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Robust interpolation of sequences with periodically stationary multiplicative seasonal increments

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We consider stochastic sequences with periodically stationary generalized multiple increments of fractional order which combines cyclostationary, multi-seasonal, integrated and fractionally integrated patterns. We solve the interpolation problem for linear functionals constructed from unobserved values of a stochastic sequence of this type based on observations of the sequence with a periodically stationary noise sequence. For sequences with known matrices of spectral densities, we obtain formulas for calculating values of the mean square errors and the spectral characteristics of the optimal interpolation of the functionals. Formulas that determine the least favorable spectral densities and the minimax (robust) spectral characteristics of the optimal linear interpolation of the functionals are proposed in the case where spectral densities of the sequences are not exactly known while some sets of admissible spectral densities are given.

Key words and phrases: periodically stationary increment, SARFIMA, fractional integration, filtering, optimal linear estimate, mean square error, least favourable spectral density matrix, minimax spectral characteristic.

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Introduction

A big amount of data arising and being collected in the past decades motivates developing new techniques for their effective processing. Among others, we pay attention to time series models which are appropriate for describing data coming from economics, finance, climatology, air pollution, signal processing etc. The particular examples can be found in the articles by A.E. Dudek and H. Hurd [6], S. Johansen and M.O. Nielsen [20], V.A. Reisen et al. [43]. Usually researchers have to deal with non-stationary and fractional behavior of data series, which in a simple case can be described by a general multiplicative model, or $SARIMA(p,d,q)\times(P,D,Q)_s$ model with integrated and seasonal factors, introduced in the book by G.E.P. Box et al. [4]:

$$\Psi(B^s)\psi(B)(1-B)^d(1-B^s)^D x_t = \Theta(B^s)\theta(B)\varepsilon_t, \tag{1}$$

where ε_t, $t\in\mathbb{Z}$, is a sequence of zero mean i.i.d. random variables, $\psi(z)$, $\theta(z)$ are polynomials of p and q degrees, and where $\Psi(z)$ and $\Theta(z)$ are polynomials of degrees of P and Q respectively which have roots outside the unit circle. The parameters d and D are allowed

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to be fractional, and under conditions $|d+D| < 1/2$ and $|D| < 1/2$ the process (1) is stationary and invertible. For an illustration of an application of a seasonal ARFIMA model see S. Porter-Hudak [41] who analyzed the monetary aggregates used by U.S. Federal Reserve. The fractional integration is also described by GARMA processes (see H.I. Gray, N.-F. Zheng and W.A. Woodward [13])

$$(1-2uB+B^2)^d x_t = \varepsilon_t, \quad |u| \leq 1. \tag{2}$$

Some resent results dedicated to the statistical inference for seasonal long-memory sequences can be found in the following three papers. H. Tsai, H. Rachinger and E.M.H. Lin [46] developed methods of estimation of parameters in case of measurement errors. R.T. Baillie, C. Kongcharoen and G. Kapetanios [2] compared MLE and semiparametric estimation procedures for prediction problems based on ARFIMA models by conducting a simulation study. They concluded a better performance of MLE predictor than the one based on the two-step local Whittle estimation. U. Hassler and M.O. Pohle [16] (see also U. Hassler [17]) assess a predictive performance of various methods of forecasting of inflation and return volatility time series and show strong evidences for models with a fractional integration component.

Periodically correlated, or cyclostationary, processes introduced by E.G. Gladyshev [11], allow to describe another type of non-stationarity – a time-dependent spectrum. They are widely used in signal processing and communications, and also can be considered as an extension of seasonal models [1, 3, 27, 39]. For a review of recent works on cyclostationarity and its applications, we refer to A. Napolitano [38].

Dealing with real data problems, a range of issues, not being covered by the classical theories, arises: the presence of outliers, measurement errors, incomplete information about the spectral, or model, structure etc. So there is an increasing interest to robust methods of estimation that are reasonable in such cases [42, 45]. Since the work by U. Grenander [14], the minimax extrapolation, interpolation and filtering problems for stationary sequences and processes have been studied by Y. Hosoya [18], S.A. Kassam [23], J. Franke [8], S.K. Vastola and H.V. Poor [47], M.P. Moklyachuk [34,37], M.M. Luz and M.P. Moklyachuk [30], Y. Liu et al. [26] and others.

This article is dedicated to the robust interpolation of stochastic sequences with periodically stationary long memory multiple seasonal increments, or sequences with periodically stationary generalized multiplicative (GM) increments, introduced by M. Luz and M. Moklyachuk [31]. The definition of processes combining a periodic structure of the covariation function and the multiple seasonal factors is motivated by the interest to the models with multiple seasonal and periodic patterns, see G. Dudek [7], P.G. Gould et al. [12] and V.A. Reisen et al. [43], H. Hurd and V. Piparas [19]. The discussed problem is a natural continuation of the researches on minimax interpolation and filtering of stationary vector-valued processes, periodically correlated processes and processes with stationary increments have been performed by M.P. Moklyachuk and O.Yu. Masyutka [33], I.I. Dubovets'ka, O.Yu. Masyutka and M.P. Moklyachuk [5], M. Luz and M. Moklyachuk [28,29] respectively. We also should mention the works by M.P. Moklyachuk, O.Yu. Masyutka and M.I. Sidei [32, 35, 36], who derive minimax estimates of stationary processes from observations with missed values, and the work by P.S. Kozak and M.P. Moklyachuk [25], who have studied interpolation problem for stochastic sequences with periodically stationary increments.

The article is organized as follows. In Section 1, we recall definitions of generalized multiple (GM) increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ and stochastic sequences $\xi(m)$ with periodically stationary (periodically correlated, cyclostationary) GM increments. The spectral theory of vector-valued GM increment sequences and the case of non-stationary fractional integration are discussed. Section 2 deals with the classical interpolation problem for the linear functional $A_N\xi$ which is constructed from unobserved values of the sequence $\xi(m)$ when the spectral densities of the sequence $\xi(m)$ and a noise sequence $\eta(m)$ are known. The Hilbert space projection technique is applied to obtain the estimates of the vector-valued sequence $\vec{\xi}(m)+\vec{\eta}(m)$ with stationary GM increments under the stationary noise sequence $\vec{\eta}(m)$ uncorrelated with $\vec{\xi}(m)$. Section 3 is dedicated to the minimax (robust) estimates in cases, where spectral densities of sequences are not exactly known while some sets of admissible spectral densities are specified. We illustrate the proposed technique on the particular types of the sets, which are generalizations of the sets of admissible spectral densities described in a survey article by S.A. Kassam and H.V. Poor [22] for stationary stochastic processes.

1 Stochastic sequences with periodically stationary generalized multiple increments

1.1 Preliminary notations and definitions

Consider a stochastic sequence $\xi(m)$, $m\in\mathbb{Z}$, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Denote by B_μ a backward shift operator with the step $\mu\in\mathbb{Z}$, such that $B_\mu\xi(m)=\xi(m-\mu)$, $B:=B_1$. Then $B_\mu^s=B_\mu B_\mu\cdots B_\mu$.

Define the incremental operator

$$\chi_{\overline{\mu},\overline{s}}^{(d)}(B):=(1-B_{\mu_1}^{s_1})^{d_1}(1-B_{\mu_2}^{s_2})^{d_2}\dots(1-B_{\mu_r}^{s_r})^{d_r}=\sum_{k=0}^{n(\gamma)}e_\gamma(k)B^k,$$

where $d:=d_1+d_2+\ldots+d_r$, $\overline{d}=(d_1,d_2,\ldots,d_r)\in(\mathbb{N}^*)^r$, $\overline{s}=(s_1,s_2,\ldots,s_r)\in(\mathbb{N}^*)^r$ and $\overline{\mu}=(\mu_1,\mu_2,\ldots,\mu_r)\in(\mathbb{N}^*)^r$ or $\in(\mathbb{Z}\setminus\mathbb{N})^r$, $n(\gamma):=\sum_{i=1}^r\mu_i s_i d_i$. Here $\mathbb{N}^*=\mathbb{N}\setminus\{0\}$. The explicit formula for the coefficients $e_\gamma(k)$ is given in [31].

Definition 1. *For a stochastic sequence $\xi(m)$, $m\in\mathbb{Z}$, the sequence*

$$\begin{aligned}\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi(m)):=\chi_{\overline{\mu},\overline{s}}^{(d)}(B)\xi(m)&=(1-B_{\mu_1}^{s_1})^{d_1}(1-B_{\mu_2}^{s_2})^{d_2}\cdots(1-B_{\mu_r}^{s_r})^{d_r}\xi(m)\\&=\sum_{l_1=0}^{d_1}\dots\sum_{l_r=0}^{d_r}(-1)^{l_1+\ldots+l_r}\binom{d_1}{l_1}\cdots\binom{d_r}{l_r}\xi(m-\mu_1s_1l_1-\ldots-\mu_rs_rl_r)\end{aligned}\tag{3}$$

is called a stochastic generalized multiple (GM) increment sequence of differentiation order d with a fixed seasonal vector $\overline{s}\in(\mathbb{N}^*)^r$ and a varying step $\overline{\mu}\in(\mathbb{N}^*)^r$ or $\in(\mathbb{Z}\setminus\mathbb{N})^r$.

Definition 2. *A stochastic GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi(m))$ is called a wide sense stationary if the mathematical expectations*

$$\mathsf{E}\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi(m_0))=c_{\overline{s}}^{(d)}(\overline{\mu}),\quad \mathsf{E}\chi_{\overline{\mu}_1,\overline{s}}^{(d)}(\xi(m_0+m))\chi_{\overline{\mu}_2,\overline{s}}^{(d)}(\xi(m_0))=D_{\overline{s}}^{(d)}(m;\overline{\mu}_1,\overline{\mu}_2)$$

exist for all $m_0,m,\overline{\mu},\overline{\mu}_1,\overline{\mu}_2$ and do not depend on m_0. The function $c_{\overline{s}}^{(d)}(\overline{\mu})$ is called a mean value and the function $D_{\overline{s}}^{(d)}(m;\overline{\mu}_1,\overline{\mu}_2)$ is called a structural function of the stationary GM increment sequence (of a stochastic sequence with stationary GM increments).

The stochastic sequence $\xi(m)$, $m \in \mathbb{Z}$, determining the stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi(m))$ by (3) *is called a stochastic sequence with stationary GM increments (or GM increment sequence of order d).*

Remark 1. *For spectral properties of one-pattern increment sequence*

$$\chi_{\mu,1}^{(n)}(\xi(m)) := \xi^{(n)}(m,\mu) = (1-B_{\mu})^n \xi(m)$$

see, e.g., [29, pp. 1–8], [9, pp. 48–60, 261–268], [48, pp. 390–430]. The corresponding results for continuous time increment process $\xi^{(n)}(t,\tau) = (1-B_{\tau})^n \xi(t)$ are described in [48, 49].

1.2 Definition and spectral representation of stochastic sequences with periodically stationary GM increment

In this section, we present definition, justification and a brief review of the spectral theory of stochastic sequences with periodically stationary multiple seasonal increments.

Definition 3. *A stochastic sequence $\xi(m)$, $m \in \mathbb{Z}$, is called a stochastic sequence with periodically stationary (periodically correlated) GM increments with period T if the mathematical expectations*

$$\mathsf{E}\chi_{\overline{\mu},T\overline{s}}^{(d)}(\xi(m+T)) = \mathsf{E}\chi_{\overline{\mu},T\overline{s}}^{(d)}(\xi(m)) = c_{T\overline{s}}^{(d)}(m,\overline{\mu}),$$
$$\mathsf{E}\chi_{\overline{\mu}_1,T\overline{s}}^{(d)}(\xi(m+T))\chi_{\overline{\mu}_2,T\overline{s}}^{(d)}(\xi(k+T)) = D_{T\overline{s}}^{(d)}(m+T,k+T;\overline{\mu}_1,\overline{\mu}_2) = D_{T\overline{s}}^{(d)}(m,k;\overline{\mu}_1,\overline{\mu}_2)$$

exist for every $m,k,\overline{\mu}_1,\overline{\mu}_2$ and $T>0$ is the least integer for which these equalities hold.

Using Definition 3, one can directly check that the sequence

$$\xi_p(m) = \xi(mT+p-1), \quad p = 1,2,\dots,T, \quad m \in \mathbb{Z}, \tag{4}$$

forms a vector-valued sequence $\vec{\xi}(m) = \{\xi_p(m)\}_{p=1,2,\dots,T}$, $m \in \mathbb{Z}$, with stationary GM increments by the relation $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m)) = \chi_{\overline{\mu},T\overline{s}}^{(d)}(\xi(mT+p-1))$, $p = 1,2,\dots,T$, where $\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))$ is the GM increment of the pth component of the vector-valued sequence $\vec{\xi}(m)$.

The following theorem describes the spectral structure of the GM increment [21, 31].

Theorem 1.

1. *The mean value and structural function of the vector-valued stochastic stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ can be represented in the form*

$$c_{\overline{s}}^{(d)}(\overline{\mu}) = c\prod_{i=1}^{r}\mu_i^{d_i}, \quad D_{\overline{s}}^{(d)}(m;\overline{\mu}_1,\overline{\mu}_2) = \int_{-\pi}^{\pi}\chi_{\overline{\mu}_1}^{(d)}(e^{-i\lambda})\chi_{\overline{\mu}_2}^{(d)}(e^{i\lambda})\frac{e^{i\lambda m}}{|\beta^{(d)}(i\lambda)|^2}\,dF(\lambda),$$

   where

$$\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda}) = \prod_{j=1}^{r}(1-e^{-i\lambda\mu_j s_j})^{d_j}, \quad \beta^{(d)}(i\lambda) = \prod_{j=1}^{r}\prod_{k_j=-[s_j/2]}^{[s_j/2]}(i\lambda - 2\pi i k_j/s_j)^{d_j},$$

   c is a vector, $F(\lambda)$ is the matrix-valued spectral function of the stationary stochastic sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$. The vector c and the matrix-valued function $F(\lambda)$ are determined uniquely by the GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$.

2. The stationary GM increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m))$ admits the spectral representation

$$\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m)) = \int_{-\pi}^{\pi} \chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})\frac{e^{im\lambda}}{\beta^{(d)}(i\lambda)}\, d\vec{Z}_{\xi^{(d)}}(\lambda),$$

where $d\vec{Z}_{\xi^{(d)}}(\lambda) = \{Z_p(\lambda)\}_{p=1}^{T}$ is a (vector-valued) stochastic process with uncorrelated increments on $[-\pi,\pi)$ connected with the spectral function $F(\lambda)$ by the relation

$$\mathsf{E}(Z_p(\lambda_2) - Z_p(\lambda_1))(\overline{Z_q(\lambda_2) - Z_q(\lambda_1)}) = F_{pq}(\lambda_2) - F_{pq}(\lambda_1),$$

where $-\pi \leq \lambda_1 < \lambda_2 < \pi$; $p,q = 1,2,\ldots,T$.

Consider another vector-valued stochastic sequence with the stationary GM increments $\vec{\zeta}(m) = \vec{\xi}(m) + \vec{\eta}(m)$, where $\vec{\eta}(m)$ is a vector-valued stationary stochastic sequence, uncorrelated with $\vec{\xi}(m)$, with a spectral representation

$$\vec{\eta}(m) = \int_{-\pi}^{\pi} e^{i\lambda m} dZ_{\eta}(\lambda), \quad Z_{\eta}(\lambda) = \{Z_{\eta,p}(\lambda)\}_{p=1}^{T}, \quad \lambda \in [-\pi,\pi),$$

is a stochastic process with uncorrelated increments, that corresponds to the spectral function $G(\lambda)$ [15]. The stochastic stationary GM increment $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\zeta}(m))$ allows the spectral representation

$$\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\zeta}(m)) = \int_{-\pi}^{\pi} \chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})\frac{e^{i\lambda m}}{\beta^{(d)}(i\lambda)}\, dZ_{\xi^{(n)}}(\lambda) + \int_{-\pi}^{\pi} e^{i\lambda m}\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})\, dZ_{\eta}(\lambda),$$

while $dZ_{\eta}(\lambda) = (\beta^{(d)}(i\lambda))^{-1} dZ_{\eta^{(n)}}(\lambda)$, $\lambda \in [-\pi,\pi)$. Therefore, in the case where the spectral functions $F(\lambda)$ and $G(\lambda)$ have the spectral densities $f(\lambda)$ and $g(\lambda)$, the spectral density $p(\lambda) = \{p_{ij}(\lambda)\}_{i,j=1}^{T}$ of the stochastic sequence $\vec{\zeta}(m)$ is determined by the formula

$$p(\lambda) = f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda).$$

1.3 Sequences with GM fractional increments

Now we extend the definition of GM increment sequence $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(m))$ of the positive integer orders $(d_1,\ldots,d_r)$ to the fractional ones. Within the subsection, we put the step $\overline{\mu} = (1,1,\ldots,1)$. Represent the increment operator $\chi^{(d)}_{\overline{s}}(B)$ in the form

$$\chi^{(R+D)}_{\overline{s}}(B) = (1-B)^{R_0+D_0}\prod_{j=1}^{r}(1-B^{s_j})^{R_j+D_j}, \tag{5}$$

where $(1-B)^{R_0+D_0}$ is an integrating component, R_j, $j = 0,1,\ldots,r$, are non-negative integer numbers, $1 < s_1 < \ldots < s_r$. Below we describe a representations $d_j = R_j + D_j$, $j = 0,1,\ldots,r$, of the increment orders d_j by stating conditions on the fractional parts D_j such that the increment sequence

$$\vec{y}(m) := (1-B)^{R_0}\prod_{j=1}^{r}(1-B^{s_j})^{R_i}\vec{\xi}(m)$$

is a stationary fractionally integrated seasonal stochastic sequence. For example, in case of single increment pattern $(1-B^{s^*})^{R^*+D^*}$, this condition is $|D^*| < 1/2$.

Definition 4. *A sequence $\chi_{\overline{s}}^{(R+D)}(\vec{\xi}(m))$ is called a fractional multiple (FM) increment sequence.*

Consider the generating function of the Gegenbauer polynomial:

$$(1-2uB+B^2)^{-d} = \sum_{n=0}^{\infty} C_n^{(d)}(u)B^n, \quad C_n^{(d)}(u) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2u)^{n-2k}\Gamma(d-k+n)}{k!(n-2k)!\Gamma(d)}.$$

The following lemma and theorem hold true [31].

Lemma 1. *Define the sets $\mathcal{M}_j = \{\nu_{k_j} = 2\pi k_j/s_j : \ k_j = 0,1,\dots,[s_j/2]\}$, $j = 0,1,\dots,r$, and $\mathcal{M} = \bigcup_{j=0}^{r} \mathcal{M}_j$. Then the multiple seasonal increment operator admits the following representation:*

$$\begin{aligned}
\chi_{\overline{s}}^{(D)}(B) &:= (1-B)^{D_0}\prod_{j=1}^{r}(1-B^{s_j})^{D_j} = \prod_{\nu\in\mathcal{M}}(1-2\cos\nu B+B^2)^{\widetilde{D}_\nu} \\
&= (1-B)^{D_0+D_1+\ldots+D_r}(1+B)^{D_\pi}\prod_{\nu\in\mathcal{M}\setminus\{0,\pi\}}(1-2\cos\nu B+B^2)^{D_\nu} \\
&= \left(\sum_{m=0}^{\infty} G_{k^*}^{+}(m)B^m\right)^{-1} = \sum_{m=0}^{\infty} G_{k^*}^{-}(m)B^m,
\end{aligned}$$

where

$$\begin{aligned}
G_{k^*}^{+}(m) &= \sum_{\substack{0\le n_1,\dots,n_{k^*}\le m,\\ n_1+\ldots+n_{k^*}=m}} \prod_{\nu\in\mathcal{M}} C_{n_\nu}^{(\widetilde{D}_\nu)}(\cos\nu), \\
G_{k^*}^{-}(m) &= \sum_{\substack{0\le n_1,\dots,n_{k^*}\le m,\\ n_1+\ldots+n_{k^*}=m}} \prod_{\nu\in\mathcal{M}} C_{n_\nu}^{(-\widetilde{D}_\nu)}(\cos\nu),
\end{aligned}$$

where $k^* = |\mathcal{M}|$, $D_\nu = \sum_{j=0}^{r} D_j \mathbb{I}\{\nu\in\mathcal{M}_j\}$, $\widetilde{D}_\nu = D_\nu$ for $\nu\in\mathcal{M}\setminus\{0,\pi\}$, $\widetilde{D}_\nu = D_\nu/2$ for $\nu = 0$ and $\nu = \pi$.

Theorem 2. *Assume that for a stochastic vector-valued sequence $\vec{\xi}(m)$ and fractional differencing orders $d_j = R_j + D_j$, $j = 0,1,\dots,r$, the FM increment sequence $\chi_{\overline{1},\overline{s}}^{(R+D)}(\vec{\xi}(m))$ generated by increment operator* (5) *is a stationary sequence with a bounded from zero and infinity spectral density $\widetilde{f}_{\overline{1}}(\lambda)$. Then for the non-negative integer numbers R_j, $j = 0,1,\dots,r$, the GM increment sequence $\chi_{\overline{1},\overline{s}}^{(R)}(\vec{\xi}(m))$ is stationary if $-1/2 < D_\nu < 1/2$ for all $\nu\in\mathcal{M}$, where D_ν are defined by real numbers D_j, $j = 0,1,\dots,r$, in Lemma 1, and it is long memory if $0 < D_\nu < 1/2$ for at least one $\nu\in\mathcal{M}$, and invertible if $-1/2 < D_\nu < 0$. The spectral density $f(\lambda)$ of the stationary GM increment sequence $\chi_{\overline{1},\overline{s}}^{(R)}(\vec{\xi}(m))$ admits the representation*

$$f(\lambda) = |\beta^{(R)}(i\lambda)|^2|\chi_{\overline{1}}^{(R)}(e^{-i\lambda})|^{-2}|\chi_{\overline{1}}^{(D)}(e^{-i\lambda})|^{-2}\widetilde{f}_{\overline{1}}(\lambda) =: |\chi_{\overline{1}}^{(D)}(e^{-i\lambda})|^{-2}\widetilde{f}(\lambda),$$

where

$$|\chi_{\overline{1}}^{(D)}(e^{-i\lambda})|^{-2} = \left|\sum_{m=0}^{\infty} G_{k^*}^{+}(m)e^{-i\lambda m}\right|^2 = \left|\sum_{m=0}^{\infty} G_{k^*}^{-}(m)e^{-i\lambda m}\right|^{-2}.$$

For further conditions on the spectral density $f(\lambda)$ and the structural function $D_{\overline{s}}^{(R)}(m,\overline{1},\overline{1})$ of a stationary GM increment sequence $\chi_{\overline{1},\overline{s}}^{(R)}(\vec{\xi}(m))$ as well as for examples of Theorem 2 application, we refer to W. Palma and P. Bondon [40], L. Giraitis and R. Leipus [10], M. Luz and M. Moklyachuk [31].

2 Hilbert space projection method of interpolation

2.1 Interpolation of vector-valued stochastic sequences with stationary GM increments

Consider a vector-valued stochastic sequence with stationary GM increments $\vec{\xi}(m)$ constructed from transformation (4) and a vector-valued stationary stochastic sequence $\vec{\eta}(m)$ uncorrelated with the sequence $\vec{\eta}(m)$. Let the stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m)) = \{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))\}_{p=1}^{T}$ and the stationary stochastic sequence $\vec{\eta}(m)$ have the absolutely continuous spectral functions $F(\lambda)$ and $G(\lambda)$ with the spectral densities $f(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^{T}$ and $g(\lambda) = \{g_{ij}(\lambda)\}_{i,j=1}^{T}$ respectively.

Without loss of generality we will assume that the mean values of the increment sequences are $\mathsf{E}\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m)) = 0$, $\mathsf{E}\vec{\eta}(m) = 0$ and $\overline{\mu} > \overline{0}$.

Interpolation problem. Consider the problem of mean square optimal linear estimation of the functional $A_N\vec{\xi} = \sum_{k=0}^{N}(\vec{a}(k))^{\top}\vec{\xi}(k)$ which depends on the unobserved values of the stochastic sequence $\vec{\xi}(k) = \{\xi_p(k)\}_{p=1}^{T}$ with stationary GM increments. Estimates are based on observations of the sequence $\vec{\zeta}(k) = \vec{\xi}(k) + \vec{\eta}(k)$ at points of the set $\mathbb{Z} \setminus \{0, 1, 2, \ldots, N\}$.

Assume that spectral densities $f(\lambda)$ and $g(\lambda)$ satisfy the minimality condition

$$\int_{-\pi}^{\pi} \mathrm{Tr}\left[\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(f(\lambda) + |\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}\right] d\lambda < \infty. \tag{6}$$

The latter condition is the necessary and sufficient one under which the mean square errors of estimates of functional $A\vec{\xi}$ is not equal to 0.

We apply the Hilbert space estimation technique proposed by A.N. Kolmogorov [24] which can be described as a 3-stage procedure:

(i) define a target element (to be estimated) of the space $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$ of random variables γ which have zero mean values and finite variances, $\mathsf{E}\gamma = 0$, $\mathsf{E}|\gamma|^2 < \infty$, endowed with the inner product $\langle \gamma_1, \gamma_2 \rangle = \mathsf{E}\gamma_1\overline{\gamma_2}$;

(ii) define a subspace of H generated by observations;

(iii) find an estimate of the target element as an orthogonal projection on the defined subspace.

Stage (i). The functional $A_N\vec{\xi}$ does not belong to the space H. With the help of the following lemma we describe representations of the functional as a sum of a functional with finite second moments belonging to H and a functional depending on observed values of the sequence $\vec{\zeta}(k)$ ("initial values") (for more details see [28, 29, 31]).

Lemma 2. *The functional $A_N\vec{\xi}$ admits the representation*

$$A_N\vec{\xi} = A_N\vec{\zeta} - A_N\vec{\eta} = H_N\vec{\xi} - V_N\vec{\zeta}, \tag{7}$$

where $H_N\vec{\xi} := B_N\chi\vec{\zeta} - A_N\vec{\eta}$,

$$A_N\vec{\zeta} = \sum_{k=0}^{N}(\vec{a}(k))^{\top}\vec{\zeta}(k), \quad A_N\vec{\eta} = \sum_{k=0}^{N}(\vec{a}(k))^{\top}\vec{\eta}(k),$$

$$B_N\chi\vec{\zeta} = \sum_{k=0}^{N}(\vec{b}_N(k))^{\top}\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\zeta}(k)), \quad V_N\vec{\zeta} = \sum_{k=-n(\gamma)}^{-1}(\vec{v}_N(k))^{\top}\vec{\zeta}(k),$$

the coefficients

$$\vec{b}_N(k)=\{b_{N,p}(k)\}_{p=1}^T,\quad k=0,1,\dots,N,$$

and

$$\vec{v}_N(k)=\{v_{N,p}(k)\}_{p=1}^T,\quad k=-1,-2,\dots,-n(\gamma),$$

are calculated by the formulas

$$\vec{v}_N(k)=\sum_{l=0}^{N\wedge k+n(\gamma)}\mathrm{diag}_T(e_\nu(l-k))\vec{b}_N(l),\quad k=-1,-2,\dots,-n(\gamma),$$
$$\vec{b}_N(k)=\sum_{m=k}^{N}\mathrm{diag}_T(d_{\overline{\mu}}(m-k))\vec{a}(m)=(D_N^{\overline{\mu}}\mathbf{a}_N)_k,\quad k=0,1,\dots,N,$$

where $D_N^{\overline{\mu}}$ is the linear transformation determined by a matrix with the entries $(D_N^{\overline{\mu}})(k,j)=\mathrm{diag}_T(d_{\overline{\mu}}(j-k))$ if $0\le k\le j\le N$, and $(D_N^{\overline{\mu}})(k,j)=0$ if $0\le j<k\le N$, $\mathrm{diag}_T(x)$ denotes a $T\times T$ diagonal matrix with x on its diagonal, $\mathbf{a}_N=((\vec{a}(0))^\top,(\vec{a}(1))^\top,\dots,(\vec{a}(N))^\top)^\top$, coefficients $\{d_{\overline{\mu}}(k):k\ge 0\}$ are determined by the relationship

$$\sum_{k=0}^{\infty}d_{\overline{\mu}}(k)x^k=\prod_{i=1}^{r}\left(\sum_{j_i=0}^{\infty}x^{\mu_i s_i j_i}\right)^{d_i}.$$

The functional $H_N\vec{\xi}$ from representation (7) has finite variance and the functional $V_N\vec{\zeta}$ depends on the known observations of the stochastic sequence $\vec{\zeta}(k)$ at points $k=-n(\gamma)$, $-n(\gamma)+1,\dots,-1$. Therefore, estimates $\widehat{A}_N\vec{\xi}$ and $\widehat{H}_N\vec{\xi}$ of the functionals $A_N\vec{\xi}$ and $H_N\vec{\xi}$ and the mean-square errors $\Delta(f,g;\widehat{A}_N\vec{\xi})=\mathsf{E}|A_N\vec{\xi}-\widehat{A}_N\vec{\xi}|^2$ and $\Delta(f,g;\widehat{H}_N\vec{\xi})=\mathsf{E}|H_N\vec{\xi}-\widehat{H}_N\vec{\xi}|^2$ of the estimates $\widehat{A}_N\vec{\xi}$ and $\widehat{H}_N\vec{\xi}$ satisfy the following relations

$$\widehat{A}_N\vec{\xi}=\widehat{H}_N\vec{\xi}-V_N\vec{\zeta},\tag{8}$$
$$\Delta(f,g;\widehat{A}_N\vec{\xi})=\mathsf{E}|A_N\vec{\xi}-\widehat{A}_N\vec{\xi}|^2=\mathsf{E}|H_N\vec{\xi}-\widehat{H}_N\vec{\xi}|^2=\Delta(f,g;\widehat{H}_N\vec{\xi}).$$

Therefore, the estimation problem for the functional $A_N\vec{\xi}$ is equivalent to the one for the functional $H_N\vec{\xi}$. This problem can be solved by applying the Hilbert space projection method proposed by A.N. Kolmogorov [24].

The functional $H_N\vec{\xi}$ admits the spectral representation

$$H_N\vec{\xi}=\int_{-\pi}^{\pi}(\vec{B}_{\overline{\mu},N}(e^{i\lambda}))^\top\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda)-\int_{-\pi}^{\pi}(\vec{A}_N(e^{i\lambda}))^\top d\vec{Z}_\eta(\lambda),$$

where

$$\vec{B}_{\overline{\mu},N}(e^{i\lambda})=\sum_{k=0}^{N}\vec{b}_{\overline{\mu},N}(k)e^{i\lambda k}=\sum_{k=0}^{N}(D_N^{\overline{\mu}}\mathbf{a}_N)_k e^{i\lambda k},\quad \vec{A}_N(e^{i\lambda})=\sum_{k=0}^{N}\vec{a}(k)e^{i\lambda k}.$$

Stage (ii). Introduce the following notations. Denote by $H^{0-}(\xi_{\overline{\mu},\overline{s}}^{(d)}+\eta_{\overline{\mu},\overline{s}}^{(d)})$ the closed linear subspace generated by values $\{\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(k))+\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\eta}(k)):k=-1,-2,-3,\dots\}$, $\overline{\mu}>\vec{0}$ of the observed GM increments in the Hilbert space $H=L_2(\Omega,\mathcal{F},\mathsf{P})$ of random variables γ with zero mean value, $\mathsf{E}\gamma=0$, finite variance, $\mathsf{E}|\gamma|^2<\infty$, and the inner product $(\gamma_1;\gamma_2)=\mathsf{E}\gamma_1\overline{\gamma_2}$. Denote by $H^{N+}(\xi_{-\overline{\mu},\overline{s}}^{(d)}+\eta_{-\overline{\mu},\overline{s}}^{(d)})$ the closed linear subspace generated by values $\{\chi_{-\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(k))+$

$\chi^{(d)}_{-\overline{\mu},\overline{s}}(\vec{\eta}(k)) : k \geq N\}$, of the observed GM increments in the Hilbert space $H = L_2(\Omega, \mathcal{F}, \mathsf{P})$. Denote by $L_2^{0-}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))$ and $L_2^{N+}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))$ the closed linear subspaces of the Hilbert space $L_2(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))$ of vector-valued functions with the inner product

$$\langle g_1; g_2\rangle = \int_{-\pi}^{\pi} (g_1(\lambda))^\top (f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))\overline{g_2(\lambda)}\, d\lambda$$

which is generated by the functions

$$e^{i\lambda k}\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})\frac{\vec{\delta}_l}{\beta^{(d)}(i\lambda)}, \quad \vec{\delta}_l = \{\delta_{lp}\}_{p=1}^T, \quad l = 1,\dots,T,$$

for $k \leq -1$ and $k \geq N+1$, respectively, where δ_{lp} is a Kronecker symbol.

Then the relation

$$\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(k)) + \chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\eta}(k)) = \int_{-\pi}^{\pi} \chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})\frac{e^{i\lambda k}}{\beta^{(d)}(i\lambda)}\, dZ_{\xi^{(d)}+\eta^{(d)}}(\lambda)$$

yields a one-to-one correspondence between elements $e^{i\lambda k}\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})\vec{\delta}_l/\beta^{(d)}(i\lambda)$ from the space

$$L_2^{0-}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda)) \oplus L_2^{N+}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))$$

and elements $\chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\xi}(k)) + \chi^{(d)}_{\overline{\mu},\overline{s}}(\vec{\eta}(k))$ from the space

$$H^{0-}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}}) \oplus H^{N+}(\xi^{(d)}_{-\overline{\mu},\overline{s}} + \eta^{(d)}_{-\overline{\mu},\overline{s}}) = H^{0-}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}}) \oplus H^{(N+n(\gamma))+}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}}).$$

Relation (8) implies that every linear estimate $\widehat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ can be represented in the form

$$\widehat{A}_N\vec{\xi} = \int_{-\pi}^{\pi} (\vec{h}_{\overline{\mu},N}(\lambda))^\top d\vec{Z}_{\xi^{(d)}+\eta^{(d)}}(\lambda) - \sum_{k=-\mu n}^{-1} (\vec{v}_N(k))^\top (\vec{\xi}(k)+\vec{\eta}(k)), \tag{9}$$

where $\vec{h}_{\overline{\mu},N}(\lambda) = \{h_p(\lambda)\}_{p=1}^T$ is the spectral characteristic of the optimal estimate $\widehat{H}_N\vec{\xi}$.

Stage (iii). At this stage we find the mean square optimal estimate $\widehat{H}_N\vec{\xi}$ as a projection of the element $H_N\vec{\xi}$ on the subspace $H^{0-}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}}) \oplus H^{(N+n(\gamma))+}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}})$. This projection is determined by two conditions:

1) $\widehat{H}_N\vec{\xi} \in H^{0-}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}}) \oplus H^{(N+n(\gamma))+}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}})$;

2) $(H_N\vec{\xi} - \widehat{H}_N\vec{\xi}) \perp H^{0-}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}}) \oplus H^{(N+n(\gamma))+}(\xi^{(d)}_{\overline{\mu},\overline{s}} + \eta^{(d)}_{\overline{\mu},\overline{s}})$.

The second condition implies the following relation which holds true for all $k \leq -1$ and $k \geq N+n(\gamma)+1$

$$\int_{-\pi}^{\pi} \Bigg[\left(\vec{B}_{\overline{\mu},N}(e^{i\lambda})^\top \frac{\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} - \vec{h}_{\overline{\mu},N}(\lambda)\right)^\top (f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))$$
$$- (\vec{A}_N(e^{i\lambda}))^\top g(\lambda)\overline{\beta^{(d)}(i\lambda)}\Bigg] \overline{\frac{\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}}\, e^{-i\lambda k} d\lambda = 0.$$

This relation allows us to derive the spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda)$ of the estimate $\widehat{H}_N\vec{\xi}$ which can be represented in the form

$$(\vec{h}_{\overline{\mu},N}(\lambda))^{\top} = (\vec{B}_{\overline{\mu},N}(e^{i\lambda}))^{\top}\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)} - (\vec{C}_{\overline{\mu},N}(e^{i\lambda}))^{\top}\frac{\overline{\beta^{(d)}(i\lambda)}}{\overline{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}}$$
$$\times (f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1} - (\vec{A}_N(e^{i\lambda}))^{\top} g(\lambda)\overline{\beta^{(d)}(i\lambda)}$$
$$\times (f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}, \tag{10}$$

where $\vec{C}_{\overline{\mu},N}(e^{i\lambda}) = \sum_{k=0}^{N+n(\gamma)} \vec{c}_{\overline{\mu},N}(k)e^{ik\lambda}$, $\vec{c}_{\overline{\mu},N}(k) = \{c_{\overline{\mu},N,p}(k)\}_{p=1}^{T}, k = 0,1,\ldots,N+n(\gamma)$, are unknown coefficients to be found.

It follows from condition 1) that the following equations should be satisfied for $0 \le j \le N+n(\gamma)$

$$\int_{-\pi}^{\pi}\Bigg[(\vec{B}_{\overline{\mu},N}(e^{i\lambda}))^{\top} - (\vec{A}_N(e^{i\lambda}))^{\top} g(\lambda)\frac{|\beta^{(d)}(i\lambda)|^2}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}$$
$$- (\vec{C}_{\overline{\mu},N}(e^{i\lambda}))^{\top}\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda))^{-1}\Bigg]e^{-ij\lambda}d\lambda = 0. \tag{11}$$

Define for $0 \le k,j \le N+n(\gamma)$ the Fourier coefficients of the corresponding functions

$$T_{k,j}^{\overline{\mu}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda(j-k)}\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[g(\lambda)\left(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda)\right)^{-1}\right]^{\top} d\lambda;$$

$$P_{k,j}^{\overline{\mu}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda(j-k)}\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[\left(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda)\right)^{-1}\right]^{\top} d\lambda;$$

$$Q_{k,j} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda(j-k)}\left[f(\lambda)\left(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda)\right)^{-1} g(\lambda)\right]^{\top} d\lambda.$$

Making use of the defined Fourier coefficients, relation (11) can be presented as a system of $N+n(\gamma)+1$ linear equations determining the unknown coefficients $\vec{c}_{\overline{\mu},N}(k)$, $0 \le k \le N+n(\gamma)$,

$$\vec{b}_{\overline{\mu},N}(j) - \sum_{m=0}^{N+n(\gamma)} T_{j,m}^{\overline{\mu}}\vec{a}_{\overline{\mu},N}(m) = \sum_{k=0}^{N+n(\gamma)} P_{j,k}^{\overline{\mu}}\vec{c}_{\overline{\mu},N}(k), \quad 0 \le j \le N, \tag{12}$$
$$- \sum_{m=0}^{N+n(\gamma)} T_{j,m}^{\overline{\mu}}\vec{a}_{\overline{\mu},N}(m) = \sum_{k=0}^{N+n(\gamma)} P_{j,k}^{\overline{\mu}}\vec{c}_{\overline{\mu},N}(k), \quad N+1 \le j \le N+n(\gamma), \tag{13}$$

where coefficients $\{\vec{a}_{\overline{\mu},N}(m) : 0 \le m \le N+n(\gamma)\}$ are calculated by the formula

$$\vec{a}_{\overline{\mu},N}(m) = \vec{a}_{-\overline{\mu},N}(m-n(\gamma)), \quad 0 \le m \le N+n(\gamma),$$
$$\vec{a}_{-\overline{\mu},N}(m) = \sum_{l=\max\{m,0\}}^{\min\{m+n(\gamma),N\}} e_{\gamma}(l-m)\vec{a}(l), \quad -n(\gamma) \le m \le N.$$

Denote by $[D^{\overline{\mu}}_N\mathbf{a}_N]_{+n(\gamma)}$ a vector of dimension $(N+n(\gamma)+1)T$ which is constructed by adding $n(\gamma)T$ zeros to the vector $D^{\overline{\mu}}_N\mathbf{a}_N$ of dimension $(N+1)T$. Making use of this definition the system (12)–(13) can be represented in the matrix form $[D^{\overline{\mu}}_N\mathbf{a}_N]_{+n(\gamma)}-\mathbf{T}^{\overline{\mu}}_N\mathbf{a}^{\overline{\mu}}_N=\mathbf{P}^{\overline{\mu}}_N\mathbf{c}^{\overline{\mu}}_N$, where

$$\mathbf{a}^{\overline{\mu}}_N=((\vec{a}_{\overline{\mu},N}(0))^\top,(\vec{a}_{\overline{\mu},N}(1))^\top,(\vec{a}_{\overline{\mu},N}(2))^\top,\dots,(\vec{a}_{\overline{\mu},N}(N+n(\gamma)))^\top)^\top,$$
$$\mathbf{c}^{\overline{\mu}}_N=((\vec{c}_{\overline{\mu},N}(0))^\top,(\vec{c}_{\overline{\mu},N}(1))^\top,(\vec{c}_{\overline{\mu},N}(2))^\top,\dots,(\vec{c}_{\overline{\mu},N}(N+n(\gamma)))^\top)^\top$$

are vectors of dimension $(N+n(\gamma)+1)T$, $\mathbf{P}^{\overline{\mu}}_N$ and $\mathbf{T}^{\overline{\mu}}_N$ are matrices of dimension $(N+n(\gamma)+1)T\times(N+n(\gamma)+1)T$ with $T\times T$ matrix elements $(\mathbf{P}^{\overline{\mu}}_N)_{j,k}=P^{\overline{\mu}}_{j,k}$ and $(\mathbf{T}^{\overline{\mu}}_N)_{j,k}=T^{\overline{\mu}}_{j,k}$, $0\leq j,k\leq N+n(\gamma)$.

Thus, the coefficients $\vec{c}_{\overline{\mu},N}(k)$, $0\leq k\leq N+n(\gamma)$, are determined by the formula

$$\vec{c}_{\overline{\mu},N}(k)=((\mathbf{P}^{\overline{\mu}}_N)^{-1}[D^{\overline{\mu}}_N\mathbf{a}_N]_{+n(\gamma)}-(\mathbf{P}^{\overline{\mu}}_N)^{-1}\mathbf{T}^{\overline{\mu}}_N\mathbf{a}^{\overline{\mu}}_N)_k,\quad 0\leq k\leq N+n(\gamma),$$

where $((\mathbf{P}^{\overline{\mu}}_N)^{-1}[D^{\overline{\mu}}_N\mathbf{a}_N]_{+n(\gamma)}-(\mathbf{P}^{\overline{\mu}}_N)^{-1}\mathbf{T}^{\overline{\mu}}_N\mathbf{a}^{\overline{\mu}}_N)_k$, $0\leq k\leq N+n(\gamma)$, is the kth element of the vector $(\mathbf{P}^{\overline{\mu}}_N)^{-1}[D^{\overline{\mu}}_N\mathbf{a}_N]_{+n(\gamma)}-(\mathbf{P}^{\overline{\mu}}_N)^{-1}\mathbf{T}^{\overline{\mu}}_N\mathbf{a}^{\overline{\mu}}_N$.

The existence of the inverse matrix $(\mathbf{P}^{\mu}_N)^{-1}$ was shown in [29] under condition (6).

The spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda)$ of the estimate $\widehat{H}_N\xi$ of the functional $H_N\xi$ is calculated by formula (10), where

$$\vec{C}_{\overline{\mu},N}(e^{i\lambda})=\sum_{k=0}^{N+n(\gamma)}((\mathbf{P}^{\overline{\mu}}_N)^{-1}[D^{\overline{\mu}}_N\mathbf{a}_N]_{+n(\gamma)}-(\mathbf{P}^{\overline{\mu}}_N)^{-1}\mathbf{T}^{\overline{\mu}}_N\mathbf{a}^{\overline{\mu}}_N)_k e^{i\lambda k}. \tag{14}$$

The value of the mean-square errors of the estimates $\widehat{A}_N\vec{\xi}$ and $\widehat{H}_N\vec{\xi}$ can be calculated by the formula

$$\begin{aligned}\Delta(f,g;\widehat{A}_N\vec{\xi})&=\Delta(f,g;\widehat{H}_N\vec{\xi})=\mathsf{E}|H_N\vec{\xi}-\widehat{H}_N\vec{\xi}|^2\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}[\chi^{(d)}_{\overline{\mu}}(e^{i\lambda})(\vec{A}_N(e^{i\lambda}))^\top g(\lambda)+(\vec{C}_{\overline{\mu},N}(e^{i\lambda}))^\top]\\&\quad\times(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}f(\lambda)(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}\\&\quad\times[\chi^{(d)}_{\overline{\mu}}(e^{i\lambda})\vec{A}_N(e^{-i\lambda})g(\lambda)+(\vec{C}_{\overline{\mu},N}(e^{-i\lambda}))]d\lambda\\&\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\chi^{(d)}_{\overline{\mu}}(e^{i\lambda})(\vec{A}_N(e^{i\lambda}))^\top f(\lambda)-|\beta^{(d)}(i\lambda)|^2(\vec{C}_{\overline{\mu},N}(e^{i\lambda}))^\top}{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}\\&\quad\times(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}g(\lambda)(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}\\&\quad\times[\chi^{(d)}_{\overline{\mu}}(e^{i\lambda})\vec{A}_N(e^{-i\lambda})f(\lambda)-|\beta^{(d)}(i\lambda)|^2(\vec{C}_{\overline{\mu},N}(e^{-i\lambda}))]d\lambda\\&=\langle[D^{\overline{\mu}}_N\mathbf{a}_N]_{+n(\gamma)}-\mathbf{T}^{\overline{\mu}}_N\mathbf{a}^{\overline{\mu}}_N,(\mathbf{P}^{\overline{\mu}}_N)^{-1}[D^{\overline{\mu}}_N\mathbf{a}_N]_{+n(\gamma)}-(\mathbf{P}^{\overline{\mu}}_N)^{-1}\mathbf{T}^{\overline{\mu}}_N\mathbf{a}^{\overline{\mu}}_N\rangle\\&\quad+\langle\mathbf{Q}_N\mathbf{a}_N,\mathbf{a}_N\rangle,\end{aligned} \tag{15}$$

where $\mathbf{Q}_N$ is a matrix of the dimension $(N+1)T\times(N+1)T$ with the $T\times T$ matrix elements $(\mathbf{Q}_N)_{j,k}=Q_{j,k}$, $0\leq j,k\leq N$.

The following theorem holds true.

Theorem 3. *Let $\{\vec{\xi}(m)\colon m\in\mathbb{Z}\}$ be a stochastic sequence which defines the stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))=\{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))\}_{p=1}^{T}$ with the absolutely continuous spectral function $F(\lambda)$ which has spectral density $f(\lambda)$. Let $\{\vec{\eta}(m)\colon m\in\mathbb{Z}\}$ be an uncorrelated with the sequence $\vec{\xi}(m)$ stationary stochastic sequence with an absolutely continuous spectral function $G(\lambda)$ which has spectral density $g(\lambda)$. Let the minimality condition* (6) *be satisfied. The optimal linear estimate $\widehat{A}_N\vec{\xi}$ of the functional $A_N\vec{\xi}$ which depends on the unknown values of elements $\vec{\xi}(k)$, $k=0,1,2,\dots,N$, from observations of the sequence $\vec{\xi}(m)+\vec{\eta}(m)$ at points of the set $Z\setminus\{0,1,2,\dots,N\}$ is calculated by formula* (9)*. The spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda)$ of the optimal estimate $\widehat{A}_N\vec{\xi}$ is calculated by formulas* (10), (14)*. The value of the mean-square error $\Delta(f,g;\widehat{A}_N\vec{\xi})$ is calculated by formula* (15).

Corollary 1. *The spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda)$* (10) *admits the representation*

$$\vec{h}_{\overline{\mu},N}(\lambda)=\vec{h}^{1}_{\overline{\mu},N}(\lambda)-\vec{h}^{2}_{\overline{\mu},N}(\lambda),$$

where

$$\begin{aligned}(\vec{h}^{1}_{\overline{\mu},N}(\lambda))^{\top}&=(\vec{B}_{\overline{\mu},N}(e^{i\lambda}))^{\top}\frac{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}{\beta^{(d)}(i\lambda)}-\frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}\left(\sum_{k=0}^{N+n(\gamma)}((\mathbf{P}_N^{\overline{\mu}})^{-1}[D_N^{\overline{\mu}}\mathbf{a}_N]_{+n(\gamma)})_k e^{i\lambda k}\right)^{\top}\\&\quad\times(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1},\\(\vec{h}^{2}_{\overline{\mu},N}(\lambda))^{\top}&=(\vec{A}_N(e^{i\lambda}))^{\top}\overline{\beta^{(d)}(i\lambda)}g(\lambda)(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}\\&\quad-\frac{\overline{\beta^{(d)}(i\lambda)}}{\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})}\left(\sum_{k=0}^{N+n(\gamma)}((\mathbf{P}_N^{\overline{\mu}})^{-1}\mathbf{T}_N^{\overline{\mu}}\mathbf{a}_N^{\overline{\mu}})_k e^{i\lambda k}\right)^{\top}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2g(\lambda))^{-1}.\end{aligned}$$

Here $\vec{h}^{1}_{\overline{\mu},N}(\lambda)$ and $\vec{h}^{2}_{\overline{\mu},N}(\lambda)$ are the spectral characteristics of the optimal estimates $\widehat{B}_N\chi\vec{\zeta}$ and $\widehat{A}_N\vec{\eta}$ of the functionals $B_N\chi\vec{\zeta}$ and $A_N\vec{\eta}$ respectively based on observations $\vec{\xi}(k)+\vec{\eta}(k)$ at points of the set $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$.

Remark 2. *The interpolation problem for stochastic sequences with fractional multiple (FM) increments can be solved with the help of results described in Theorem 3 under the conditions of Theorem 2 on the increment orders d_i.*

2.2 Interpolation of stochastic sequences with periodically stationary GM increments

Consider the problem of mean square optimal linear estimation of the functional $A_M\vartheta=\sum_{k=0}^{N}a^{(\vartheta)}(k)\vartheta(k)$ which depend on unobserved values of the stochastic sequence $\vartheta(m)$ with periodically stationary GM increments. Estimates are based on observations of the sequence $\zeta(m)=\vartheta(m)+\eta(m)$ at points of the set $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$, where the periodically stationary noise sequence $\eta(m)$ is uncorrelated with $\vartheta(m)$.

The functional $A_M\vartheta$ can be represented in the form

$$\begin{aligned}A_M\vartheta&=\sum_{k=0}^{M}a^{(\vartheta)}(k)\vartheta(k)=\sum_{m=0}^{N}\sum_{p=1}^{T}a^{(\vartheta)}(mT+p-1)\vartheta(mT+p-1)\\&=\sum_{m=0}^{N}\sum_{p=1}^{T}a_p(m)\xi_p(m)=\sum_{m=0}^{N}(\vec{a}(m))^{\top}\vec{\xi}(m)=A_N\vec{\xi},\end{aligned}$$

where $N=[M/T]$, the sequence $\vec{\xi}(m)$ is determined by the formula

$$\vec{\xi}(m)=(\xi_1(m),\xi_2(m),\dots,\xi_T(m))^\top,\quad \xi_p(m)=\vartheta(mT+p-1),\quad p=1,2,\dots,T, \tag{16}$$

$$\begin{aligned}(\vec{a}(m))^\top &= (a_1(m),a_2(m),\dots,a_T(m))^\top,\\ a_p(m)&=a^{\vartheta}(mT+p-1),\quad 0\leq m\leq N,\ 1\leq p\leq T,\ mT+p-1\leq M,\\ a_p(N)&=0,\quad M+1\leq NT+p-1\leq (N+1)T-1,\ 1\leq p\leq T.\end{aligned} \tag{17}$$

Making use of the introduced notations and statements of Theorem 3 we can claim that the following theorem holds true.

Theorem 4. *Let a stochastic sequence $\vartheta(k)$ with periodically stationary GM increments generate by formula* (16) *a vector-valued stochastic sequence $\vec{\xi}(m)$ which determine a stationary GM increment sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))=\{\chi_{\overline{\mu},\overline{s}}^{(d)}(\xi_p(m))\}_{p=1}^{T}$ with the spectral density matrix $f(\lambda)=\{f_{ij}(\lambda)\}_{i,j=1}^{T}$. Let $\{\vec{\eta}(m)\colon m\in\mathbb{Z}\}$, $\vec{\eta}(m)=(\eta_1(m),\eta_2(m),\dots,\eta_T(m))^\top$, $\eta_p(m)=\eta(mT+p-1)$, $p=1,2,\dots,T$, be uncorrelated with the sequence $\vec{\xi}(m)$ stationary stochastic sequence with an absolutely continuous spectral function $G(\lambda)$ which has spectral density $g(\lambda)$. Let the minimality condition* (6) *be satisfied. Let coefficients $\vec{a}(k), k\geqslant 0$, be determined by formula* (17). *The optimal linear estimate $\widehat{A}_M\zeta$ of the functional $A_M\zeta=A_N\vec{\xi}$ based on observations of the sequence $\zeta(m)=\vartheta(m)+\eta(m)$ at points of the set $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$ is calculated by formula* (9). *The spectral characteristic $\vec{h}_{\overline{\mu},N}(\lambda)=\{h_{\overline{\mu},N,p}(\lambda)\}_{p=1}^{T}$ and the value of the mean square error $\Delta(f;\widehat{A}_M\zeta)$ are calculated by formulas* (10), (14), *and* (15), *respectively.*

3 Minimax (robust) method of interpolation

The values of the mean square errors and the spectral characteristics of the optimal estimate of the functional $A_N\vec{\xi}$ depending on the unobserved values of a stochastic sequence $\vec{\xi}(m)$ which determine a stationary GM increments sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ with the spectral density matrix $f(\lambda)$ based on observations of the sequence $\vec{\xi}(m)+\vec{\eta}(m)$ at points $\mathbb{Z}\setminus\{0,1,2,\dots,N\}$ can be calculated by formulas (10), (14), (15) respectively, in the case where the spectral density matrices $f(\lambda)$ and $g(\lambda)$ of the target sequence and the noise are exactly known.

In practical cases, however, complete information about the spectral density matrices is not available in most cases. If in such cases a set $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ of admissible spectral densities is defined, the minimax-robust approach to estimation of linear functionals depending on unobserved values of stochastic sequences with stationary increments may be applied.

This method consists in finding an estimate that minimizes the maximal values of the mean square errors for all spectral densities from a given class $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ of admissible spectral densities simultaneously. This method will be applied in the case of concrete classes of spectral densities.

To formalize this approach we recall the following definitions [37].

Definition 5. *For a given class of spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ the spectral densities $f_0(\lambda)\in\mathcal{D}_f$, $g_0(\lambda)\in\mathcal{D}_g$ are called least favorable in the class $\mathcal{D}$ for the optimal linear estimation of the functional $A_N\vec{\xi}$ if the following relation holds true:*

$$\Delta(f_0,g_0)=\Delta(h(f_0,g_0);f_0,g_0)=\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h(f,g);f,g).$$

Definition 6. *For a given class of spectral densities $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ the spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A_N\vec{\xi}$ is called minimax-robust if there are satisfied the conditions*

$$h^0(\lambda) \in H_{\mathcal{D}} = \bigcap_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} L_2^{0-}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda)) \oplus L_2^{(N+n(\nu))+}(f(\lambda)+|\beta^{(d)}(i\lambda)|^2 g(\lambda)),$$

$$\min_{h\in H_{\mathcal{D}}} \max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} \Delta(h;f,g) = \max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} \Delta(h^0;f,g).$$

Taking into account the introduced definitions and the relations derived in the previous sections we can verify that the following lemma holds true.

Lemma 3. *The spectral densities $f^0 \in \mathcal{D}_f$, $g^0 \in \mathcal{D}_g$ which satisfy the minimality condition (6) are least favorable in the class $\mathcal{D} = \mathcal{D}_f \times \mathcal{D}_g$ for the optimal linear estimation of the functional $A_N\vec{\xi}$ based on observations of the sequence $\xi(m)+\eta(m)$ at points $m \in \mathbb{Z}\setminus\{0,1,2,\dots,N\}$ if the matrices $(\mathbf{T}_N^{\overline{\mu}})^0$, $(\mathbf{P}_N^{\overline{\mu}})^0$, $(\mathbf{Q}_N)^0$ whose elements are defined by the Fourier coefficients of the functions*

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[g^0(\lambda)\left(f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2 g^0(\lambda)\right)^{-1}\right]^{\top},$$

$$\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}\left[\left(f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2 g^0(\lambda)\right)^{-1}\right]^{\top}, \quad \left[f^0(\lambda)\left(f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2 g^0(\lambda)\right)^{-1} g^0(\lambda)\right]^{\top}$$

determine a solution of the constrained optimisation problem

$$\begin{aligned}\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g} &(\langle [D_N^{\overline{\mu}}\mathbf{a}_N]_{+n(\gamma)} - \mathbf{T}_N^{\overline{\mu}}\mathbf{a}_{\overline{\mu}}, (\mathbf{P}_N^{\overline{\mu}})^{-1}[D_N^{\overline{\mu}}\mathbf{a}_N]_{+n(\gamma)} - (\mathbf{P}_N^{\overline{\mu}})^{-1}\mathbf{T}_N^{\overline{\mu}}\mathbf{a}_N^{\overline{\mu}}\rangle + \langle \mathbf{Q}_N\mathbf{a}_N, \mathbf{a}_N\rangle)\\ &= \langle [D_N^{\overline{\mu}}\mathbf{a}_N]_{+n(\gamma)} - (\mathbf{T}_N^{\overline{\mu}})^0\mathbf{a}_N^{\overline{\mu}}, ((\mathbf{P}_N^{\overline{\mu}})^0)^{-1}[D_N^{\overline{\mu}}\mathbf{a}_N]_{+n(\gamma)}\\ &\quad - ((\mathbf{P}_N^{\overline{\mu}})^0)^{-1}(\mathbf{T}_N^{\overline{\mu}})^0\mathbf{a}_N^{\overline{\mu}}\rangle + \langle \mathbf{Q}_N^0\mathbf{a}_N, \mathbf{a}_N\rangle.\end{aligned} \tag{18}$$

The minimax spectral characteristic $h^0 = \vec{h}_{\overline{\mu},N}(f^0,g^0)$ is calculated by formula (10) *if $\vec{h}_{\overline{\mu},N}(f^0,g^0) \in H_{\mathcal{D}}$.*

The more detailed analysis of properties of the least favorable spectral densities and the minimax-robust spectral characteristics shows that the minimax spectral characteristic h^0 and the least favourable spectral densities f^0 and g^0 form a saddle point of the function $\Delta(h;f,g)$ on the set $H_{\mathcal{D}}\times\mathcal{D}$. The saddle point inequalities

$$\Delta(h;f^0,g^0) \geq \Delta(h^0;f^0,g^0) \geq \Delta(h^0;f,g) \quad \forall (f,g)\in\mathcal{D}, \quad \forall h\in H_{\mathcal{D}}$$

hold true if $h^0 = \vec{h}_{\overline{\mu},N}(f^0,g^0)$, $\vec{h}_{\overline{\mu}}(f^0,g^0) \in H_{\mathcal{D}}$ and (f^0,g^0) is a solution of the constrained optimization problem

$$\widetilde{\Delta}(f,g) = -\Delta(\vec{h}_{\overline{\mu}}(f^0,g^0);f,g) \to \inf, \quad (f,g)\in\mathcal{D}, \tag{19}$$

where the functional $\Delta(\vec{h}_{\overline{\mu},N}(f^0,g^0);f,g)$ is calculated by the formula

$$\Delta(\vec{h}_{\overline{\mu},N}(f^0,g^0);f,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\beta^{(d)}(i\lambda)|^2}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}[\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})(\vec{A}_N(e^{i\lambda}))^{\top}g^0(\lambda)+(\vec{C}^0_{\mu,N}(e^{i\lambda}))^{\top}]$$
$$\times(f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2g^0(\lambda))^{-1}f(\lambda)(f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2g^0(\lambda))^{-1}$$
$$\times[\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})\vec{A}_N(e^{-i\lambda})g^0(\lambda)+\vec{C}^0_{\mu,N}(e^{-i\lambda})]d\lambda$$
$$+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{[\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})(\vec{A}_N(e^{i\lambda}))^{\top}f^0(\lambda)-|\beta^{(d)}(i\lambda)|^2(\vec{C}^0_{\mu,N}(e^{i\lambda}))^{\top}]}{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}$$
$$\times(f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2g^0(\lambda))^{-1}g(\lambda)(f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2g^0(\lambda))^{-1}$$
$$\times[\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})\vec{A}_N(e^{-i\lambda})f^0(\lambda)-|\beta^{(d)}(i\lambda)|^2\vec{C}^0_{\mu,N}(e^{-i\lambda})]d\lambda,$$

where

$$\vec{C}^0_{\overline{\mu},N}(e^{i\lambda})=\sum_{k=0}^{N+n(\gamma)}(((\mathbf{P}_N^{\overline{\mu}})^0)^{-1}[D_N^{\overline{\mu}}\mathbf{a}_N]_{+n(\gamma)}-((\mathbf{P}_N^{\overline{\mu}})^0)^{-1}(\mathbf{T}_N^{\overline{\mu}})^0\mathbf{a}_N^{\overline{\mu}})_k e^{ik\lambda}.$$

The constrained optimization problem (19) is equivalent to the unconstrained optimization problem

$$\Delta_{\mathcal{D}}(f,g)=\widetilde{\Delta}(f,g)+\delta(f,g|\mathcal{D})\to\inf, \tag{20}$$

where $\delta(f,g|\mathcal{D})$ is the indicator function of the set $\mathcal{D}$, namely $\delta(f,g|\mathcal{D})=0$ if $(f,g)\in\mathcal{D}$ and $\delta(f,g|\mathcal{D})=+\infty$ if $(f,g)\notin\mathcal{D}$. The condition $0\in\partial\Delta_{\mathcal{D}}(f^0,g^0)$ characterizes a solution (f^0,g^0) of the stated unconstrained optimization problem. This condition is the necessary and sufficient condition under which the point (f^0,g^0) belongs to the set of minimums of the convex functional $\Delta_{\mathcal{D}}(f,g)$ [34,44]. Thus, it allows us to find equations which determine the least favourable spectral densities in some special classes of spectral densities $\mathcal{D}$.

The form of the functional $\Delta(\vec{h}_{\overline{\mu}}(f^0,g^0);f,g)$ is suitable for application of the Lagrange method of indefinite multipliers to the constrained optimization problem (19). Thus, the complexity of the problem is reduced to finding the subdifferential of the indicator function of the set of admissible spectral densities. We illustrate the solving of the problem (20) for concrete sets admissible spectral densities in the following subsections.

3.1 Least favorable spectral density in classes $\mathcal{D}_0\times\mathcal{D}_\varepsilon$

Consider the minimax interpolation problem for the functional $A_N\vec{\xi}$ depending on the unobserved values of the stochastic sequence $\vec{\xi}(m)$ which determine a stationary GM increments sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ for the following sets of admissible spectral densities $\mathcal{D}_0^k$, $k=1,2,3,4$,

$$\mathcal{D}_0^1=\left\{f(\lambda)\colon\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}f(\lambda)\,d\lambda=P\right\},$$

$$\mathcal{D}_0^2=\left\{f(\lambda)\colon\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\mathrm{Tr}[f(\lambda)]\,d\lambda=p\right\},$$

$$\mathcal{D}_0^3=\left\{f(\lambda)\colon\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}f_{kk}(\lambda)\,d\lambda=p_k,\ k=\overline{1,T}\right\},$$

$$\mathcal{D}_0^4=\left\{f(\lambda)\colon\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\langle B_1,f(\lambda)\rangle\,d\lambda=p\right\},$$

where $p, p_k, k=\overline{1,T}$ are given numbers, P, B_1 are given positive-definite Hermitian matrices, and sets of admissible spectral densities $\mathcal{D}_\varepsilon^k$, $k=1,2,3,4$ for the stationary noise sequence $\vec{\eta}(m)$

$$\mathcal{D}_\varepsilon^1=\left\{g(\lambda)\colon\ \mathrm{Tr}[g(\lambda)]=(1-\varepsilon)\mathrm{Tr}[g_1(\lambda)]+\varepsilon\mathrm{Tr}[W(\lambda)],\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}[g(\lambda)]d\lambda=q\right\},$$
$$\mathcal{D}_\varepsilon^2=\left\{g(\lambda)\colon g_{kk}(\lambda)=(1-\varepsilon)g_{kk}^1(\lambda)+\varepsilon w_{kk}(\lambda),\ \frac{1}{2\pi}\int_{-\pi}^{\pi}g_{kk}(\lambda)d\lambda=q_k,\ k=\overline{1,T}\right\},$$
$$\mathcal{D}_\varepsilon^3=\left\{g(\lambda)\colon \langle B_2,g(\lambda)\rangle=(1-\varepsilon)\langle B_2,g_1(\lambda)\rangle+\varepsilon\langle B_2,W(\lambda)\rangle,\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\langle B_2,g(\lambda)\rangle d\lambda=q\right\},$$
$$\mathcal{D}_\varepsilon^4=\left\{g(\lambda)\colon g(\lambda)=(1-\varepsilon)g_1(\lambda)+\varepsilon W(\lambda),\ \frac{1}{2\pi}\int_{-\pi}^{\pi}g(\lambda)d\lambda=Q\right\},$$

where $g_1(\lambda)$ is a fixed spectral density, $W(\lambda)$ is an unknown spectral density, $q, q_k, k=\overline{1,T}$, are given numbers, Q is a given positive-definite Hermitian matrix.

In the following we will use the next notations:

$$C_{\overline{\mu},N}^{f0}(e^{i\lambda}):=\chi_{\overline{\mu}}^{(d)}(e^{i\lambda})\vec{A}_N(e^{-i\lambda}))^\top g^0(\lambda)$$
$$-\left(\sum_{k=0}^{N+n(\gamma)}(((\mathbf{P}_N^{\overline{\mu}})^0)^{-1}[D_N^{\overline{\mu}}\mathbf{a}_N]_{+n(\gamma)}-((\mathbf{P}_N^{\overline{\mu}})^0)^{-1}(\mathbf{T}_N^{\overline{\mu}})^0\mathbf{a}_N^{\overline{\mu}})_k)e^{ik\lambda}\right)^\top,$$
$$C_{\overline{\mu},N}^{g0}(e^{i\lambda}):=\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(\vec{A}_N(e^{-i\lambda}))^\top f^0(\lambda)$$
$$+\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})\left(\sum_{k=0}^{N+n(\gamma)}(((\mathbf{P}_N^{\overline{\mu}})^0)^{-1}[D_N^{\overline{\mu}}\mathbf{a}_N]_{+n(\gamma)}-((\mathbf{P}_N^{\overline{\mu}})^0)^{-1}(\mathbf{T}_N^{\overline{\mu}})^0\mathbf{a}_N^{\overline{\mu}})_k)e^{ik\lambda}\right)^\top,$$
$$p_\chi^0(\lambda)=\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}(f^0(\lambda)+|\beta^{(d)}(i\lambda)|^2g^0(\lambda)).$$

From the condition $0\in\partial\Delta_{\mathcal{D}}(f^0,g^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first pair of the sets of admissible spectral densities $\mathcal{D}_{f0}^1\times\mathcal{D}_\varepsilon^1$ we have equations

$$\begin{aligned}(C_{\overline{\mu},N}^{f0}(e^{i\lambda}))(C_{\overline{\mu},N}^{f0}(e^{i\lambda}))^*&=p_\chi^0(\lambda)\vec{\alpha}\cdot\vec{\alpha}^*p_\chi^0(\lambda),\\(C_{\overline{\mu},N}^{g0}(e^{i\lambda}))(C_{\overline{\mu},N}^{g0}(e^{i\lambda}))^*&=(\alpha^2+\gamma_1(\lambda))(p_\chi^0(\lambda))^2,\end{aligned}\tag{21}$$

where α^2, $\vec{\alpha}$ are Lagrange multipliers, the function $\gamma_1(\lambda)\le0$ and $\gamma_1(\lambda)=0$ if $\mathrm{Tr}[g_0(\lambda)]>(1-\varepsilon)\mathrm{Tr}[g_1(\lambda)]$.

For the second pair of the sets of admissible spectral densities $\mathcal{D}_{f0}^2\times\mathcal{D}_\varepsilon^2$ we have equation

$$\begin{aligned}(C_{\overline{\mu},N}^{f0}(e^{i\lambda}))(C_{\overline{\mu},N}^{f0}(e^{i\lambda}))^*&=\alpha^2(p_\chi^0(\lambda))^2,\\(C_{\overline{\mu},N}^{g0}(e^{i\lambda}))(C_{\overline{\mu},N}^{g0}(e^{i\lambda}))^*&=(p_\chi^0(\lambda))\{(\alpha_k^2+\gamma_k^1(\lambda))\delta_{kl}\}_{k,l=1}^T(p_\chi^0(\lambda)),\end{aligned}\tag{22}$$

where α^2, α_k^2 are Lagrange multipliers, functions $\gamma_k^1(\lambda)\le0$ and $\gamma_k^1(\lambda)=0$ if $g_{kk}^0(\lambda)>(1-\varepsilon)g_{kk}^1(\lambda)$.

For the third pair of the sets of admissible spectral densities $\mathcal{D}_{f0}^3 \times \mathcal{D}_{\varepsilon}^3$ we have equation

$$\begin{aligned}(C_{\overline{\mu},N}^{f0}(e^{i\lambda}))(C_{\overline{\mu},N}^{f0}(e^{i\lambda}))^* &= (p_{\chi}^0(\lambda))\{\alpha_k^2\delta_{kl}\}_{k,l=1}^T(p_{\chi}^0(\lambda)),\\ (C_{\overline{\mu},N}^{g0}(e^{i\lambda}))(C_{\overline{\mu},N}^{g0}(e^{i\lambda}))^* &= (\alpha^2+\gamma_1'(\lambda))p_{\chi}^0(\lambda))B_2^{\top}(p_{\chi}^0(\lambda)),\end{aligned} \tag{23}$$

where α_k^2, α^2 are Lagrange multipliers, function $\gamma_1'(\lambda) \leq 0$ and $\gamma_1'(\lambda) = 0$ if $\langle B_2, g_0(\lambda)\rangle > (1-\varepsilon)\langle B_2, g_1(\lambda)\rangle$, δ_{kl} is the Kronecker symbol.

For the fourth pair of the sets of admissible spectral densities $\mathcal{D}_{f0}^4 \times \mathcal{D}_{\varepsilon}^4$ we have equation

$$\begin{aligned}(C_{\overline{\mu},N}^{f0}(e^{i\lambda}))(C_{\overline{\mu},N}^{f0}(e^{i\lambda}))^* &= \alpha^2(p_{\chi}^0(\lambda))B_1^{\top}(p_{\chi}^0(\lambda)),\\ (C_{\overline{\mu},N}^{g0}(e^{i\lambda}))(C_{\overline{\mu},N}^{g0}(e^{i\lambda}))^* &= (p_{\chi}^0(\lambda))(\vec{\alpha}\cdot\vec{\alpha}^*+\Gamma(\lambda))(p_{\chi}^0(\lambda)),\end{aligned} \tag{24}$$

where α^2, $\vec{\alpha}$ are Lagrange multipliers, function $\Gamma(\lambda) \leq 0$ and $\Gamma(\lambda) = 0$ if $g_0(\lambda) > (1-\varepsilon)g_1(\lambda)$.

The following theorem holds true.

Theorem 5. *The least favorable spectral densities $f^0(\lambda)$ and $g^0(\lambda)$ in the classes $\mathcal{D}_0^k \times \mathcal{D}_{\varepsilon}^k$, $k = 1,2,3,4$, for the optimal linear estimation of the functional $A_N\vec{\xi}$ are determined by pairs of equations* (21)–(24)*, the minimality condition* (6)*, the constrained optimization problem* (18) *and restrictions on densities from the corresponding classes $\mathcal{D}_0^k \times \mathcal{D}_{\varepsilon}^k$, $k = 1,2,3,4$. The minimax-robust spectral characteristic $\vec{h}_{\overline{\mu},N}(f^0, g^0)$ of the optimal estimate of the functional $A_N\vec{\xi}$ is determined by the formula* (10)*.*

3.2 Least favorable spectral density in classes $\mathcal{D}_{1\delta} \times \mathcal{D}_V^U$

Consider the minimax interpolation problem for the functional $A_N\vec{\xi}$ depending on the unobserved values of the stochastic sequence $\vec{\xi}(m)$ which determine a stationary GM increments sequence $\chi_{\overline{\mu},\overline{s}}^{(d)}(\vec{\xi}(m))$ for the following sets of admissible spectral densities $\mathcal{D}_{1\delta}^k$, $k = 1,2,3,4$,

$$\mathcal{D}_{1\delta}^1 = \left\{f(\lambda)\colon \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}|\mathrm{Tr}(f(\lambda)-f_1(\lambda))|\,d\lambda \leq \delta\right\},$$

$$\mathcal{D}_{1\delta}^2 = \left\{f(\lambda)\colon \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}|f_{kk}(\lambda)-f_{kk}^1(\lambda)|\,d\lambda \leq \delta_k,\ k=\overline{1,T}\right\},$$

$$\mathcal{D}_{1\delta}^3 = \left\{f(\lambda)\colon \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}|\langle B_1, f(\lambda)-f_1(\lambda)\rangle|\,d\lambda \leq \delta\right\},$$

$$\mathcal{D}_{1\delta}^4 = \left\{f(\lambda)\colon \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi_{\overline{\mu}}^{(d)}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}|f_{ij}(\lambda)-f_{ij}^1(\lambda)|\,d\lambda \leq \delta_i^j,\ i,j=\overline{1,T}\right\},$$

where $f_1(\lambda)$ is a fixed spectral density, B_1 is a given positive-definite Hermitian matrix, $\delta, \delta_k, k=\overline{1,T}$, $\delta_i^j, i,j=\overline{1,T}$, are given numbers, and sets of admissible spectral densities $\mathcal{D}_V^{U^k}$, $k = 1,2,3,4$, for the stationary noise sequence $\vec{\eta}(m)$:

$$\mathcal{D}_V^{U^1} = \left\{g(\lambda)\colon V(\lambda) \leq g(\lambda) \leq U(\lambda), \frac{1}{2\pi}\int_{-\pi}^{\pi}g(\lambda)\,d\lambda = Q\right\},$$

$$\mathcal{D}_V^{U^2} = \left\{g(\lambda)\colon \mathrm{Tr}[V(\lambda)] \leq \mathrm{Tr}[g(\lambda)] \leq \mathrm{Tr}[U(\lambda)],\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Tr}[g(\lambda)]\,d\lambda = q\right\},$$

$$\mathcal{D}_V^{U^3} = \left\{g(\lambda)\colon v_{kk}(\lambda) \leq g_{kk}(\lambda) \leq u_{kk}(\lambda),\ \frac{1}{2\pi}\int_{-\pi}^{\pi}g_{kk}(\lambda)\,d\lambda = q_k,\ k=\overline{1,T}\right\},$$

$$\mathcal{D}_V^{U^4} = \left\{g(\lambda)\colon \langle B_2, V(\lambda)\rangle \leq \langle B_2, g(\lambda)\rangle \leq \langle B_2, U(\lambda)\rangle,\ \frac{1}{2\pi}\int_{-\pi}^{\pi}\langle B_2, g(\lambda)\,\rangle d\lambda = q\right\},$$

where the spectral densities $V(\lambda), U(\lambda)$ are known and fixed, q, q_k, $k=\overline{1,T}$, are given numbers, Q, B_2 are given positive definite Hermitian matrices.

From the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0, g^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities.

For the first pair of the sets of admissible spectral densities $\mathcal{D}_{1\delta}^1 \times \mathcal{D}_V^{U1}$ we have equations

$$\begin{aligned}(C^{f0}_{\overline{\mu},N}(e^{i\lambda}))(C^{f0}_{\overline{\mu},N}(e^{i\lambda}))^* &= \beta^2\gamma_2(\lambda)(p^0_\chi(\lambda))^2,\\ (C^{g0}_{\overline{\mu},N}(e^{i\lambda}))(C^{g0}_{\overline{\mu},N}(e^{i\lambda}))^* &= (p^0_\chi(\lambda))(\vec{\beta}\cdot\vec{\beta}^* + \Gamma_1(\lambda)+\Gamma_2(\lambda))(p^0_\chi(\lambda)),\end{aligned} \tag{25}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}\left|\mathrm{Tr}(f_0(\lambda)-f_1(\lambda))\right| d\lambda = \delta, \tag{26}$$

where β^2, $\vec{\beta}$ are Lagrange multipliers, the function $\Gamma_1(\lambda)\leq 0$ and $\Gamma_1(\lambda)=0$ if $g^0(\lambda) > V(\lambda)$, the function $\Gamma_2(\lambda)\geq 0$ and $\Gamma_2(\lambda)=0$ if $g^0(\lambda)<U(\lambda)$, the function $|\gamma_2(\lambda)|\leq 1$ and

$$\gamma_2(\lambda) = \mathrm{sign}(\mathrm{Tr}(f_0(\lambda)-f_1(\lambda))): \quad \mathrm{Tr}(f_0(\lambda)-f_1(\lambda))\neq 0.$$

For the second pair of the sets of admissible spectral densities $\mathcal{D}_{1\delta}^2 \times \mathcal{D}_V^{U2}$ we have equations

$$\begin{aligned}(C^{f0}_{\overline{\mu},N}(e^{i\lambda}))(C^{f0}_{\overline{\mu},N}(e^{i\lambda}))^* &= (p^0_\chi(\lambda))\{\beta_k^2\gamma_k^2(\lambda)\delta_{kl}\}_{k,l=1}^T(p^0_\chi(\lambda)),\\ (C^{g0}_{\overline{\mu},N}(e^{i\lambda}))(C^{g0}_{\overline{\mu},N}(e^{i\lambda}))^* &= (\beta^2+\gamma_1(\lambda)+\gamma_2(\lambda))(p^0_\chi(\lambda))^2,\end{aligned} \tag{27}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}|f^0_{kk}(\lambda)-f^1_{kk}(\lambda)|d\lambda = \delta_k, \quad k=\overline{1,T}, \tag{28}$$

where β^2, β_k^2 are Lagrange multipliers, δ_{kl} is the Kronecker symbol, the function $\gamma_1(\lambda)\leq 0$ and $\gamma_1(\lambda)=0$ if $\mathrm{Tr}[g^0(\lambda)] > \mathrm{Tr}[V(\lambda)]$, the function $\gamma_2(\lambda)\geq 0$ and $\gamma_2(\lambda)=0$ if $\mathrm{Tr}[g^0(\lambda)] < \mathrm{Tr}[U(\lambda)]$, the functions $|\gamma_k^2(\lambda)|\leq 1$ and

$$\gamma_k^2(\lambda) = \mathrm{sign}(f^0_{kk}(\lambda)-f^1_{kk}(\lambda)): \quad f^0_{kk}(\lambda)-f^1_{kk}(\lambda)\neq 0, \quad k=\overline{1,T}.$$

For the third pair of the sets of admissible spectral densities $\mathcal{D}_{1\delta}^3 \times \mathcal{D}_V^{U3}$ we have equations

$$\begin{aligned}(C^{f0}_{\overline{\mu},N}(e^{i\lambda}))(C^{f0}_{\overline{\mu},N}(e^{i\lambda}))^* &= \beta^2\gamma_2'(\lambda)(p^0_\chi(\lambda))B_1^\top(p^0_\chi(\lambda)),\\ (C^{g0}_{\overline{\mu},N}(e^{i\lambda}))(C^{g0}_{\overline{\mu},N}(e^{i\lambda}))^* &= (p^0_\chi(\lambda))\{(\beta_k^2+\gamma_{1k}(\lambda)+\gamma_{2k}(\lambda))\delta_{kl}\}_{k,l=1}^T(p^0_\chi(\lambda)),\end{aligned} \tag{29}$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2}|\langle B_1, f_0(\lambda)-f_1(\lambda)\rangle|d\lambda = \delta, \tag{30}$$

where β^2, β_k^2 are Lagrange multipliers, δ_{kl} is the Kronecker symbol, the function $\gamma_{1k}(\lambda)\leq 0$ and $\gamma_{1k}(\lambda)=0$ if $g^0_{kk}(\lambda) > v_{kk}(\lambda)$, the function $\gamma_{2k}(\lambda)\geq 0$ and $\gamma_{2k}(\lambda)=0$ if $g^0_{kk}(\lambda)<u_{kk}(\lambda)$, the function $|\gamma_2'(\lambda)|\leq 1$ and

$$\gamma_2'(\lambda) = \mathrm{sign}\langle B_1, f_0(\lambda)-f_1(\lambda)\rangle: \quad \langle B_1, f_0(\lambda)-f_1(\lambda)\rangle\neq 0.$$

For the fourth pair of the sets of admissible spectral densities $\mathcal{D}_{1\delta}^{4} \times \mathcal{D}_{V}^{U^4}$ we have equations

$$(C^{f0}_{\overline{\mu},N}(e^{i\lambda}))(C^{f0}_{\overline{\mu},N}(e^{i\lambda}))^{*} = (p^0_{\chi}(\lambda))\{\beta_{ij}(\lambda)\gamma_{ij}(\lambda)\}_{i,j=1}^{T}(p^0_{\chi}(\lambda)),$$
$$(C^{g0}_{\overline{\mu},N}(e^{i\lambda}))(C^{g0}_{\overline{\mu},N}(e^{i\lambda}))^{*} = (\beta^2 + \gamma_1'(\lambda) + \gamma_2'(\lambda))(p^0_{\chi}(\lambda))B_2^{\top}(p^0_{\chi}(\lambda)), \tag{31}$$

end

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|\chi^{(d)}_{\overline{\mu}}(e^{-i\lambda})|^2}{|\beta^{(d)}(i\lambda)|^2} |f^0_{ij}(\lambda) - f^1_{ij}(\lambda)| d\lambda = \delta_i^j, \quad i,j = \overline{1,T}, \tag{32}$$

where β^2, β_{ij} are Lagrange multipliers, the function $\gamma_1'(\lambda) \leq 0$ and $\gamma_1'(\lambda) = 0$ if $\langle B_2, g^0(\lambda)\rangle > \langle B_2, V(\lambda)\rangle$, the function $\gamma_2'(\lambda) \geq 0$ and $\gamma_2'(\lambda) = 0$ if $\langle B_2, g^0(\lambda)\rangle < \langle B_2, U(\lambda)\rangle$, functions $|\gamma_{ij}(\lambda)| \leq 1$ and

$$\gamma_{ij}(\lambda) = \frac{f^0_{ij}(\lambda) - f^1_{ij}(\lambda)}{|f^0_{ij}(\lambda) - f^1_{ij}(\lambda)|} : \quad f^0_{ij}(\lambda) - f^1_{ij}(\lambda) \neq 0, \quad i,j = \overline{1,T}.$$

The following theorem holds true.

Theorem 6. *The least favorable spectral densities $f^0(\lambda)$ and $g^0(\lambda)$ in the classes $\mathcal{D}_{1\delta}^{k} \times \mathcal{D}_{V}^{U^k}$, $k = 1,2,3,4$, for the optimal linear estimation of the functional $A_N\vec{\xi}$ are determined by pairs of equations* (25)–(32)*, the minimality condition* (6)*, the constrained optimization problem* (18) *and restrictions on densities from the corresponding classes $\mathcal{D}_{1\delta}^{k} \times \mathcal{D}_{V}^{U^k}$, $k = 1,2,3,4$. The minimax-robust spectral characteristic $\vec{h}_{\overline{\mu},N}(f^0, g^0)$ of the optimal estimate of the functional $A_N\vec{\xi}$ is determined by the formula* (10)*.*

4 Conclusions

In this article, we present methods of solution of the interpolation problem for stochastic sequences with periodically stationary long memory multiple seasonal increments, or sequences with periodically stationary general multiplicative (GM) increments, introduced in the article by M. Luz and M. Moklyachuk [31]. These non-stationary stochastic sequences combine periodic structure of covariation functions of sequences as well as multiple seasonal factors, including the integrating one. A short review of the spectral theory of vector-valued generalized multiple increment sequences is presented. We describe methods of solution of the interpolation problem in the case where the spectral densities of the sequence $\xi(m)$ and a noise sequence $\eta(m)$ are exactly known. Estimates are obtained by applying the Hilbert space projection technique to the vector sequence $\vec{\xi}(m) + \vec{\eta}(m)$ with stationary GM increments under the stationary noise sequence $\vec{\eta}(m)$ uncorrelated with $\vec{\xi}(m)$. The case of non-stationary fractional integration is discussed as well. The minimax-robust approach to interpolation problem is applied in the case of spectral uncertainty where the spectral densities of sequences are not exactly known while, instead, sets of admissible spectral densities are specified. We propose a representation of the mean square error in the form of a linear functional in L_1 space with respect to spectral densities, which allows us to solve the corresponding constrained optimization problem and describe the minimax (robust) estimates of the functionals. Relations are described which determine the least favourable spectral densities and the minimax spectral characteristics of the optimal estimates of linear functionals for a collection of specific classes of admissible spectral densities. These sets are generalizations of the sets of admissible spectral densities described in a survey article by S.A. Kassam and H.V. Poor [22] for stationary stochastic processes.

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Луз М.М., Моклячук М.П. *Робастна інтерполяція послідовностей із періодично стаціонарними кратними сезонними приростами* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 105–126.

Ми розглядаємо послідовності з періодично стаціонарними узагальненими кратними приростами дробового порядку, які поєднують циклостаціонарні, багатосезонні, інтегровані та дробово інтегровані структури. Ми досліджуємо задачу оптимального оцінювання функціоналів, що залежать від невідомих значень стохастичної послідовності цього типу на основі спостережень за послідовністю з періодично стаціонарним шумом. Для послідовностей з відомими матрицями спектральних щільностей ми встановили формули для обчислення значень середньоквадратичних похибок та спектральних характеристик оптимальних оцінок функціоналів. Формули, що визначають найменш сприятливі спектральні щільності та мінімаксну (надійну) спектральну характеристики оптимальної лінійної інтерполяції функціоналів пропонуються у випадку, коли спектральні щільності послідовностей точно невідомі, тоді як наведені деякі набори допустимих спектральних щільностей.

Ключові слова і фрази: послідовність із періодично стаціонарними приростами, мініміксна оцінка, робастна оцінка, середньоквадратична похибка, найменш сприятлива спектральна щільність, мінімаксна спектральна характеристика.