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On fixed points of some multivalued mappings under certain function classes

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It is well known that the Banach contraction principle implies the existence of fixed points of single-valued mappings. On the other hand, S.B. Nadler has solved the problem that guarantees the existence of fixed point for multivalued mapping. However, we have to emphasize that similar methods are not applied for nonexpansive multivalued mappings. The aim of this study is to investigate the existence of a fixed point on nonexpansive multivalued mappings with the help of function sequences and functions having shifting distance property. In addition, some hypothesis of this work were explained with an interesting example.

Key words and phrases: fixed point, multivalued mapping, shifting distance function, function sequence.

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Introduction

It is a fact that multivalued mappings are more suitable for applying to real-life problems than single-valued mappings. It was first noticed and studied by S.B. Nadler [20] because of the suitability of multivalued mappings to real-life models. Nadler's approach was to apply the Banach contraction principle [4], which is valid for single-valued mappings, to multivalued mappings with the help of Hausdorff metric. Hence, he showed the existence of fixed points of multivalued mappings. Later, many authors examined some generalizations of the mappings and developed these results (see [1–3, 8, 10, 12, 15, 16, 23, 24]).

Another important concept that we use in this study is functions and function sequences. In fixed point theory, the existence of fixed point is studied by using some special function classes. In this sense, M. Berzig [5] defined a contraction using shifting distance functions. In addition, A. Samadi and M.B. Ghaemi [22] generalized the Darbo fixed point theorem [9] by changing distance functions. Subsequently, they defined a new contraction using the definition given in [5]. Naturally, in the generalization of the above studies, some similar functions were also used in different works. Consequently, using this new approach for multivalued mappings, many researchers have made different generalizations of multivalued mappings using function classes. We refer the reader to [6, 7, 19, 21].

The main purpose of the present paper is to obtain fixed point of nonexpansive multivalued mappings with relations between function classes and function sequences under certain conditions. In the literature, W.A. Kirk [17], W.A. Kirk, H.K. Xu [18] and S.B. Nadler [11] used

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function sequences to determine fixed points of nonexpansive multivalued mappings. Quite recently, V. Karakaya et al. [13, 14] studied on the function sequences, both to obtain new contraction and to show the existence of fixed points of mappings by using the uniform convergence property of function sequences. In multivalued mappings, the Nadler's sense constriction mapping ensures the existence of the fixed point of multivalued mapping. However, when the multivalued mapping type is a nonexpansive mapping, it is not possible to guarantee the existence of the fixed point of the mapping. Our aim is to obtain fixed point in nonexpansive multivalued mappings using both function classes and function sequences satisfying the conditions such as uniform convergence of function sequences and shifting distance properties of functions.

1 Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ denote natural and real numbers, respectively. Let (X, d) be a metric space. We denote by $P(X)$ the family of all nonempty subsets of X and by $CB(X)$ the family of all nonempty closed bounded subsets of X. We define the Hausdorff metric H on $CB(X)$ by

$$H(A, B) := \max\left\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\right\},$$

for all $A, B \in CB(X)$, where for $x \in X$ and $C \subset X$, $D(x, C) := \inf\left\{d(x, y) : y \in C\right\}$ is the distance from the point x to the subset C.

Definition 1 ([20]). *A multivalued mapping $T : X \to CB(X)$ is said to be a contraction if there exists a constant $\lambda \in [0, 1)$ such that*

$$H(Tx, Ty) \leq \lambda d(x, y) \quad \text{for all} \quad x, y \in X.$$

A multivalued mapping $T : X \to CB(X)$ is said to be a nonexpansive if

$$H(Tx, Ty) \leq d(x, y) \quad \text{for all} \quad x, y \in X.$$

Definition 2 ([20]). *A point $x_0 \in X$ is called a fixed point of a multivalued mapping $T : X \to CB(X)$ if $x_0 \in Tx_0$.*

Theorem 1 ([20]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ satisfies*

$$H(Tx, Ty) \leq \lambda d(x, y) \quad \text{for all} \quad x, y \in X, \quad \text{where} \quad 0 \leq \lambda < 1.$$

Then T has a fixed point.

Lemma 1 ([20]). *Let (X, d) be a metric space and $A, B \subset CB(X)$.Then for each $a \in A$ and $\epsilon > 0$ there exists $b \in B$ such that*

$$d\left(a, b\right) \leq H\left(A, B\right) + \epsilon.$$

Definition 3 ([5]). *Let $\psi, \phi : [0, \infty) \to \mathbb{R}$ be two functions. The pair (ψ, ϕ) is said to be a pair of shifting distance functions, if the following conditions hold:*

(i) for $u, v \in [0, \infty)$ if $\psi(u) \leq \phi(v)$ then $u \leq v$,

(ii) for $\{u_k\}$, $\{v_k\} \subset [0, \infty)$ with $\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = w$, if $\psi(u_k) \leq \phi(v_k)$ for all $k \in \mathbb{N}$ then $w = 0$.

Definition 4 ([13]). *Let $\psi_n, \phi_n : [0,\infty) \to \mathbb{R}$ be two function sequences. The pair (ψ_n, ϕ_n) is said to be a pair of function sequences with shifting distance properties which satisfy the following conditions:*

(i) for $u, v \in [0,\infty)$ if $\psi_n(u) \to \psi(u)$ and $\phi_n(v) \to \phi(v)$ uniformly in n and also $\psi_n(u) \leq \phi_n(v)$, then $u \leq v$,

(ii) for $\{u_k\}, \{v_k\} \subset [0,\infty)$ with $\lim\limits_{k\to\infty} u_k = \lim\limits_{k\to\infty} v_k = w$, if $\psi_n(u_k) \to \psi(u_k)$, $\phi_n(v_k) \to \phi(v_k)$ uniformly in n and $\psi_n(u_k) \leq \phi_n(v_k)$ for all $k \in \mathbb{N}$, then $w = 0$.

Definition 5 ([13]). *The pair (ψ_n, ϕ_n) is said to be having shifting distance property if $(\psi_n, \phi_n) \to (\psi, \phi)$ uniformly in n and the pair (ψ, ϕ) is shifting distance functions.*

Lemma 2 ([13]). *Let $\psi_n, \phi_n : [0,\infty) \to \mathbb{R}$ be two function sequences. Assume that the following conditions hold:*

(i) if (ψ_n) upper semi-continuous function sequences and $\psi_n \leq \psi_{n+1}$, then $\psi_n \to \psi$ is uniform convergence according to n,

(ii) if (ϕ_n) lower semi-continuous function sequences and $\phi_n \geq \phi_{n+1}$, then $\phi_n \to \phi$ is uniform convergence according to n.

Then the pair (ψ_n, ϕ_n) is function sequences having shifting distance property.

2 Main Results

In this section, we discuss some properties of the multivalued mappings defined by both functions and function sequences.

Theorem 2. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued mapping. Suppose that there exists a pair of shifting distance functions (ψ, ϕ) such that*

$$\psi\big(H(Tx, Ty)\big) \leq \phi\big(d(x, y)\big) \quad \text{for all} \quad x, y \in X. \tag{1}$$

Then T has a fixed point in X.

Proof. Let $\alpha < 1$ and let $x_0 \in X$. Let us take $x_1 \in Tx_0$. Under the condition of Lemma 1, we can consider the iteration process as follows:

$$\begin{array}{ll}
\exists\, x_2 \in Tx_1 & d(x_1, x_2) \leq H(Tx_0, Tx_1) + \alpha, \\
\exists\, x_3 \in Tx_2 & d(x_2, x_3) \leq H(Tx_1, Tx_2) + \alpha^2, \\
\exists\, x_4 \in Tx_3 & d(x_3, x_4) \leq H(Tx_2, Tx_3) + \alpha^3, \\
\ldots & \ldots \\
\exists\, x_{k+1} \in Tx_k & d(x_k, x_{k+1}) \leq H(Tx_{k-1}, Tx_k) + \alpha^k
\end{array}$$

for all $k \geq 1$. Firstly, if we choose $x = x_k$ and $y = x_{k+1}$ in inequality (1), we get

$$\psi\big(H(Tx_k, Tx_{k+1})\big) \leq \phi\big(d(x_k, x_{k+1})\big)$$

for all $k \geq 1$. From the condition (i) of Definition 3, we can write

$$\psi(u_k) \leq \phi(v_k) \implies u_k \leq v_k, \tag{2}$$

where $u_k = H(Tx_k, Tx_{k+1})$ and $v_k = d(x_k, x_{k+1})$. If Lemma 1 and the inequality (2) are combined, we have

$$\begin{aligned} d(x_k, x_{k+1}) &\le H(Tx_{k-1}, Tx_k) + \alpha^k \le d(x_{k-1}, x_k) + \alpha^k, \\ d(x_k, x_{k+1}) &\le d(x_{k-1}, x_k) + \alpha^k. \end{aligned} \tag{3}$$

Considering the inequalities (3), let us do the following calculations

$$\begin{aligned} d(x_k, x_{k+1}) &\le d(x_{k-1}, x_k) + \alpha^k, \\ d(x_{k-1}, x_k) &\le d(x_{k-2}, x_{k-1}) + \alpha^{k-1}, \\ d(x_{k-2}, x_{k-1}) &\le d(x_{k-3}, x_{k-2}) + \alpha^{k-2}, \\ &\dots \\ d(x_1, x_2) &\le d(x_0, x_1) + \alpha. \end{aligned}$$

Hence, we have for all $k \in \mathbb{N}$

$$d(x_k, x_{k+1}) \le d(x_0, x_1) + \alpha + \alpha^2 + \cdots + \alpha^k \le d(x_0, x_1) + \alpha\left(\frac{1-\alpha^k}{1-\alpha}\right).$$

Since $1 - \alpha^k < 1$ for all $k \in \mathbb{N}$, we can write

$$d(x_k, x_{k+1}) \le d(x_0, x_1) + \frac{\alpha}{1-\alpha}.$$

Therefore, we infer that $\{v_k\}$ is a bounded sequence. According to the Bolzano-Weierstrass Theorem, $\{v_k\}$ has at least a convergent subsequence $\{v_{k_r}\}$ such that $\lim\limits_{r\to\infty} v_{k_r} = \ell$. That is, $\lim\limits_{r\to\infty} d(x_{k_r}, x_{k_{r+1}}) = \ell$. According to (3), it is easy to see that $\lim\limits_{r\to\infty} H(Tx_{k_r}, Tx_{k_{r+1}}) = \ell$. So, by condition (ii) of Definition 3, we have

$$\lim_{r\to\infty} d(x_{k_r}, x_{k_{r+1}}) = 0. \tag{4}$$

Let us prove that $\{x_{k_r}\}$ is a Cauchy sequence. Assume that $\{x_{k_r}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequences $\{x_{k_{r(p)}}\}$ and $\{x_{k_{m(p)}}\}$ of $\{x_{k_r}\}$ with $k_{r(p)} > k_{m(p)} > p$ such that for all $p \in \mathbb{N}$

$$d\left(x_{k_{r(p)}}, x_{k_{m(p)}}\right) \ge \varepsilon \quad \text{and} \quad d\left(x_{k_{r(p)-1}}, x_{k_{m(p)}}\right) < \varepsilon.$$

Under the condition above, we get

$$\varepsilon \le d\big(x_{k_{r(p)}}, x_{k_{m(p)}}\big) \le d\big(x_{k_{r(p)}}, x_{k_{r(p)-1}}\big) + d\big(x_{k_{r(p)-1}}, x_{k_{m(p)}}\big) \le \varepsilon + d\big(x_{k_{r(p)}}, x_{k_{r(p)-1}}\big) \tag{5}$$

for all $p \in \mathbb{N}$. If we apply limit to both sides for $p \to \infty$, we get

$$\lim_{p\to\infty} d\big(x_{k_{r(p)}}, x_{k_{m(p)}}\big) = \varepsilon. \tag{6}$$

Then, we have

$$\begin{aligned} \big|d\big(x_{k_{r(p)}}, x_{k_{m(p)}}\big) - d\big(x_{k_{r(p)}}, x_{k_{m(p)-1}}\big)\big| &\le d\big(x_{k_{m(p)}}, x_{k_{m(p)-1}}\big) \\ \big|d\big(x_{k_{r(p)}}, x_{k_{m(p)-1}}\big) - d\big(x_{k_{r(p)-1}}, x_{k_{m(p)-1}}\big)\big| &\le d\big(x_{k_{r(p)}}, x_{k_{r(p)-1}}\big). \end{aligned}$$

Considering together with (4) and (5), we have

$$\lim_{p\to\infty} d\left(x_{k_{r(p)-1}}, x_{k_{m(p)-1}}\right) = \varepsilon. \tag{7}$$

From Lemma 1, there exists $p_0 \in \mathbb{N}$ such that

$$0 < \frac{\varepsilon}{2} < d\left(x_{k_{r(p)}}, x_{k_{m(p)}}\right) \leq H\left(Tx_{k_{r(p)-1}}, Tx_{k_{m(p)-1}}\right) + \alpha^{k_{r(p)}} \leq d\left(x_{k_{r(p)-1}}, x_{k_{m(p)-1}}\right) + \alpha^{k_{r(p)}}$$

for all $p > p_0$. Letting $p \to \infty$ in the above inequality, we get

$$\lim_{p\to\infty} H\left(Tx_{k_{r(p)-1}}, Tx_{k_{m(p)-1}}\right) = \varepsilon, \tag{8}$$

where $\lim\limits_{p\to\infty} \alpha^{k_{r(p)}} = 0$ for $\alpha < 1$. By taking $x = x_{k_{r(p)-1}}$ and $y = x_{k_{m(p)-1}}$ in the inequality (1), we obtain

$$\psi\left(g_{k_r}\right) \leq \phi\left(h_{k_r}\right), \tag{9}$$

where $g_{k_r} = H\left(Tx_{k_{r(p)-1}}, Tx_{k_{m(p)-1}}\right)$ and $h_{k_r} = d\left(x_{k_{r(p)-1}}, x_{k_{m(p)-1}}\right)$. Therefore, if considering (9) together with (7) and (8), the condition (ii) of Definition 3, we get $\varepsilon = 0$. Hence, this is a contradiction. It follows that $\{x_{k_r}\}$ is a Cauchy sequence in X.

Since (X, d) is a complete metric space, then $\{x_{k_r}\}$ converges to a point $x^* \in X$, that is, $\lim\limits_{r\to\infty} d\left(x_{k_r}, x^*\right) = 0$. Taking $x = x_{k_r}$ and $y = x^*$ in inequality (1), we have

$$\psi\left(H(Tx_{k_r}, Tx^*)\right) \leq \phi(d(x_{k_r}, x^*)).$$

By using the condition (i) of Definition 3, we have

$$H(Tx_{k_r}, Tx^*) \leq d(x_{k_r}, x^*).$$

Since $x_{k_{r+1}} \in Tx_{k_r}$, we can write that

$$D(x_{k_{r+1}}, Tx^*) \leq d(x_{k_r}, x^*).$$

Passing to limit as $r \to \infty$, we obtain

$$D(x^*, Tx^*) \leq d(x^*, x^*).$$

We conclude that $D(x^*, Tx^*) = 0$, hence $x^* \in Tx^*$. Therefore, x^* is a fixed point of T. □

Theorem 3. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued mapping. Suppose that there exists a pair (ψ_n, ϕ_n) of function sequences having shifting distance property such that*

$$\psi_n\left(H(Tx, Ty)\right) \leq \phi_n(d(x, y)) \tag{10}$$

for all $x, y \in X$ and for all $n \in \mathbb{N}$, where $\psi_n, \phi_n : [0, \infty) \to \mathbb{R}$ are two function sequences. Then T has a fixed point in X.

Proof. We suppose that $\{x_k\}$ is a sequence such that $x_0 \in X$ and $x_{k+1} \in Tx_k$ for $k \geq 1$. Let (ψ_n, ϕ_n) be a pair of function sequences with shifting distance property satisfied conditions of Lemma 2. Let (ψ_n) be increasing function sequence bounded by ψ and (ϕ_n) be decreasing function sequence bounded by ϕ for all $n \in \mathbb{N}$. By using (1), we can write

$$\begin{aligned}\psi_1\big(H(Tx_k, Tx_{k+1})\big) &< \psi_2\big(H(Tx_k, Tx_{k+1})\big) < \cdots \\ &< \psi_n\big(H(Tx_k, Tx_{k+1})\big) < \psi\big(H(Tx_k, Tx_{k+1})\big) \\ &\leq \phi\big(d(x_k, x_{k+1})\big) < \phi_n\big(d(x_k, x_{k+1})\big) < \cdots \\ &< \phi_2\big(d(x_k, x_{k+1})\big) < \phi_1\big(d(x_k, x_{k+1})\big)\end{aligned}$$

for all $k, n \in \mathbb{N}$. It follows that

$$\psi_n\big(H(Tx_k, Tx_{k+1})\big) \leq \phi_n\big(d(x_k, x_{k+1})\big) \tag{11}$$

for all $k, n \in \mathbb{N}$. From the condition (i) of Definition 4, we can write

$$\psi_n(u_k) \leq \phi_n(v_k) \implies u_k \leq v_k,$$

where $u_k = H(Tx_k, Tx_{k+1})$ and $v_k = d(x_k, x_{k+1})$.

From Lemma 2, we know that $\psi_n \to \psi$ and $\phi_n \to \phi$ uniformly according to n. Hence, taking limit on both sides of (11) as $n \to \infty$, we get

$$\psi\big(H(Tx_k, Tx_{k+1})\big) \leq \phi\big(d(x_k, x_{k+1})\big) \tag{12}$$

for $k \geq 1$. Since the inequality (12) proved in Theorem 2, we have to show that the mapping T has a fixed point.

In (10), taking $x = x_{k_r}$ and $y = x^*$, we get

$$\psi_n(H\big(Tx_{k_r}, Tx^*)\big) \leq \phi_n\big(d(x_{k_r}, x^*)\big).$$

Letting $n \to \infty$ in the above inequality and by Lemma 2, we have

$$\psi\big(H(Tx_{k_r}, Tx^*)\big) \leq \phi\big(d(x_{k_r}, x^*)\big).$$

By using condition (i) of Definition 3 and taken limit as $r \to \infty$, it follows that

$$D(x^*, Tx^*) = 0.$$

Therefore, we obtain $x^* \in Tx^*$, so we achieves the desired result. □

Remark 1. *While we examine the contraction of multivalued mappings according to Nadler's definition, we have to show that there exists $\lambda \in [0, 1)$ to ensure the inequality*

$$H(Tx, Ty) \leq \lambda d(x, y).$$

However, if $\lambda = 1$, the mapping that provides this inequality is called the nonexpansive mapping. All the same, it is known that nonexpansive mappings do not have to have a fixed point. In this paper, the inequality $u \leq v$ obtained due to both (1) and (10) has a form of nonexpansive mapping in the Nadler's sense [20]. Therefore, the fixed point of mapping T has been obtained with the help of function sequences and functions having shifting distance property.

Example 1. *Let $T : X \to CB(X)$. Also, let us give two function sequences according to the conditions of Definition 4*

$$\psi_n(u) = \frac{2n(1+u) + 3\,(2u+1)}{n+3}, \quad \phi_n(v) = \frac{n^2(1+v) + n^2 + 1}{n^2}.$$

It is clear that $\psi_n(u) \le \phi_n(v)$ for all $n \in \mathbb{N}$ and $u, v \in [0, \infty)$. Besides, the pair $(\psi_n, \phi_n) \to (\psi, \phi)$ are shifting distance functions. After that, it can be seen that

$$\lim_{n\to\infty} \frac{2n(1+u) + 3\,(2u+1)}{n+3} = 2 + 2u \le 2 + v = \lim_{n\to\infty} \frac{n^2(2+v) + 1}{n^2}.$$

Therefore, the pair (ψ, ϕ) is shifting distance functions.

Now, we suppose that $u = H(Tx, Ty)$ and $v = d(x, y)$. Since

$$\frac{2n\big(1 + H(Tx, Ty)\big) + 3\big(2H(Tx, Ty) + 1\big)}{n+3} \le \frac{n^2\big(2 + d(x, y)\big) + 1}{n^2},$$

we have

$$2H(Tx, Ty) - d(x, y) \le \frac{3n^2 + n + 3}{n^2(n+3)}. \tag{13}$$

If limit goes to infinity in (13), we obtain

$$2H(Tx, Ty) - d(x, y) \le 2H(Tx, Ty) \le d(x, y)H(Tx, Ty) \le \frac{1}{2}d(x, y).$$

As a result, according to condition of Nadler's fixed point theorem [20], T has a fixed point under continuous function sequences.

Let (I_n) be a unit function sequence. In Theorem 3, if we take $(\psi_n) = (I_n)$ such that $\lim_{n\to\infty} I_n = I$ uniformly, we obtain the following result.

Corollary 1. *Let (X, d) be a complete metric space. Suppose that $T : X \to CB(X)$ is a multi-valued mapping such that*

$$I_n\big(H(Tx, Ty)\big) \le \phi_n\big(d(x, y)\big),$$

for all $x, y \in X$ and $n \in \mathbb{N}$, where $\phi_n : [0, \infty) \to \mathbb{R}$ is a function sequence such that

(a) for $u, v \in [0, \infty)$ if $I_n\,(u) \le \phi_n(v)$, then $u \le v$,

(b) for $\{u_k\}, \{v_k\} \subset [0, \infty)$ with $\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = w$ if $I_n\,(u_k) \le \phi_n(v_k)$ for all $n, k \in \mathbb{N}$, then $w = 0$.

Then T has a fixed point in X.

Theorem 4. *Let (X, d) be a complete metric space. Suppose that $T : X \to CB(X)$ is a continuous mapping such that*

$$\psi_n\big(H(Tx, Ty)\big) \le \psi_n\big(d(x, y)\big) - \phi_n\big(d(x, y)\big) \tag{14}$$

for all $x, y \in X$ and $n \in \mathbb{N}$, where $\psi_n, \phi_n : [0, \infty) \to \mathbb{R}^+$ is a pair having shifting distance property. Also, let ψ, ϕ be two nondecreasing and continuous functions satisfying $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then T has a fixed point in X.

Proof. Suppose that (14) holds. In this inequality, if we take $x = x_k$ and $y = x_{k+1}$, we get

$$\psi_n\big(H(Tx_k, Tx_{k+1})\big) \leq \psi_n\big(d(x_k, x_{k+1})\big) - \phi_n\big(d(x_k, x_{k+1})\big) \quad \text{for all} \quad k, n \in \mathbb{N}.$$

By Lemma 2, taking limit as $n \to \infty$, we have

$$\psi\big(H(Tx_k, Tx_{k+1})\big) \leq \psi\big(d(x_k, x_{k+1})\big) - \phi\big(d(x_k, x_{k+1})\big). \tag{15}$$

Besides, by using hypothesis in statement of Theorem 4, we assume that

$$\psi\big(d(x_k, x_{k+1})\big) = \phi\big(d(x_k, x_{k+1})\big).$$

If the above equation is substituted in inequality (15), it is clear that $\psi\big(H(Tx_k, Tx_{k+1})\big) = 0$, hence

$$H(Tx_k, Tx_{k+1}) = 0.$$

Since $x_{k+1} \in Tx_k$, we conclude that $D\big(x_{k+1}, Tx_{k+1}\big) = 0$. Thus, $x_{k+1} \in Tx_{k+1}$.

Conversely, let $d(x_k, x_{k+1}) = 0$. If we evaluate together with the inequality (15), then we get $H(Tx_k, Tx_{k+1}) = 0$ and $x_{k+1} \in Tx_{k+1}$. As a result, we have found that T has a fixed point. ☐

In Theorem 3, if we take $(\psi_n) = (I_n)$ and $(\phi_n) = \lambda\,(I_n)$ for $\lambda \in [0, 1)$, such that $I_n \to I$ uniformly according to n, we obtain the following corollary, known as Nadler's fixed point theorem [20].

Corollary 2 ([20]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ satisfies*

$$H(Tx, Ty) \leq \lambda d(x, y) \quad \textit{for all} \quad x, y \in X,$$

where $0 \leq \lambda < 1$.

Then T has a fixed point.

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Каракая В., Секман Д. *Про нерухомі точки деяких багатозначних відображень у певних функціональних класах* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 128–136.

Добре відомо, що з теореми Банаха про нерухому точку випливає існування нерухомих точок однозначних відображень. З іншого боку, С.Б. Наддер довів теорему, що гарантує існування нерухомої точки для багатозначного відображення. Однак слід відзначити, що подібні методи не застосовні для нерозширюючих багатозначних відображень. Метою цієї статті є дослідження існування нерухомої точки нерозширюючих багатозначних відображень за допомогою функціональних послідовностей та функцій, що мають властивість зсувної відстані. Додатково деякі гіпотези цієї роботи були роз'яснені на цікавому прикладі.

Ключові слова і фрази: нерухома точка, багатозначне відображення, функція зсувної відстані, функціональна послідовність.