k^4) + 4j_k^4, \\
b_5 &= -3(-2)^kL_kQ_kj_k\big(2^k(L_k^2 + Q_k^2) + j_k^2 - 5(-2)^k\big), \\
b_6 &= 2^{2k+1}\big((-2)^kL_k^2Q_k^2 + (-2)^{k+2} - (2^{k+1} - (-1)^kj_k^2)(L_k^2 + Q_k^2) - 2j_k^2\big), \\
b_7 &= -(-8)^kL_kQ_kj_k.
\end{aligned}$$

Note that (7) reduces to (5) when $k = 1$. Taking $k = 2$ in (7) yields the following identity relating products of the respective half-sequences

$$\begin{aligned}
4F_{2n}P_{2n}J_{2n} &- 315F_{2n-2}P_{2n-2}J_{2n-2} + 6081F_{2n-4}P_{2n-4}J_{2n-4} - 41625F_{2n-6}P_{2n-6}J_{2n-6} \\
&+ 103746F_{2n-8}P_{2n-8}J_{2n-8} - 99900F_{2n-10}P_{2n-10}J_{2n-10} \\
&+ 32432F_{2n-12}P_{2n-12}J_{2n-12} - 2880F_{2n-14}P_{2n-14}J_{2n-14} \\
= L_{2n-2}&Q_{2n-2}j_{2n-2} - 193L_{2n-6}Q_{2n-6}j_{2n-6} + 720L_{2n-8}Q_{2n-8}j_{2n-8} \\
&- 772L_{2n-10}Q_{2n-10}j_{2n-10} + 64L_{2n-14}Q_{2n-14}j_{2n-14}.
\end{aligned}$$

The other identities above may be comparably generalized.

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Received 01.05.2021

Фрончак Р., Гой Т., Шатук М. *Тотожності з шести представників узагальненої послідовності Фібоначчі* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 6–19.

У цій роботі ми встановили декілька тотожностей, які пов'язують суму добутків трьох різних представників загального лінійного рекурентного співвідношення другого порядку з сумою, що складається з членів трьох супутніх послідовностей. Ці тотожності отримані нами як окремі випадки формул для двох лінійних комбінацій добутків трьох узагальнених послідовностей Фібоначчі або Люка. Останні формули, у свою чергу, випливають з більш загального результату для твірної функції добутку трьох доданків — представників лінійних рекурентних послідовностей другого порядку з довільними початковими даними. Серед послідовностей, для яких встановлені згадані тотожності, є послідовності Фібоначчі, Пелля, Якобсталя та Мерсенна, а також пов'язані з ними супутні послідовності Люка.

Ключові слова і фрази: послідовність Горадама, послідовність Фібоначчі, послідовність Люка, послідовність Пелля, послідовність Якобсталя, генератриса.