



On unconditionally convergent series in topological rings

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We define a topological ring R to be *Hirsch*, if for any unconditionally convergent series $\sum_{n \in \omega} x_n$ in R and any neighborhood U of the additive identity 0 of R there exists a neighborhood $V \subseteq R$ of 0 such that $\sum_{n \in F} a_n x_n \in U$ for any finite set $F \subset \omega$ and any sequence $(a_n)_{n \in F} \in V^F$. We recognize Hirsch rings in certain known classes of topological rings. For this purpose we introduce and develop the technique of seminorms on actogroups. We prove, in particular, that a topological ring R is Hirsch provided R is locally compact or R has a base at the zero consisting of open ideals or R is a closed subring of the Banach ring $C(K)$, where K is a compact Hausdorff space. This implies that the Banach ring ℓ_∞ and its subrings c_0 and c are Hirsch. Applying a recent result of Banakh and Kadets, we prove that for a real number $p \geq 1$ the commutative Banach ring ℓ_p is Hirsch if and only if $p \leq 2$. Also for any $p \in (1, \infty)$, the (noncommutative) Banach ring $L(\ell_p)$ of continuous endomorphisms of the Banach ring ℓ_p is not Hirsch.

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Introduction

We define a topological ring R to be *Hirsch* if for any unconditionally convergent series $\sum_{n \in \omega} x_n$ in R and any neighborhood $U \subseteq R$ of the additive identity 0 of R there exists a neighborhood $V \subseteq R$ of 0 such that $\sum_{n \in F} a_n x_n \in U$ for any finite set $F \subset \omega$ and any sequence $(a_n)_{n \in F} \in V^F$.

This notion was suggested by a question [8] of Dylan Hirsch who was motivated by the idea of generalizing the standard theory of Lebesgue measure and integral to functions with values in topological rings.

In this paper we consider the following general problem.

Problem 1 (see [8]). *Recognize Hirsch rings in known classes of topological rings.*

Exploring Problem 1, we shall introduce and develop the technique of seminorms on actogroups, see Section 3. We shall show, in particular, that a topological ring R is Hirsch provided R has a base at the zero consisting of open ideals (see Proposition 3) or R is locally compact (see Theorem 9) or R is a subring of the topological rings $C_{\mathcal{K}}(X, S)$ of continuous functions from a topological space X to a locally compact topological ring S , endowed with the topology of uniform convergence on sets from an ideal \mathcal{K} of compact Hausdorff subsets

of X (see Theorem 10). This implies that the Banach ring ℓ_∞ and its subrings c_0 and c are Hirsch, see Theorem 15. We prove also that for a number $p \in [1, \infty]$ the commutative Banach ring ℓ_p is Hirsch if and only if $p \in [1, 2] \cup \{\infty\}$, see Corollary 7. Also for any $p \in (1, \infty)$, the (noncommutative) Banach ring $L(\ell_p)$ of continuous endomorphisms of the Banach ring ℓ_p is not Hirsch, see Propositions 6 and 7.

1 Unconditional convergence in topological groups

A *group* in this paper means an Abelian topological group. We shall use the additive notation for denoting the group operations. For a group G its identity element is denoted by 0_G (so, $x + 0_G = x = 0_G + x$ for every $x \in G$).

Let I be a countable infinite set and $(x_i)_{i \in I}$ be a sequence of elements of a group G . The sequence $(x_i)_{i \in I}$

- *converges* to an element $x \in G$ if for any neighborhood U of the identity of G there exists a finite set $F \subset I$ such that $x_i \in x + U$ for any $i \in I \setminus F$;
- is *Cauchy* if for any neighborhood U of the identity of G there exists a finite set $F \subset I$ such that $x_i - x_j \in U$ for any $i, j \in I \setminus F$.

It is easy to show that any convergent sequence is Cauchy. A group G is *sequentially complete* if any Cauchy sequence in G converges.

The series $\sum_{i \in I} x_i$

- *unconditionally converges* to an element $x \in G$ if for any neighborhood U of the identity of G there exists a finite set $F \subset I$ such that $\sum_{i \in J} x_i \in x + U$ for any finite set $J \subset I$ containing F (if the set J is empty then we put $\sum_{i \in J} x_i = 0$);
- is *unconditionally Cauchy* if for any neighborhood U of the identity of G there exists a finite set $F \subset I$ such that $\sum_{i \in K} x_i \in U$ for any finite set $K \subset I \setminus F$.

It is easy to show that any unconditionally convergent series is unconditionally Cauchy. On the other hand we have the following assertion.

Proposition 1. *Let $\sum_{i \in I} x_i$ be an unconditionally Cauchy series in a group G and $(I_n)_{n \in \omega}$ be a nondecreasing sequence of finite sets such that $\bigcup_{n \in \omega} I_n = I$. If the sequence $(\sum_{i \in I_n} x_i)_{n \in \omega}$ converges to a point $x \in G$ then the series $\sum_{i \in I} x_i$ unconditionally converges to x .*

Proof. Let U be any neighborhood of the identity of G . Pick a neighborhood V of the identity of G such that $V + V \subseteq U$. Since the sequence $(\sum_{i \in I_n} x_i)_{n \in \omega}$ converges to x , there exists $m \in \omega$, such that $\sum_{i \in I_n} x_i \in x + V$ for any $n \geq m$. Since the series $\sum_{i \in I} x_i$ is unconditionally Cauchy, there exists a finite set $J \subset I$ such that $\sum_{i \in K} x_i \in V$ for any finite set $K \subset I \setminus J$. Since the sequence $(I_n)_{n \in \omega}$ is nondecreasing and $\bigcup_{n \in \omega} I_n = I$, there exists a number $k \geq m$ such that $I_k \supseteq J$. Let $F \subset I$ be any finite set containing I_k . Then

$$\sum_{i \in F} x_i = \sum_{i \in I_k} x_i + \sum_{i \in F \setminus I_k} x_i \in x + V + V \subseteq x + U.$$

□

Proposition 2. Any unconditionally Cauchy series $\sum_{i \in I} x_i$ in a sequentially complete group G is unconditionally convergent.

Proof. Fix any increasing sequence $(I_n)_{n \in \omega}$ of finite subsets of I such that $\bigcup_{n \in \omega} I_n = I$. Let U be any neighborhood of the identity of G such that $U = -U$. Since the series $\sum_{i \in I} x_i$ is unconditionally Cauchy, there exists a finite set $F \subset I$ such that $\sum_{i \in K} x_i \in U$ for any finite set $K \subset I \setminus F$. Since the sequence $(I_n)_{n \in \omega}$ is increasing and $\bigcup_{n \in \omega} I_n = I$, there exists $k \in \omega$ such that $I_k \supseteq F$. Let $m, n \in \omega$ be any numbers with $k \leq m \leq n$. Since $I_n \setminus I_m \subset I \setminus F$, we have

$$\sum_{i \in I_n} x_i - \sum_{i \in I_m} x_i = \sum_{i \in I_n \setminus I_m} x_i \in U.$$

Thus the sequence $(\sum_{i \in I_n} x_i)_{n \in \omega}$ is Cauchy. Since the group G is sequentially complete, the sequence $(\sum_{i \in I_n} x_i)_{n \in \omega}$ converges to a point $a \in G$. By Proposition 1, the series $\sum_{i \in I} x_i$ unconditionally converges to x . □

An endomorphism of a group G is a continuous homomorphism from G to itself.

Lemma 1. Let $\sum_{i \in I} x_i$ be an unconditionally Cauchy series in a group G and $(v_i)_{i \in I}$ be a sequence of endomorphisms of G such that the set $D = \{v_i\}_{i \in I}$ is finite. Then the series $\sum_{i \in I} v_i(x_i)$ is unconditionally Cauchy.

Proof. Let U be any neighborhood of the identity of G . Pick a neighborhood V of the identity of G such that the sum of $|D|$ summands V is contained in U . Let $v \in D$ be any element. Put $I_v = \{i \in I : v_i = v\}$ and pick a neighborhood W_v of the identity of G such that $v(W_v) \subseteq V$. Put $W = \bigcap_{v \in D} W_v$. Since the series $\sum_{i \in I} x_i$ is unconditionally Cauchy, there exists a finite set $F \subset I$ such that $\sum_{i \in K} x_i \in W$ for any finite set $K \subset I \setminus F$. Then for any finite set $K \subset I \setminus F$ we have

$$\sum_{i \in K} v_i(x_i) = \sum_{v \in D} \sum_{i \in K \cap I_v} v(x_i) = \sum_{v \in D} v\left(\sum_{i \in K \cap I_v} x_i\right) \in \sum_{v \in D} v(W) \subseteq \sum_{v \in D} v(W_v) \subseteq \sum_{v \in D} V \subseteq U,$$

witnessing that the series $\sum_{i \in I} v_i(x_i)$ is unconditionally Cauchy. □

2 Topological rings that have a local base of open ideals

A topological ring is a group R endowed with a continuous associative binary operation $\cdot : R \times R \rightarrow R$ which is distributive, that is

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz$$

for all $x, y, z \in R$. A nonempty subset J of a ring R is called an ideal in R if

$$J + J \subseteq J \quad \text{and} \quad JR \cup RJ \subseteq J.$$

Proposition 3. Every topological ring R that has a neighborhood base at zero consisting of open ideals is Hirsch.

Proof. Let $\sum_{i \in \omega} x_i$ be any (unconditionally) convergent series in R and U be any neighborhood of the zero 0 of R . Let J be an arbitrary open ideal contained in U . Then for every finite set $I \subset \omega$ we have $\sum_{i \in I} Jx_i \subseteq J \subseteq U$ as J is an ideal in R . □

3 Hirsch actogroups

It is natural to explore Problem 1 in a wider context of active groups, i.e., groups endowed with an action of a topological space with zero.

Definition 1. An active group (briefly an actogroup) $A \curvearrowright G$ is a topological group G endowed with a continuous action $A \times G \rightarrow G$, $(a, x) \mapsto ax$, of a topological space A with a distinguished point $0_A \in A$. The action should satisfy two axioms:

- (i) $0_A \cdot x = 0_G$ for every $x \in G$;
- (ii) $a(x + x) = ax + ax$ for every $a \in A$ and $x \in G$.

The topological space A is called the acting space of the actogroup $A \curvearrowright G$ and the distinguished point 0_A is called the zero of A . If the acting space A and the action $A \times G \rightarrow G$ are clear from the context, then we shall write G instead of $A \curvearrowright G$.

Example 1. Each topological ring R is an actogroup $R \curvearrowright R$ with respect to the continuous action $R \times R \rightarrow R$, $(a, x) \mapsto ax$, assigning to a pair of elements $a, x \in R$ their product ax in the ring R .

Definition 2. An actogroup $A \curvearrowright G$ is defined to be Hirsch if for every unconditionally convergent series $\sum_{i \in \omega} x_i$ in G and every neighborhood $U \subseteq G$ of 0_G there exists a neighborhood V of zero 0_A in the acting space A of G such that for every finite set $F \subseteq \omega$ and sequence $(a_i)_{i \in F} \in V^F$ we have $\sum_{i \in F} a_i x_i \in U$.

It follows that a topological ring is Hirsch if and only if it is Hirsch as an actogroup. So, Problem 1 is a partial case of the following more general one.

Problem 2. Recognize Hirsch actogroups in known classes of actogroups.

In the following subsections we shall present examples of Hirsch actogroups among actogroups whose topology is generated by some specific seminorms.

3.1 Seminorms on groups

Definition 3. A seminorm on a (topological) group G is a (continuous) function $\|\cdot\| : G \rightarrow [0, \infty)$ such that

$$\|0_G\| = 0, \quad \|-x\| = \|x\| \quad \text{and} \quad \|x + y\| \leq \|x\| + \|y\|$$

for every $x, y \in G$.

A seminorm $\|\cdot\| : G \rightarrow [0, \infty)$ is a norm if

- $\|x\| > 0$ for any $x \in G \setminus \{0_G\}$ and
- $\|nx\| = n\|x\|$ for any $x \in G$ and $n \in \omega$.

Here nx is the element of G , defined by the recursive formula:

$$0x = 0_G \quad \text{and} \quad (n + 1)x = nx + x \quad \text{for every } n \in \omega.$$

We say that the topology of a group G is *generated by a family* \mathcal{S} of seminorms if for every neighborhood $U \subseteq G$ of 0_G there exists a seminorm $\|\cdot\| \in \mathcal{S}$ such that $\{x \in G : \|x\| < \varepsilon\} \subseteq U$ for some $\varepsilon > 0$ (by the definition, all seminorms on a group are continuous). If the family \mathcal{S} consists of a single seminorm $\|\cdot\|$, then we say that the topology of G is *generated by the seminorm* $\|\cdot\|$.

A seminorm $\|\cdot\| : G \rightarrow [0, \infty)$ is said to be *bounded* if $\sup_{x \in G} \|x\| < \infty$. Theorem 3.3.9 from [1] implies that the topology of any group is generated by a suitable family of bounded seminorms.

3.2 Locally 2-homogeneous seminorms on groups

Definition 4. A seminorm $\|\cdot\|$ on a group G is called

- 2-homogeneous if $\|2x\| = 2\|x\|$ for every $x \in G$;
- locally 2-homogeneous if there exists $\varepsilon > 0$ such that $\|2x\| = 2\|x\|$ for every $x \in G$ with $\|x\| \leq \varepsilon$.

Each 2-homogeneous seminorm on a group is locally 2-homogeneous.

Example 2. Each norm on a group is 2-homogeneous. In particular, the norm $\|\cdot\| : \mathbb{R} \rightarrow [0, \infty)$, $\|\cdot\| : x \mapsto |x|$, on the additive group of real numbers \mathbb{R} is 2-homogeneous.

Next we explore some operations over (locally) 2-homogeneous seminorms. The following two lemmas can be easily derived from the definitions.

Lemma 2. Let $h : H \rightarrow G$ be a continuous homomorphism between groups. If $\|\cdot\|_G$ is a (locally) 2-homogeneous seminorm on G , then the seminorm

$$\|\cdot\|_H : H \rightarrow [0, \infty), \quad \|\cdot\|_H : x \mapsto \|h(x)\|_G,$$

on the group H is (locally) 2-homogeneous.

We recall that for a group G and a subgroup $H \subseteq G$ the quotient group G/H is the group whose elements are cosets $x + H$ where $x \in G$.

Lemma 3. Let $\|\cdot\|_G$ and $\|\cdot\|_H$ be seminorms on groups G and H , respectively. If the seminorms $\|\cdot\|_G$ and $\|\cdot\|_H$ are (locally) 2-homogeneous, then so is the seminorm $\|\cdot\| : G \times H \rightarrow [0, \infty)$, $\|\cdot\| : (x, y) \mapsto \max\{\|x\|_G, \|y\|_H\}$.

Lemma 4. Let $\|\cdot\|_G$ be a locally 2-homogeneous seminorm on a group G and H be a subgroup of G such that $\{x \in H : \|x\|_G < 5\varepsilon\} = \{0_H\}$ for some $\varepsilon > 0$. Then the seminorm

$$\|\cdot\| : G/H \rightarrow [0, \infty), \quad \|y\| = \inf\{\|x\|_G : x \in y\}$$

is locally 2-homogeneous.

Proof. Replacing $\varepsilon > 0$ by a smaller positive number, we can assume that $\|2x\|_G = 2\|x\|_G$ for all $x \in G$ with $\|x\|_G < \frac{3}{2}\varepsilon$. We claim that for every $y \in G/H$ with $\|y\| \leq \varepsilon$ we have $\|2y\| = 2\|y\|$. Indeed, assume for a contradiction that $\|2y\| \neq 2\|y\|$. Since $\|2y\| \leq \|y\| + \|y\| = 2\|y\|$, we have $\|2y\| < 2\|y\|$, so there exists $z \in 2y$ such that $\|z\|_G < 2\|y\| \leq 2\varepsilon$. Since $\|y\| \leq \varepsilon$, there

exists $x \in y$ such that $\|x\|_G < \frac{3}{2}\varepsilon$ and then $\|2x\|_G = 2\|x\|_G < 3\varepsilon$. It follows from $z, 2x \in 2y$ that $z - 2x \in H$. The triangle inequality implies that

$$\|2x - z\|_G \leq \|2x\|_G + \|z\|_G < 3\varepsilon + 2\varepsilon = 5\varepsilon$$

and hence $2x - z \in \{h \in H : \|h\| < 5\varepsilon\} = \{0_H\}$ and finally $z = 2x$. Then

$$2\|y\| \leq 2\|x\|_G = \|2x\|_G = \|z\|_G < 2\|y\|,$$

which is a required contradiction. □

Corollary 1. *The seminorm $\|\cdot\| : \mathbb{R}/\mathbb{Z} \rightarrow [0, \infty)$, $\|\cdot\| : y \mapsto \min\{|x| : x \in y\}$, on the quotient group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is locally 2-homogeneous.*

Lemma 5. *Let H be an open subgroup of a group G . If a seminorm $\|\cdot\|_H : H \rightarrow [0, \infty)$ on the group H is locally 2-homogeneous, then the seminorm*

$$\|\cdot\|_G : G \rightarrow [0, \infty), \quad \|\cdot\|_G \mapsto \begin{cases} \min\{1, \|x\|_H\}, & \text{if } x \in H, \\ 1, & \text{if } x \in G \setminus H \end{cases}$$

on the group G is locally 2-homogeneous, too.

Proof. Since $\|\cdot\|_H$ is locally 2-homogeneous, there exists a positive $\varepsilon < \frac{1}{2}$ such that $\|2x\|_H = 2\|x\|_H$ for every $x \in H$ with $\|x\|_H \leq \varepsilon$. The definition of the seminorm $\|\cdot\|_G$ ensures that for every $x \in G$ with $\|x\|_G \leq \varepsilon$ we will have $x \in H$ and hence $1 > \varepsilon \geq \|x\|_G = \min\{1, \|x\|_H\}$ implies that $\|x\|_H = \|x\|_G$ and $\|2x\|_H \leq 2\|x\|_H \leq 2\varepsilon < 1$. Then

$$\|2x\|_G = \min\{1, \|2x\|_H\} = \|2x\|_H = 2\|x\|_H = \min\{2, 2\|x\|_H\} = 2 \min\{1, \|x\|_H\} = 2\|x\|_G,$$

so the seminorm $\|\cdot\|_G$ is locally 2-homogeneous. □

3.3 Lipschitz and contracting seminorms on actogroups

By a *seminorm* on an actogroup $A \curvearrowright G$ we understand a seminorm on the group G .

Definition 5. *A seminorm $\|\cdot\| : G \rightarrow [0, \infty)$ on an actogroup $A \curvearrowright G$ is called*

- *Lipschitz if there exist positive real numbers L, ε and a neighborhood $V \subseteq A$ of 0_A such that $\|ax\| \leq L\|x\|$ for every $a \in V$ and every $x \in G$ with $\|x\| \leq \varepsilon$;*
- *contracting if for every $\lambda > 0$ there exist $\varepsilon > 0$ and a neighborhood $V \subseteq A$ of 0_A such that $\|ax\| \leq \lambda\|x\|$ for every $a \in V$ and every $x \in G$ with $\|x\| \leq \varepsilon$.*

It is clear that each contracting seminorm on an actogroup is Lipschitz.

Theorem 1. Let $\|\cdot\| : G \rightarrow [0, \infty)$ be a locally 2-homogeneous seminorm on an actogroup $A \curvearrowright G$ and assume that there exists $\varepsilon > 0$ such that the closed ε -ball $B = \{x \in G : \|x\| \leq \varepsilon\}$ is compact. Then the seminorm $\|\cdot\|$ is contracting.

Proof. Since $\|\cdot\|$ is locally 2-homogeneous, we can replace ε by a smaller positive number and assume that $\varepsilon \leq 1$ and $\|2x\| = 2\|x\|$ for every $x \in B$. To prove that $\|\cdot\|$ is contracting, take any positive real number $\lambda \leq 1$.

The continuity of the action $A \times G \rightarrow G, (a, x) \mapsto ax$, implies that $W = \{(a, x) \in A \times B : \|ax\| < \frac{1}{4}\lambda\varepsilon\}$ is an open neighborhood of the compact set $\{0_A\} \times B$ in $A \times B$. Using the compactness of B , we can find a neighborhood $V \subseteq A$ of 0_A such that $V \times B \subseteq W$ and hence $\|ax\| < \frac{1}{4}\lambda\varepsilon$ for every $a \in V$ and $x \in B$.

To see that $\|\cdot\|$ is contracting, it remains to show that $\|ax\| \leq \lambda\|x\|$ for every $a \in V$ and $x \in B$. Suppose for a contradiction that there exist $a \in V$ and $x \in B$ such that $\|ax\| > \lambda\|x\|$.

It follows from $x \in B$ that $\|x\| \leq \varepsilon$ and hence $0 \leq \lambda\|x\| < \|ax\| < \frac{1}{4}\lambda\varepsilon \leq \varepsilon$. Then there exists a unique number $k \in \omega$ such that $2^k\|ax\| \leq \varepsilon < 2^{k+1}\|ax\|$.

Claim 1. For every nonnegative integer $i \leq k + 1$ we have $\|a(2^i x)\| = 2^i\|ax\|$.

Proof. We prove the claim by induction on i . For $i = 0$ the equality $\|a(2^0 x)\| = 2^0\|ax\|$ is trivial. Assume that $\|a(2^i x)\| = 2^i\|ax\|$ for some $i < k + 1$. Then

$$\|a(2^i x)\| = 2^i\|ax\| \leq 2^k\|ax\| \leq \varepsilon$$

and hence

$$\|a(2^{i+1} x)\| = \|a(2^i x + 2^i x)\| = \|a(2^i x) + a(2^i x)\| = 2\|a(2^i x)\| = 2^{i+1}\|ax\|$$

by the inductive hypothesis. □

Claim 2. $\|x\| > 0$.

Proof. Assume on contrary that $\|x\| = 0$. Claim 1 and the choice of k imply $\|a(2^{k+1} x)\| = 2^{k+1}\|ax\| > \varepsilon$. On the other hand, $\|x\| = 0$ implies $\|2^{k+1} x\| = 0$ and hence $2^{k+1} x \in B$ and $\|a(2^{k+1} x)\| < \frac{1}{4}\lambda\varepsilon \leq \varepsilon$ by the choice of the neighborhood $V \ni a$. This contradiction shows that $\|x\| > 0$. □

Since $0 < \|x\| \leq \varepsilon$, there exists a unique number $n \in \omega$ such that $2^n\|x\| \leq \varepsilon < 2^{n+1}\|x\|$. By analogy with Claim 1 we can prove that $\|2^i x\| = 2^i\|x\|$ for all nonnegative integers $i \leq n + 1$.

Since $\lambda \leq 1$, there exists a unique number $d \in \omega$ such that $2^d\lambda \leq 1 < 2^{d+1}\lambda$.

Observe that for every $i \leq n$ we have $\|2^i x\| \leq 2^i\|x\| \leq 2^n\|x\| \leq \varepsilon$ and hence $2^i x \in B$ and $\|a(2^i x)\| < \frac{1}{4}\lambda\varepsilon \leq \varepsilon < 2^{k+1}\|ax\| = \|a(2^{k+1} x)\|$ by the choice of the neighborhood $V \ni a$ and the number k . Also for every integer $i \leq d + 2$, by the triangle inequality, $\|a(2^{n+i} x)\| \leq 2^i\|a(2^n x)\| < 2^{d+2}\frac{1}{4}\lambda\varepsilon \leq \varepsilon < 2^{k+1}\|ax\| = \|a(2^{k+1} x)\|$. Therefore, for every $i \leq n + d + 2$ we have $\|a(2^i x)\| \leq \varepsilon < \|a(2^{k+1} x)\|$, which implies that $n + d + 2 < k + 1$. Then Claim 1 and the choice of k and n ensure that

$$2^{n+d+2}\|ax\| = \|a(2^{n+d+2} x)\| \leq \varepsilon < 2^{n+1}\|x\|$$

and finally $\|ax\| < \frac{1}{2^{d+1}}\|x\| < \lambda\|x\|$, which contradicts our assumption. □

3.4 Absolute seminorms

Definition 6. A seminorm $\|\cdot\|$ on a group G is defined to be

- C -absolute for some positive real number C if there exists $\varepsilon > 0$ such that for any finite set F and sequence $(x_i)_{i \in F} \in G^F$ with $M := \max_{E \subseteq F} \|\sum_{i \in E} x_i\| \leq \varepsilon$ we have $\sum_{i \in F} \|x_i\| \leq CM$;
- absolute if $\|\cdot\|$ is C -absolute for some positive real number C .

Example 3. The norm $|\cdot| : \mathbb{R} \rightarrow [0, \infty)$, $|\cdot| : x \mapsto |x|$, on the additive group \mathbb{R} of real numbers is 2-absolute.

Proof. Given any finite set F and sequence $(x_i)_{i \in F} \in \mathbb{R}^F$, consider the sets $F_+ = \{i \in F : x_i \geq 0\}$ and $F_- = F \setminus F_+$. It follows from

$$\sum_{i \in F} |x_i| = \sum_{i \in F_+} x_i - \sum_{i \in F_-} x_i = \left| \sum_{i \in F_+} x_i \right| + \left| \sum_{i \in F_-} x_i \right|$$

that

$$\sum_{i \in F} |x_i| \leq 2 \max \left\{ \left| \sum_{i \in F_+} x_i \right|, \left| \sum_{i \in F_-} x_i \right| \right\} \leq 2 \max_{E \subseteq F} \left| \sum_{i \in E} x_i \right|,$$

witnessing that the norm $|\cdot|$ on \mathbb{R} is 2-absolute. □

A result of I. Netuka and J. Veselý [10] implies the following general fact.

Example 4. For every positive integer n , the standard Euclidean norm $\|\cdot\| : \mathbb{R}^n \rightarrow [0, \infty)$ on the additive group \mathbb{R}^n is C -absolute for the constant

$$C = 2\sqrt{\pi} \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})},$$

which is the best possible. Here $\Gamma(z)$ is the well-known gamma function, see [4, §2.1].

As partial cases of Example 4 we have two next examples.

Example 5. The norm $\|\cdot\| : \mathbb{C} \rightarrow [0, \infty)$, $\|\cdot\| : z \mapsto |z|$, on the additive group of complex numbers \mathbb{C} is π -absolute.

Example 6. The norm $\|\cdot\| : \mathbb{H} \rightarrow [0, \infty)$, $\|\cdot\| : q \mapsto |q|$, on the additive group of quaternion numbers \mathbb{H} is 3π -absolute.

Lemma 6. Let G, H be two groups and $\|\cdot\|_G, \|\cdot\|_H$ be seminorms on G, H , respectively. If for some positive real number C these seminorms are C -absolute, then the norm

$$\|\cdot\| : G \times H \rightarrow [0, \infty), \quad \|\cdot\| : (g, h) \mapsto \max\{\|g\|_G, \|h\|_H\},$$

is $2C$ -absolute.

Proof. Let F be a finite set and $(z_i)_{i \in F}$ be a sequence in $G \times H$. Write each z_i as (x_i, y_i) for some $x_i \in G$ and $y_i \in H$. For the sets $I = \{i \in F : \|x_i\|_G \leq \|y_i\|_H\}$ and $J = \{i \in F : \|x_i\|_G > \|y_i\|_H\}$ we have

$$\sum_{i \in F} \|z_i\| = \sum_{i \in F} \max\{\|x_i\|_G, \|y_i\|_H\} = \sum_{i \in J} \|x_i\|_G + \sum_{i \in I} \|y_i\|_H$$

and hence

$$\max \left\{ \sum_{i \in J} \|x_i\|_G, \sum_{i \in I} \|y_i\|_H \right\} \geq \frac{1}{2} \sum_{i \in F} \|z_i\|.$$

Since the norms $\|\cdot\|_G, \|\cdot\|_H$ are C -absolute, there are subsets $I' \subseteq I$ and $J' \subseteq J$ such that

$$\left\| \sum_{i \in I'} x_i \right\|_G \geq \frac{1}{C} \sum_{i \in I} \|x_i\|_G \quad \text{and} \quad \left\| \sum_{i \in J'} y_i \right\|_H \geq \frac{1}{C} \sum_{i \in J} \|y_i\|_H.$$

Then

$$\max_{E \subseteq F} \left\| \sum_{i \in E} z_i \right\| \geq \max \left\{ \left\| \sum_{i \in I'} x_i \right\|_G, \left\| \sum_{i \in J'} y_i \right\|_H \right\} \geq \frac{1}{C} \max \left\{ \sum_{i \in I} \|x_i\|_G, \sum_{i \in J} \|y_i\|_H \right\} \geq \frac{1}{2C} \sum_{i \in F} \|z_i\|,$$

witnessing that the norm $\|\cdot\|$ on $G \times H$ is $2C$ -absolute. \square

Lemma 6 implies the following result.

Corollary 2. *Let G, H be two groups and $\|\cdot\|_G, \|\cdot\|_H$ be seminorms on G, H , respectively. If the seminorms $\|\cdot\|_G, \|\cdot\|_H$ are absolute, then so is the seminorm*

$$\|\cdot\| : G \times H \rightarrow [0, \infty), \quad \|\cdot\| : (g, h) \mapsto \max\{\|g\|_G, \|h\|_H\}.$$

Lemma 7. *Let C be a positive real number, $\|\cdot\|_G : G \rightarrow [0, \infty)$ be a C -absolute seminorm on a group G , and H be a subgroup of G such that $\{x \in H : \|x\|_G < 5\varepsilon\} = \{0_H\}$ for some $\varepsilon > 0$. Then the seminorm*

$$\|\cdot\| : G/H \rightarrow [0, \infty), \quad \|\cdot\| : y \mapsto \inf\{\|x\|_G : x \in y\},$$

is C -absolute.

Proof. Replacing the number ε by a smaller number, if necessary, we can assume that for any finite set F and any sequence $(x_i)_{i \in F} \in G^F$ with $M := \max_{A \subseteq F} \left\| \sum_{i \in A} x_i \right\|_G \leq \varepsilon$ we have $\sum_{i \in F} \|x_i\|_G \leq CM$. To show that the seminorm $\|\cdot\|$ on the group G/H is C -absolute, take any finite set F and sequence $(y_i)_{i \in F} \in (G/H)^F$ such that $M := \max_{A \subseteq F} \left\| \sum_{i \in A} y_i \right\| \leq \varepsilon$. By the definition of the seminorm $\|\cdot\|$ on G/H , for every $i \in F$ there exists an element $x_i \in y_i$ such that $\|y_i\| \leq \|x_i\|_G < \|y_i\| + \frac{1}{2}\varepsilon$.

Claim 3. *For every $k \in \{0, \dots, |F|\}$ and any set $E \subseteq F$ of cardinality $|E| = k$ we have*

$$\left\| \sum_{i \in E} x_i \right\|_G = \left\| \sum_{i \in E} y_i \right\|.$$

Proof. The claim will be proved by induction on k . For $k = 0$ and the empty set $E = \emptyset$ we have

$$\left\| \sum_{i \in E} x_i \right\|_G = \|0_G\|_G = 0 = \|0_{G/H}\| = \left\| \sum_{i \in E} y_i \right\|.$$

Now assume that the equality in the claim has been proved for some nonnegative integer $k < |F|$. Take any set $E \subseteq F$ of cardinality $|E| = k + 1$. Choose any index $j \in E$ and consider the set $E' = E \setminus \{j\}$. By the inductive assumption, $\left\| \sum_{i \in E'} x_i \right\|_G = \left\| \sum_{i \in E'} y_i \right\| \leq M \leq \varepsilon$.

Assuming that $\|\sum_{i \in E} x_i\|_G \neq \|\sum_{i \in E} y_i\|_G$ and taking into account that $\sum_{i \in E} x_i \in \sum_{i \in E} y_i$, we conclude that $\|\sum_{i \in E} y_i\|_G < \|\sum_{i \in E} x_i\|_G$. Then we can find an element $h \in H$ such that

$$\|h + \sum_{i \in E} x_i\|_G < \|\sum_{i \in E} x_i\|_G.$$

The triangle inequality implies that

$$\begin{aligned} \|h\|_G &= \|h + \sum_{i \in E} x_i - \sum_{i \in E} x_i\|_G \leq \|h + \sum_{i \in E} x_i\|_G + \|\sum_{i \in E} x_i\|_G \\ &< \|\sum_{i \in E} x_i\|_G + \|\sum_{i \in E} x_i\|_G \leq 2(\|x_j\|_G + \|\sum_{i \in E'} x_i\|_G) \\ &< 2(\|y_j\|_G + \frac{1}{2}\varepsilon + \|\sum_{i \in E'} y_i\|_G) \leq 2(M + \frac{1}{2}\varepsilon + M) \leq 5\varepsilon. \end{aligned}$$

Since $\{x \in H : \|x\|_G < 5\varepsilon\} = \{0_H\}$, the element h equals 0_H and we obtain a required contradiction

$$\|\sum_{i \in E} x_i\|_G = \|h + \sum_{i \in E} x_i\|_G < \|\sum_{i \in E} x_i\|_G,$$

showing that $\|\sum_{i \in E} x_i\|_G = \|\sum_{i \in E} y_i\|_G$ and completing the proof of the claim. \square

By Claim 3,

$$\max_{E \subseteq F} \|\sum_{i \in E} x_i\|_G = \max_{E \subseteq F} \|\sum_{i \in E} y_i\|_G = M \leq \varepsilon.$$

Now the C -absoluteness of the seminorm $\|\cdot\|_G$ and the choice of ε ensure that

$$\sum_{i \in F} \|y_i\|_G \leq \sum_{i \in F} \|x_i\|_G \leq CM,$$

witnessing that the seminorm $\|\cdot\|_G$ on G/H is C -absolute. \square

Example 3 and Proposition 7 imply the next assertion.

Lemma 8. *The seminorm $\|\cdot\| : \mathbb{R}/\mathbb{Z} \rightarrow [0, \infty)$, $\|\cdot\| : y \mapsto \min\{|x| : x \in y\}$, on the compact group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is 2-absolute.*

The following two lemmas can be easily deduced from the definitions.

Lemma 9. *Let H be an open subgroup of a group G . If $\|\cdot\|_H : H \rightarrow [0, \infty)$ is a C -absolute seminorm on the group H , then the seminorm*

$$\|\cdot\|_G : G \rightarrow [0, \infty), \quad \|x\|_G = \begin{cases} \min\{1, \|x\|_H\}, & \text{if } x \in H, \\ 1, & \text{if } x \in G \setminus H \end{cases}$$

on the group G is C -absolute, too.

Lemma 10. *Let $h : H \rightarrow G$ be a continuous homomorphism between groups. If a seminorm $\|\cdot\|_G$ on the group G is C -absolute for some positive real number C , then the seminorm*

$$\|\cdot\|_H : H \rightarrow [0, \infty), \quad \|\cdot\|_H : x \mapsto \|h(x)\|_G,$$

is C -absolute.

3.5 Locally Euclidean and locally compact groups

A group G is called *locally Euclidean* if some open neighborhood of 0_G in G is homeomorphic to the Euclidean space \mathbb{R}^n for some $n \in \omega$. In particular, every discrete group is locally Euclidean.

Proposition 4. *The topology of every locally Euclidean group G is generated by a locally 2-homogeneous absolute seminorm.*

Proof. By [11, Example 75], the locally Euclidean group G contains an open subgroup H , which is topologically isomorphic to the product $\mathbb{R}^n \times \mathbb{T}^k$ for some $n, k \in \omega$. By Examples 2 and 3, the norm $\|\cdot\|_{\mathbb{R}} : \mathbb{R} \rightarrow [0, \infty)$, $\|\cdot\|_{\mathbb{R}} : x \mapsto |x|$, is locally 2-homogeneous and absolute. By Corollary 1 and Lemma 8, the seminorm $\|\cdot\|_{\mathbb{T}} : \mathbb{T} \rightarrow [0, \infty)$, $\|\cdot\|_{\mathbb{T}} : y \mapsto \min\{|x| : x \in y\}$, is locally 2-homogeneous and absolute. By Lemma 3 and Corollary 2, the seminorms

$$\begin{aligned} \|\cdot\|_{\mathbb{R}^n} : \mathbb{R}^n \rightarrow [0, \infty), \quad \|\cdot\|_{\mathbb{R}^n} : (x_i)_{i \in n} \mapsto \max_{i \in n} \|x_i\|_{\mathbb{R}}, \\ \|\cdot\|_{\mathbb{T}^k} : \mathbb{T}^k \rightarrow [0, \infty), \quad \|\cdot\|_{\mathbb{T}^k} : (x_i)_{i \in k} \mapsto \max_{i \in k} \|x_i\|_{\mathbb{T}}, \end{aligned}$$

and

$$\|\cdot\| : \mathbb{R}^n \times \mathbb{T}^k \rightarrow [0, \infty), \quad \|\cdot\| : (x, y) \mapsto \max\{\|x\|_{\mathbb{R}^n}, \|y\|_{\mathbb{T}^k}\}$$

are locally 2-homogeneous and absolute. It is clear that the topology of the group $\mathbb{R}^n \times \mathbb{T}^k$ is generated by the seminorm $\|\cdot\|$. Since H is topologically isomorphic to $\mathbb{R}^n \times \mathbb{T}^k$, the topology of the group H is generated by a locally 2-homogeneous absolute seminorm $\|\cdot\|_H$. By Lemmas 5 and 9, the seminorm

$$\|\cdot\|_G : G \rightarrow [0, \infty), \quad \|\cdot\|_G : x \mapsto \begin{cases} \min\{1, \|x\|_H\}, & \text{if } x \in H, \\ 1, & \text{if } x \in G \setminus H \end{cases}$$

is locally 2-homogeneous and absolute. It remains to observe that the seminorm $\|\cdot\|_G$ generates the topology of the group G . \square

An actogroup $A \curvearrowright G$ is *locally compact* if the group G is locally compact, that is every element of G has a compact neighborhood in G .

Proposition 5. *The topology of any locally compact actogroup $A \curvearrowright G$ is generated by a family of contracting absolute seminorms.*

Proof. Let $U \subseteq G$ be a neighborhood of 0_G in G . Since G is locally compact, we can replace U by a smaller neighborhood, if necessary, and assume that U has compact closure \bar{U} in G . By [9, Corollary 7.54], there exists a surjective continuous homomorphism $h : G \rightarrow L$ onto a locally Euclidean group L such that $h^{-1}[V] \subseteq U$ for some neighborhood $V \subseteq L$ of 0_L . By Proposition 4, the topology of the locally Euclidean group L is generated by a locally 2-homogeneous absolute seminorm $\|\cdot\|_L : L \rightarrow [0, \infty)$. So, $\{x \in L : \|x\|_L \leq \varepsilon\} \subseteq V$ for some $\varepsilon > 0$. By Lemmas 2 and 10, the seminorm

$$\|\cdot\|_G : G \rightarrow [0, \infty), \quad \|\cdot\|_G : x \mapsto \|h(x)\|_L$$

on the group G is locally 2-homogeneous and absolute. Since the closed neighborhood

$$\{x \in G : \|x\|_G \leq \varepsilon\} \subseteq \{x \in G : \|h(x)\|_L \leq \varepsilon\} \subseteq \{x \in G : h(x) \in V\} \subseteq U \subseteq \bar{U}$$

is compact, we can apply Theorem 1 and conclude that the seminorm $\|\cdot\|_G$ is contracting. \square

3.6 Obsolete seminorms

Definition 7. A seminorm $\|\cdot\| : G \rightarrow [0, \infty)$ on a group G is defined to be obsolete if for every unconditionally convergent series $\sum_{i \in \omega} x_i$ in G the series $\sum_{i \in \omega} \|x_i\|$ converges.

Lemma 11. Each absolute seminorm $\|\cdot\|$ on a group G is obsolete.

Proof. Being absolute, the seminorm $\|\cdot\|$ is C -absolute for some positive real number C . This means that there exists $\varepsilon > 0$ such that for any finite set F and sequence $(x_i)_{i \in F} \in G^F$ with $M := \max_{E \subseteq F} \|\sum_{i \in E} x_i\| \leq \varepsilon$ we have $\sum_{i \in F} \|x_i\| \leq CM$.

To show that $\|\cdot\|$ is obsolete, fix any unconditionally convergent series $\sum_{i \in \omega} x_i$ in G . The unconditional convergence of $\sum_{i \in \omega} x_i$ yields a finite set $K \subseteq \omega$ such that $\|\sum_{i \in F} x_i\| < \varepsilon$ for any finite set $F \subset \omega \setminus K$.

Suppose for a contradiction that the series $\sum_{i \in \omega} \|x_i\|$ diverges in the real line. Then we can find a finite set $F \subset \omega \setminus K$ such that $\sum_{i \in F} \|x_i\| > C\varepsilon$. The choice of the sets K and F ensures that

$$M := \max_{E \subseteq F} \left\| \sum_{i \in E} x_i \right\| < \varepsilon$$

and hence $\sum_{i \in F} \|x_i\| \leq CM \leq C\varepsilon$, a contradiction. \square

The obsolescence is preserved by standard operations over seminorms.

Lemma 12. Let $h : H \rightarrow G$ be a continuous homomorphism between groups. If $\|\cdot\|_G$ is an obsolete seminorm on the group G , then the seminorm

$$\|\cdot\|_H : H \rightarrow [0, \infty), \quad \|\cdot\|_H : x \mapsto \|h(x)\|_G$$

is an obsolete seminorm on the group H .

Proof. To prove that the seminorm $\|\cdot\|_H$ is obsolete, take any unconditionally convergent series $\sum_{i \in \omega} x_i$ in the group H . The continuity of the homomorphism h implies that $\sum_{i \in \omega} h(x_i)$ is an unconditionally convergent series in the group G . Since the seminorm $\|\cdot\|_G$ is obsolete, the series $\sum_{i \in \omega} \|h(x_i)\|_G = \sum_{i \in \omega} \|x_i\|_H$ converges, witnessing that the seminorm $\|\cdot\|_H$ is obsolete. \square

Lemma 13. For two obsolete seminorms $\|\cdot\|_G : G \rightarrow [0, \infty)$ and $\|\cdot\|_H : H \rightarrow [0, \infty)$ on groups G, H , the seminorm

$$\|\cdot\| : G \times H \rightarrow [0, \infty), \quad \|\cdot\| : (g, h) \mapsto \max\{\|g\|_G, \|h\|_H\}$$

on $G \times H$ is obsolete.

Proof. Let $\sum_{i \in \omega} z_i$ be any unconditionally convergent series in the group $G \times H$. For every $i \in \omega$ write the element $z_i \in G \times H$ as a pair (x_i, y_i) for some $x_i \in G$ and $y_i \in H$. The unconditional convergence of the series $\sum_{i \in \omega} z_i$ implies the unconditional convergence of the series $\sum_{i \in \omega} x_i$ and $\sum_{i \in \omega} y_i$. By the obsolescence of the seminorms $\|\cdot\|_G$ and $\|\cdot\|_H$, the series $\sum_{i \in \omega} \|x_i\|_G$ and $\sum_{i \in \omega} \|y_i\|_H$ converge in the real line. Then

$$\sum_{i \in \omega} \|z_i\| = \sum_{i \in \omega} \max\{\|x_i\|_G, \|y_i\|_H\} \leq \sum_{i \in \omega} (\|x_i\|_G + \|y_i\|_H) \leq \sum_{i \in \omega} \|x_i\|_G + \sum_{i \in \omega} \|y_i\|_H < \infty,$$

which means that the series $\sum_{i \in \omega} \|z_i\|$ converges, and therefore the seminorm $\|\cdot\|$ on $G \times H$ is obsolete. \square

3.7 Unconditional seminorms on actogroups

Definition 8. A seminorm $\|\cdot\| : G \rightarrow [0, \infty)$ on an actogroup $A \curvearrowright G$ is called

- *C-unconditional* for some real number C if there exist a positive real number ε and a neighborhood $V \subseteq A$ of 0_A such that for any finite set F and sequences $(x_i)_{i \in F} \in G^F$ and $(a_i)_{i \in F} \in V^F$ with $M := \max_{E \subseteq F} \left\| \sum_{i \in E} x_i \right\| \leq \varepsilon$ we have $\left\| \sum_{i \in F} a_i x_i \right\| \leq CM$;
- *unconditional* if $\|\cdot\|$ is C -unconditional for some positive real number C .

Lemma 14. If a seminorm $\|\cdot\|$ on an actogroup $A \curvearrowright G$ is absolute and Lipschitz, then $\|\cdot\|$ is unconditional.

Proof. Assuming that $\|\cdot\|$ is absolute and Lipschitz, find positive real numbers L, C, ε and a neighborhood $V \subseteq A$ of 0_A such that

- $\sum_{i \in F} \|x_i\| \leq C \cdot \max_{E \subseteq F} \left\| \sum_{i \in E} x_i \right\|$ for any finite set F and sequence $(x_i)_{i \in F} \in G^F$ with $\max_{E \subseteq F} \left\| \sum_{i \in E} x_i \right\| \leq \varepsilon$;
- $\|ax\| \leq L\|x\|$ for any $a \in V$ and $x \in G$ with $\|x\| \leq \varepsilon$.

Then for any finite set F and sequences $(x_i)_{i \in F} \in G^F$ and $(t_i)_{i \in F} \in V^F$ with $\max_{E \subseteq F} \left\| \sum_{i \in E} x_i \right\| \leq \varepsilon$ we have

$$\left\| \sum_{i \in F} t_i x_i \right\| \leq \sum_{i \in F} \|t_i x_i\| \leq \sum_{i \in F} L \|x_i\| \leq LC \max_{E \subseteq F} \left\| \sum_{i \in E} x_i \right\|,$$

which means that the seminorm $\|\cdot\|$ is LC -unconditional and hence unconditional. \square

3.8 Absolutely and unconditionally seminormable actogroups

Definition 9. An actogroup $A \curvearrowright G$ is called

- 1) *absolutely seminormable* if the topology of G is generated by a family of Lipschitz absolute seminorms;
- 2) *obsoletely seminormable* if the topology of G is generated by a family of Lipschitz obsolete seminorms;
- 3) *unconditionally seminormable* if the topology of G is generated by a family of unconditional seminorms.

Lemmas 11 and 14 imply the following result.

Lemma 15. Every absolutely seminormable actogroup is obsoletely and unconditionally seminormable.

Theorem 2. A actogroup $A \curvearrowright G$ is Hirsch if one of the following conditions holds:

- 1) $A \curvearrowright G$ is unconditionally seminormable;
- 2) $A \curvearrowright G$ is obsoletely seminormable;
- 3) $A \curvearrowright G$ is absolutely seminormable.

Proof. To prove that the actogroup $A \curvearrowright G$ is Hirsch, fix any unconditionally convergent series $\sum_{n \in \omega} x_i$ in G and any neighborhood U of 0_G in G .

1) First we assume that $A \curvearrowright G$ is unconditionally seminormable. Then there exists an unconditional seminorm $\|\cdot\|$ on G such that $\{x \in G : \|x\| \leq 2\varepsilon\} \subseteq U$ for some $\varepsilon > 0$. Since $\|\cdot\|$ is unconditional, there exist positive numbers C, δ and a neighborhood $V \subseteq A$ of 0_A such that for every finite set F and any sequences $(y_i)_{i \in F} \in G^F$ and $(a_i)_{i \in F} \in V^F$ with $M = \max_{E \subseteq F} \|\sum_{i \in E} y_i\| \leq \delta$, we have $\|\sum_{i \in F} a_i y_i\| \leq CM$.

Since the series $\sum_{i \in \omega} x_i$ is unconditionally convergent, there exists a finite set $E \subseteq \omega$ such that $\|\sum_{i \in F} x_i\| < \min\{\delta, \frac{\varepsilon}{C}\}$ for every finite set $F \subseteq \omega \setminus E$.

By the continuity of the action $A \times G \rightarrow G$, for every $i \in E$, there exists a neighborhood $V_i \subseteq A$ of 0_A such that $\sup_{a \in V_i} \|ax_i\| \leq \frac{\varepsilon}{|E|}$. Replacing the neighborhood V by the neighborhood $V \cap \bigcap_{i \in E} V_i$, we can additionally assume that $\max_{i \in E} \sup_{a \in V} \|ax_i\| \leq \frac{\varepsilon}{|E|}$.

Now observe that for every finite set $F \subset \omega$ and a sequence $(a_i)_{i \in F} \in V^F$ we have

$$\left\| \sum_{i \in F} a_i x_i \right\| \leq \sum_{i \in F \cap E} \|a_i x_i\| + \left\| \sum_{i \in F \setminus E} a_i x_i \right\| \leq \sum_{i \in F \cap E} \frac{\varepsilon}{|E|} + C \max_{K \subseteq F \setminus E} \left\| \sum_{i \in K} x_i \right\| \leq \frac{\varepsilon |F \cap E|}{|E|} + C \frac{\varepsilon}{C} \leq 2\varepsilon$$

and hence $\sum_{i \in F} a_i x_i \in U$.

2) Next, assume that $A \curvearrowright G$ is obsoletely seminormable. Then there exists a Lipschitz obsolete seminorm $\|\cdot\|$ on G such that $\{x \in G : \|x\| \leq 2\varepsilon\} \subseteq U$ for some $\varepsilon > 0$. By the obsoleteness of $\|\cdot\|$, we have $\sum_{n \in \omega} \|x_i\| < \infty$. Since the seminorm $\|\cdot\|$ is Lipschitz, there exist positive numbers $L \geq 1, \varepsilon' \leq \varepsilon$ and a neighborhood $V' \subseteq A$ of 0_A such that $\|ax\| \leq L\|x\|$ for every $a \in V'$ and $x \in G$ with $\|x\| \leq \varepsilon'$. Since $\sum_{i \in \omega} \|x_i\| < \infty$, there exists a nonempty finite set $E \subset \omega$ such that $\sum_{i \in \omega \setminus E} \|x_i\| \leq \frac{\varepsilon'}{L}$. For every $i \in \omega \setminus E$ we have $\|x_i\| \leq \sum_{j \in \omega \setminus E} \|x_j\| \leq \frac{\varepsilon'}{L}$ and hence $\sup_{a \in V'} \|ax_i\| \leq L\|x_i\|$. By the continuity of the seminorm $\|\cdot\|$ and continuity of the action $A \times G \rightarrow G$, for every $i \in E$, there exists a neighborhood $V_i \subseteq A$ of 0_A such that $\sup_{a \in V_i} \|ax_i\| \leq \frac{\varepsilon}{|E|}$. We claim that the neighborhood $V = V' \cap \bigcap_{i \in E} V_i$ witnesses that G is Hirsch. Take any finite set $F \subseteq \omega$ and any sequence $(a_i)_{i \in F} \in V^F$. It follows that

$$\left\| \sum_{i \in F} a_i x_i \right\| \leq \sum_{i \in F \cap E} \|a_i x_i\| + \sum_{i \in F \setminus E} \|a_i x_i\| \leq \sum_{i \in F \cap E} \frac{\varepsilon}{|E|} + \sum_{i \in F \setminus E} L\|x_i\| \leq \frac{\varepsilon |F \cap E|}{|E|} + L \frac{\varepsilon}{L} \leq 2\varepsilon$$

and hence $\sum_{i \in F} a_i x_i \in U$.

3) If $A \curvearrowright G$ is absolutely seminormable, then by Lemma 15, $A \curvearrowright G$ is both obsoletely seminormable and unconditionally seminormable and hence $A \curvearrowright G$ is Hirsch by any of the preceding cases. \square

Lemmas 11, 14 and Theorem 2 imply that for any actogroup we have the following implications.

$$\begin{array}{ccc} \text{absolutely seminormable} & \implies & \text{obsoletely seminormable} \\ \Downarrow & & \Downarrow \\ \text{unconditionally seminormable} & \implies & \text{Hirsch} \end{array}$$

Since every contracting seminorm of an actogroup is Lipschitz, Proposition 5 and Theorem 2 imply the next result.

Theorem 3. *Every locally compact actogroup is absolutely seminormable and Hirsch.*

3.9 Tychonoff products of actogroups

Definition 10. The Tychonoff product $\prod_{i \in I} (A_i \curvearrowright G_i)$ of a nonempty family of actogroups $(A_i \curvearrowright G_i)_{i \in I}$ is the group $\prod_{i \in I} G_i$ endowed with the action

$$\prod_{i \in I} A_i \times \prod_{i \in I} G_i \rightarrow \prod_{i \in I} G_i, \quad ((a_i)_{i \in I}, (x_i)_{i \in I}) \mapsto (a_i x_i)_{i \in I}$$

of the topological space $\prod_{i \in I} A_i$ whose distinguished point is the unique point of the singleton $\prod_{i \in I} \{0_{A_i}\}$.

If all the actogroups $A_i \curvearrowright G_i$ are equal to some fixed actogroup $A \curvearrowright G$, then the product $\prod_{i \in I} (A_i \curvearrowright G_i)$ will be denoted by $(A \curvearrowright G)^I$.

Theorem 4. The Tychonoff product $\prod_{i \in I} (A_i \curvearrowright G_i)$ of a nonempty family of absolutely (resp. obsoletely) seminormable actogroups is absolutely (resp. obsoletely) seminormable and Hirsch.

Proof. For a subset $F \subseteq I$ let $G_F = \prod_{i \in F} G_i$ and $\text{pr}_F : G \rightarrow G_F$, $\text{pr}_F : (x_i)_{i \in I} \mapsto (x_i)_{i \in F}$ be the projection homomorphism. It follows that $\prod_{i \in I} G_i = G_I$. Given any neighborhood $U \subseteq G_I$ of 0_{G_I} , find a finite set $F \subseteq I$ and a neighborhood W of 0_{G_F} in the group G_F such that $\text{pr}_F^{-1}[W] \subseteq U$. For every $i \in F$ find a neighborhood W_i of 0_{G_i} in G_i such that $\prod_{i \in F} W_i \subseteq W$. Since the actogroup $A_i \curvearrowright G_i$ is absolutely (resp. obsoletely) seminormable, there exists a Lipschitz absolute (resp. obsolete) seminorm $\|\cdot\|_i : G_i \rightarrow [0, \infty)$ such that $\{x \in G_i : \|x\|_i \leq \varepsilon_i\} \subseteq W_i$ for some $\varepsilon_i > 0$. By Lemma 6 (resp. 13), the seminorm

$$\|\cdot\|_F : G_F \rightarrow [0, \infty), \quad \|\cdot\|_F : (x_i)_{i \in F} \mapsto \max_{i \in F} \|x\|_i$$

is absolute (resp. obsolete). By Lemma 10 (resp. 12), the seminorm

$$\|\cdot\| : G_I \rightarrow [0, \infty), \quad \|\cdot\| : x \mapsto \|\text{pr}_F(x)\|_F$$

is absolute (resp. obsolete). It is clear that

$$\{x \in G_I : \|x\| \leq \min_{i \in F} \varepsilon_i\} \subseteq \text{pr}_F^{-1} \left[\prod_{i \in F} W_i \right] \subseteq \text{pr}_F^{-1}[W] \subseteq U.$$

It remains to prove that the seminorm $\|\cdot\|$ is Lipschitz. For every $i \in I$ the Lipschitz property of the seminorm $\|\cdot\|_i$ yields positive numbers L_i, δ_i and a neighborhood $V_i \subseteq A_i$ of 0_{A_i} such that $\sup_{a \in V_i} \|ax\|_i \leq L_i \|x\|_i$ for any $x \in G_i$ with $\|x\|_i \leq \delta_i$. Let $L = \max_{i \in F} L_i$, $\delta = \min_{i \in F} \delta_i$ and $V = \bigcap_{i \in F} \{(a_j)_{j \in I} \in \prod_{j \in I} A_j : a_i \in V_i\}$. It is easy to see that for every $a = (a_j)_{j \in I} \in V$ and $x = (x_j)_{j \in I} \in G$ with $\|(x_j)_{j \in I}\| = \max_{i \in F} \|x_i\|_i \leq \delta = \min_{i \in F} \delta_i$ we have

$$\|ax\| = \max_{i \in F} \|a_i x_i\|_i \leq \max_{i \in F} L_i \|x_i\|_i \leq L \max_{i \in F} \|x_i\|_i = L \|x\|,$$

which means that the seminorm $\|\cdot\|$ is Lipschitz. Therefore, the actogroup $\prod_{i \in I} (A_i \curvearrowright G_i)$ is absolutely (resp. obsoletely) seminormable. By Theorem 2, this actogroup is Hirsch. \square

Corollary 3. For every nonempty set I and every absolutely (resp. obsoletely) seminormable actogroup $A \curvearrowright G$ the actogroup $(A \curvearrowright G)^I$ is absolutely (resp. obsoletely) seminormable and Hirsch.

Theorem 5. For any nonempty set I and any locally compact actogroup $A \curvearrowright G$, the actogroup $(A \curvearrowright G)^I$ is absolutely seminormable and Hirsch.

3.10 Function actogroups

A family \mathcal{K} of compact subsets of a topological space X is called an *ideal of compact sets* if for any sets $A, B \in \mathcal{K}$ and any closed subset $K \subseteq A \cup B$ we have $K \in \mathcal{K}$.

For topological spaces X, Y and an ideal \mathcal{K} of compact subsets of X , let $C_{\mathcal{K}}(X, Y)$ be the space of continuous functions from X to Y , endowed with the topology generated by the subbase consisting of the sets

$$[K; U] = \{f \in C_{\mathcal{K}}(X, Y) : f(K) \subseteq U\},$$

where $K \in \mathcal{K}$ and U is an open subset of Y .

If \mathcal{K} is the ideal of all compact (finite) subsets of X , then the space $C_{\mathcal{K}}(X, Y)$ is denoted by $C_k(X, Y)$ (resp. $C_p(X, Y)$).

If X is a discrete space then $C_p(X, Y)$ is equal to the power Y^X , endowed with the Tychonoff product topology.

Let X be a topological space and \mathcal{K} be an ideal of compact Hausdorff subspaces of X . Given topological spaces Y, Z, T and a continuous map $\alpha : Y \times Z \rightarrow T$, consider the map $\tilde{\alpha} : C_{\mathcal{K}}(X, Y) \times C_{\mathcal{K}}(X, Z) \rightarrow C_{\mathcal{K}}(X, T)$ assigning to every pair of functions $(f, g) \in C_{\mathcal{K}}(X, Y) \times C_{\mathcal{K}}(X, Z)$ the function $\tilde{\alpha}(f, g) \in C_{\mathcal{K}}(X, T)$ defined by $\tilde{\alpha}(f, g)(x) = \alpha(f(x), g(x))$ for $x \in X$. Since the family \mathcal{K} consists of compact Hausdorff subspaces of X , we can apply Proposition 4.2 of [3] and conclude that the map $\tilde{\alpha}$ is continuous. This implies that for every actogroup $A \curvearrowright G$ we can consider the actogroup $C_{\mathcal{K}}(X, A) \curvearrowright C_{\mathcal{K}}(X, G)$ endowed with the action

$$C_{\mathcal{K}}(X, A) \times C_{\mathcal{K}}(X, G) \rightarrow C_{\mathcal{K}}(X, G), \quad (a, g) \mapsto ag,$$

where $ag : x \mapsto a(x)g(x)$. The distinguished point of the topological space $C_{\mathcal{K}}(X, A)$ is the constant function $X \rightarrow \{0_A\} \subseteq A$.

Theorem 6. *Let X be a topological space and \mathcal{K} be an ideal of compact Hausdorff subspaces of X . If $A \curvearrowright G$ is an unconditionally seminormable actogroup, then the actogroup $C_{\mathcal{K}}(X, A) \curvearrowright C_{\mathcal{K}}(X, G)$ is unconditionally seminormable and Hirsch.*

Proof. Given any neighborhood \mathcal{U} of zero in the group $C_{\mathcal{K}}(X, G)$, find a set $K \in \mathcal{K}$ and a neighborhood U of zero in G such that $[K, U] \subseteq \mathcal{U}$. Since the actogroup $A \curvearrowright G$ is unconditionally seminormable, there exists an unconditional seminorm $\|\cdot\|_G$ on G such that $\{x \in G : \|x\|_G \leq \varepsilon\} \subseteq U$ for some $\varepsilon > 0$. By the unconditionality of $\|\cdot\|_G$, there exist positive real numbers C, δ and a neighborhood $V \subseteq A$ of zero 0_A such that for any finite set F and any sequences $(x_i)_{i \in F} \in G^F$ and $(a_i)_{i \in F} \in V^F$ with $\max_{E \subseteq F} \|\sum_{i \in E} x_i\|_G \leq \delta$ we have $\|\sum_{i \in F} a_i x_i\|_G \leq C \cdot \max_{E \subseteq F} \|\sum_{i \in E} x_i\|_G$.

Consider the seminorm

$$\|\cdot\|_K : C_{\mathcal{K}}(X, G), \quad \|\cdot\|_K : f \mapsto \|f\|_K = \sup_{x \in K} \|f(x)\|_G$$

on the group $C_{\mathcal{K}}(X, G)$. The compactness of K implies that the seminorm $\|\cdot\|_K$ is well-defined and continuous. Observe that

$$\{f \in C_{\mathcal{K}}(X, G) : \|f\|_K \leq \varepsilon\} \subseteq [K; U] \subseteq \mathcal{U}.$$

It remains to show that the seminorm $\|\cdot\|_K$ is unconditional. The definition of the topology on the space $C_{\mathcal{K}}(X, T)$ ensures that $[K; V] = \{f \in C(X, A) : f[K] \subseteq V\}$ is a neighborhood of zero in the acting topological space $C_{\mathcal{K}}(X, A)$.

Now take any finite set F and sequences $(f_i)_{i \in F} \in C_{\mathcal{K}}(X, G)^F$ and $(a_i)_{i \in F} \in [K; V]^F$ such that $M := \max_{E \subseteq F} \|\sum_{i \in E} f_i\|_K \leq \varepsilon$. Observe that for every $x \in K$ we have $\{a_i(x)\}_{i \in F} \subseteq V$ and

$$\max_{E \subseteq F} \left\| \sum_{i \in E} f_i(x) \right\|_G \leq \max_{E \subseteq F} \left\| \sum_{i \in E} f_i \right\|_K = M \leq \varepsilon$$

and hence

$$\left\| \sum_{i \in F} a_i(x) f_i(x) \right\|_G \leq C \cdot \max_{E \subseteq F} \left\| \sum_{i \in E} f_i(x) \right\|_G \leq CM.$$

Then

$$\left\| \sum_{i \in F} a_i f_i \right\|_K = \max_{x \in K} \left\| \sum_{i \in F} a_i(x) f_i(x) \right\|_G \leq CM = C \max_{E \subseteq F} \left\| \sum_{i \in E} f_i \right\|_K,$$

witnessing that the seminorm $\|\cdot\|_K$ on $C_{\mathcal{K}}(X, G)$ is unconditional.

Therefore, the topology of the group $C_{\mathcal{K}}(X, G)$ is generated by unconditional seminorms and by Theorem 2, the actogroup $C_{\mathcal{K}}(X, A) \curvearrowright C_{\mathcal{K}}(X, G)$ is Hirsch. \square

Theorem 7. *Let X be a topological space and \mathcal{K} be an ideal of compact Hausdorff subspaces of X . For every locally compact actogroup $A \curvearrowright G$, the actogroup $C_{\mathcal{K}}(X, A) \curvearrowright C_{\mathcal{K}}(X, G)$ is unconditionally seminormable and Hirsch.*

Proof. By Theorem 3 and Lemma 15, the actogroup $A \curvearrowright G$ is absolutely seminormable and unconditionally seminormable. By Theorem 6, the actogroup $C_{\mathcal{K}}(X, A) \curvearrowright C_{\mathcal{K}}(X, G)$ is unconditionally seminormable and Hirsch. \square

4 Applications to Hirsch topological rings

In this section, we apply the results of the preceding section to topological rings. First we write down Definition 9 (item 3)) for the partial case of topological rings.

Definition 11. *A topological ring R is unconditionally seminormable if for every neighborhood $U \subseteq R$ of 0_R there exist a seminorm $\|\cdot\| : R \rightarrow [0, \infty)$ on the additive group of R , a neighborhood $V \subseteq R$ of 0_R , and positive real numbers C, ε such that $\{x \in R : \|x\| < \varepsilon\} \subseteq U$ and for any finite set F and sequences $(x_i)_{i \in F} \in R^F$ and $(a_i)_{i \in F} \in V^F$ with $M := \max_{E \subseteq F} \|\sum_{i \in E} x_i\| \leq \varepsilon$ we have*

$$\left\| \sum_{i \in F} a_i x_i \right\| \leq CM.$$

The following theorems are partial cases of Theorems 2, 3, 6 and 7, respectively.

Theorem 8. *Every unconditionally seminormable topological ring is Hirsch.*

Theorem 9. *Every locally compact topological ring is unconditionally seminormable and Hirsch.*

Theorem 10. *Let X be a topological space, \mathcal{K} be an ideal of compact Hausdorff subspaces of X , and R be a topological ring which is either unconditionally seminormable or locally compact. Then the topological ring $C_{\mathcal{K}}(X, R)$ is unconditionally seminormable and Hirsch.*

Theorem 10 has two corollaries.

Corollary 4. *For every locally compact topological ring R and any nonempty set I the topological ring R^I is unconditionally seminormable and Hirsch.*

Corollary 5. *For every Hausdorff topological space X and any locally compact topological ring R the topological rings $C_p(X, R)$ and $C_k(X, R)$ are unconditionally seminormable and Hirsch.*

5 Banach actospaces

A Banach actospace $A \curvearrowright X$ is a Banach space X endowed with a continuous bilinear action $A \times X \rightarrow X, (a, x) \mapsto ax$, of a Banach space A . All Banach spaces considered in this paper are over the field \mathbb{F} of real or complex numbers. It is clear that each Banach actospace is an actogroup. Banach actospaces with the Hirsch property admit a nice characterization.

Theorem 11. *For a Banach actospace $A \curvearrowright X$ the following conditions are equivalent:*

- (1) $A \curvearrowright X$ is Hirsch;
- (2) the norm of X is unconditional;
- (3) there exists a positive real constant C such that for every finite set F and sequences $(a_n)_{n \in F} \in A^F$ and $(x_n)_{n \in F} \in X^F$ we have $\| \sum_{n \in F} a_n x_n \| \leq C \cdot \max_{n \in F} \|a_n\| \cdot \max_{E \subseteq F} \| \sum_{n \in E} x_n \|$;
- (4) for any unconditionally convergent series $\sum_{n \in \omega} x_n$ in X and any bounded sequence $(a_n)_{n \in \omega}$ in A , the series $\sum_{n \in \omega} a_n x_n$ converges (unconditionally) in X .

Proof. We shall prove the implications $(3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3) \Leftrightarrow (4)$.

$(3) \Rightarrow (2)$ Assume that condition (3) is satisfied for some positive real constant C . We claim that the norm of the Banach space X is C -unconditional. Given a positive real number ε , consider the neighborhood $V = \{a \in A : \|a\| < 1\}$ of zero in the Banach space A . By the choice of C , for any finite set F and sequences $(a_n)_{n \in F} \in V^F$ and $(x_n)_{n \in F} \in X^F$ we have

$$\| \sum_{n \in F} a_n x_n \| \leq C \max_{n \in F} \|a_n\| \cdot \max_{E \subseteq F} \| \sum_{n \in E} x_n \| \leq C \max_{E \subseteq F} \| \sum_{n \in E} x_n \|,$$

witnessing that the norm of X is C -unconditional.

The implication $(2) \Rightarrow (1)$ follows from Theorem 2 (item 1)).

$(1) \Rightarrow (3)$ Assume that the Banach actogroup $A \curvearrowright X$ is Hirsch. Consider the linear space ΣX of all sequences $(x_n)_{n \in \omega}$ in X that have finite norm

$$\|(x_n)_{n \in \omega}\|_{\Sigma X} = \sup_{F \in [\omega]^{<\omega}} \| \sum_{n \in F} x_n \|,$$

where $[\omega]^{<\omega}$ is the family of finite subsets of ω . Repeating the standard proof [5, 1.16 (i)] of the completeness of the Banach space ℓ_∞ , one can easily show that ΣX is a Banach space with respect to the norm $\| \cdot \|_{\Sigma X}$. Let $\Sigma_0 X$ be the (closed) subspace of ΣX consisting of sequences $(x_n)_{n \in \omega}$ for which the series $\sum_{n \in \omega} x_n$ converges unconditionally in X . Let $\ell_\infty[A]$ be the Banach space of all bounded sequences $(a_n)_{n \in \omega}$ in A , endowed with the norm $\|(a_n)_{n \in \omega}\|_{\ell_\infty[A]} = \sup_{n \in \omega} \|a_n\|_A$. By the Hirsch property of X , the bilinear operator

$$T : \ell_\infty[A] \times \Sigma_0 X \rightarrow \Sigma X, \quad T : ((a_n)_{n \in \omega}, (x_n)_{n \in \omega}) \mapsto (a_n x_n)_{n \in \omega}$$

is well-defined. The operator T has closed graph

$$\left\{ \left((a_n)_{n \in \omega}, (x_n)_{n \in \omega}, (y_n)_{n \in \omega} \right) \in \ell_\infty[A] \times \Sigma_0 X \times \Sigma X : \forall n \in \omega \ (a_n x_n = y_n) \right\}$$

and hence is continuous by the Closed Graph Theorem of Fernandez [6]. Then T has bounded norm

$$\|T\| = \sup \{ \|T(a, x)\|_{\Sigma X} : (a, x) \in \ell_\infty[A] \times \Sigma_0 X, \max\{\|a\|_{\ell_\infty[A]}, \|x\|_{\Sigma_0 X}\} \leq 1 \}$$

and hence

$$\left\| \sum_{n \in \omega} a_n x_n \right\| \leq \|T\| \cdot \sup_{n \in \omega} \|a_n\| \cdot \sup_{F \in [\omega]^{<\omega}} \left\| \sum_{n \in F} x_n \right\|$$

for any sequences $(a_n)_{n \in \omega} \in \ell_\infty[A]$ and $(x_n)_{n \in \omega} \in \Sigma_0 X$. The latter inequality implies that condition (3) holds with the constant $C = \|T\|$.

The equivalence (3) \Leftrightarrow (4) is proved in [2, Theorem 3.2]. \square

Theorem 11 and [2, Theorem 4.1] imply the following result.

Theorem 12. *A Banach actospace $A \curvearrowright X$ is Hirsch if A is a Hilbert space possessing an orthonormal basis $(e_n)_{n \in \omega}$ such that for every $x \in X$ the series $\sum_{n \in \omega} e_n x$ converges unconditionally in X .*

For a number $p \in [1, \infty]$ let ℓ_p be the Banach space of all functions $x : \omega \rightarrow \mathbb{F}$ to the field \mathbb{F} of real or complex numbers such that the norm

$$\|x\|_{\ell_p} = \begin{cases} \left(\sum_{n \in \omega} |x(n)|^p \right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \sup_{n \in \omega} |x(n)|, & \text{if } p = \infty \end{cases}$$

is finite. For any $p, q \in [1, \infty]$ and functions $x \in \ell_p$ and $y \in \ell_q$, the function $xy : \omega \rightarrow \mathbb{F}$, $xy : n \mapsto x(n)y(n)$, belongs to ℓ_q , witnessing that the Banach actospace $\ell_p \curvearrowright \ell_q$ is well-defined. The Hirsch properties of this actospace are described in the following corollary of Theorem 11 and [2, Theorem 1.2].

Theorem 13. *For any numbers $p, q \in [1, \infty]$ the Banach actospace $\ell_p \curvearrowright \ell_q$ is Hirsch if and only if $p \leq 2$ or $p < q$ or $q = \infty$.*

6 Banach rings

In this section, we recognize Hirsch rings among Banach rings and also present examples of Banach rings which are not Hirsch.

A *Banach ring* is a Banach space X endowed with a binary operation $\cdot : X \times X \rightarrow X$ turning X into a topological ring.

6.1 Hirsch Banach rings

For a Banach ring R and a compact Hausdorff space K , let $C(K, R)$ be the Banach ring endowed with the norm $\|\cdot\|_K : C(K, R) \rightarrow [0, \infty)$, $\|\cdot\|_K : f \mapsto \sup_{x \in K} \|f(x)\|$, where $\|\cdot\|$ is the norm of the Banach space R .

Theorem 14. *For every finite-dimensional Banach ring R and compact Hausdorff space K , the Banach ring $C(K, R)$ is Hirsch.*

Proof. Being finite-dimensional, the Banach space R is locally compact. Now Corollary 5 implies that the Banach ring $C(K, R) = C_k(K, R)$ is Hirsch. \square

By $\mathbb{R}, \mathbb{C}, \mathbb{H}$ we denote the Banach rings of real, complex and quaternion numbers, respectively. Theorem 14 implies the following assertion.

Corollary 6. *For any compact Hausdorff space K the Banach rings $C(K, \mathbb{R})$, $C(K, \mathbb{C})$ and $C(K, \mathbb{H})$ are Hirsch.*

Let ℓ_∞ be the Banach ring of all bounded sequences of real (or complex) numbers, endowed with the sup-norm $\|(x_i)_{i \in \omega}\|_{\ell_\infty} = \sup_{i \in \omega} |x_i|$ and the operation of coordinatewise multiplication. Let c be the Banach subring of ℓ_∞ consisting of all convergent sequences and c_0 be the Banach subring of c , consisting of all sequences that converge to zero.

Theorem 15. *The Banach rings c_0 , c , and ℓ_∞ are Hirsch.*

Proof. Let $\beta\omega$ be the Stone-Ćech compactification of the discrete space ω of finite ordinals and \mathbb{F} be the topological field of real (or complex) numbers. It is well-known (and easy to see) that the restriction operator $C(\beta\omega, \mathbb{F}) \rightarrow \ell_\infty, f \mapsto f \upharpoonright_\omega$, is a topological isomorphism of the Banach rings $C(K, \mathbb{F})$ onto ℓ_∞ . By Corollary 6, the Banach ring $C(K, \mathbb{F})$ is Hirsch and so is its isomorphic copy ℓ_∞ . Since the Hirschness is inherited by subrings, the subrings c and c_0 of the Hirsch ring ℓ_∞ are Hirsch. \square

Observe that for every real number $p \in [1, \infty)$ the Banach space ℓ_p endowed with the pointwise multiplication of functions is a Banach ring. Theorem 13 implies the next result.

Corollary 7. *Let $p \in [1, \infty]$. The Banach ring ℓ_p is Hirsch if and only if $p \in [1, p] \cup \{\infty\}$.*

By a *Hilbert ring* we understand a Hilbert space H endowed with a continuous associative operation $\cdot : H \times H \rightarrow H$ turning H into a topological ring. An example of a Hilbert ring is the Banach ring ℓ_2 , which is Hirsch by Corollary 7.

Problem 3. *Is every (commutative) Hilbert ring Hirsch?*

6.2 Non-Hirsch Banach rings of operators

Let $p \in [1, \infty]$ and $L(\ell_p)$ be the Banach space of bounded linear operators on ℓ_p , endowed with the operator norm $\|a\| = \sup\{\|a(x)\| : x \in \ell_p \text{ with } \|x\|_{\ell_p} \leq 1\}$. Let $L^\circ(\ell_p)$ (resp. $L^\bullet(\ell_p)$) be the Banach space $L(\ell_p)$ endowed with the multiplication \circ (resp. \bullet) defined by $(a \circ b)(x) = a(b(x))$ (resp. $(a \bullet b)(x) = b(a(x))$) for any $a, b \in L(\ell_p)$ and $x \in \ell_p$. It is easy to check that both $L^\circ(\ell_p)$ and $L^\bullet(\ell_p)$ are Banach rings.

Proposition 6. *For any $p \in [1, \infty)$, the Banach ring $L^\bullet(\ell_p)$ is not Hirsch.*

Proof. Let $(e_i)_{i \in \omega}$ be the standard basis of the Banach space ℓ_p . For every $i \in \omega$ let $a_i : \ell_p \rightarrow \ell_p$ be the operator assigning to each $(x_n)_{n \in \omega} \in \ell_p$ the vector $\frac{x_i}{\sqrt{i+1}}e_i$. Since the sequence $(\frac{1}{\sqrt{i+1}})_{i \in \omega}$ converges to zero, the series $\sum_{i \in \omega} a_i$ is unconditionally Cauchy in $L(\ell_p)$, so it unconditionally

converges by Proposition 2. Now let $U = \{a \in L(\ell_p) : \|a\| \leq 1\}$ and V be any open neighborhood of 0 in $L(\ell_p)$. Find $\varepsilon > 0$ such that $\varepsilon U \subseteq V$. Since the harmonic series $\sum_{i \in \omega} \frac{1}{i+1}$ diverges, there exists a positive integer N such that $\varepsilon^p \sum_{i=0}^N \frac{1}{i+1} > 1$. For each $i \leq N$, let $f_i : \ell_p \rightarrow \ell_p$ be the linear operator assigning to each $(x_n)_{n \in \omega} \in \ell_p$ the vector $\varepsilon x_0 e_i$. Observe that $\{f_i\}_{i=0}^N \subseteq \varepsilon U \subseteq V$ but

$$\left\| \sum_{i=0}^N f_i \bullet a_i \right\|^p \geq \left\| \left(\sum_{i=0}^N f_i \bullet a_i \right) (e_0) \right\|_{\ell_p}^p = \varepsilon^p \sum_{i=0}^N \left(\frac{1}{\sqrt[i]{i+1}} \right)^p > 1.$$

So $\sum_{i=0}^N f_i \bullet a_i \notin U$ and the Banach ring $L^\bullet(\ell_p)$ is not Hirsch. □

Proposition 7. *For any $p \in (1, \infty]$, the ring $L^\circ(\ell_p)$ is not Hirsch.*

Proof. Let $(e_i)_{i \in \omega}$ be the standard basis of the Banach space ℓ_p . For every $i \in \omega$, let $a_i : \ell_p \rightarrow \ell_p$ be the operator assigning to each sequence $x = (x_n)_{n \in \omega} \in \ell_p$ the vector $a_i(x) = \frac{x_i}{\ln(i+2)} e_i$. Since the sequence $(\frac{1}{\ln(i+2)})_{i \in \omega}$ converges to zero, the series $\sum_{i \in \omega} a_i$ is unconditionally Cauchy, so it unconditionally converges by Proposition 2.

Let $U = \{f \in L(\ell_p) : \|f\| \leq 1\}$ be the closed unit ball in the Banach space $L(\ell_p)$. Given any neighborhood of zero $V \subseteq L(\ell_p)$, find $\varepsilon > 0$ such that $\varepsilon U \subseteq V$.

Let $e = \sum_{n \in \omega} \frac{1}{n+2} e_n$. Since $p > 1$, $e \in \ell_p$, see, for instance, [7, 367.5].

Since the series $\sum_{i=2}^\infty \frac{1}{i \ln i}$ diverges (see, for instance, [7, 367.6]), there exists an integer number N such that $\varepsilon \sum_{i=0}^N \frac{1}{(i+2) \ln(i+2)} > \|e\|_{\ell_p}$. For each $i \leq N$ let $f_i : \ell_p \rightarrow \ell_p$ be the operator assigning to each vector $x = (x_n) \in \ell_p$ the vector $f_i(x) = \varepsilon x_i e_0$. Then $\{f_i\}_{i=0}^N \subseteq \varepsilon U \subseteq V$ and

$$\left\| \sum_{i \in \omega} f_i \circ a_i \right\| \cdot \|e\|_{\ell_p} \geq \left\| \left(\sum_{i \in \omega} f_i \circ a_i \right) (e) \right\|_{\ell_p} = \varepsilon \sum_{i=0}^N \frac{1}{(i+2) \ln(i+2)} > \|e\|_{\ell_p}.$$

So $\sum_{i \in \omega} f_i \circ a_i \notin U$ and hence $L^\circ(\ell_p)$ is not Hirsch. □

Question 1. *Are the Banach rings $L^\bullet(\ell_\infty)$ and $L^\circ(\ell_1)$ Hirsch?*

6.3 Non-Hirsch commutative Banach rings

In this subsection, we present a simple example of a commutative Banach ring, which is not Hirsch. For a real number $p \in [1, \infty)$ we denote by ℓ_p the real Banach space of functions $x : \omega \rightarrow \mathbb{R}$ with finite norm

$$\|x\|_{\ell_p} = \left(\sum_{k \in \omega} |x(k)|^p \right)^{\frac{1}{p}} < \infty.$$

Let $\text{Re} : \mathbb{C} \rightarrow \mathbb{R}$ and $\text{Im} : \mathbb{C} \rightarrow \mathbb{R}$ be functions of taking the real and imaginary parts of a complex number, respectively.

Given two real numbers $p, q \in [1, \infty)$, consider the subring

$$\ell_p \oplus i\ell_q = \{z \in \mathbb{C}^\omega : \text{Re } \circ z \in \ell_p, \text{ Im } \circ z \in \ell_q\}$$

of the commutative ring \mathbb{C}^ω of all complex-valued sequences. The ring $\ell_p \oplus i\ell_q$ is a Banach ring with respect to the norm

$$\|z\| = \max\{\|\text{Re } \circ z\|_p, \|\text{Im } \circ z\|_q\}.$$

Theorem 16. For any distinct real numbers $p, q \in [1, \infty)$, the commutative Banach ring $\ell_p \oplus i\ell_q$ is not Hirsch.

Proof. For every $n \in \omega$ let $e_n : \omega \rightarrow \{0, 1\}$ be the unique function such that $e_n^{-1}(1) = \{n\}$.

Let $m = \min\{p, q\}$ and $M = \max\{p, q\}$. Let

$$b = \begin{cases} 1, & \text{if } p < q, \\ 0, & \text{if } q < p. \end{cases}$$

Then $i^b = i$ if $p < q$ and $i^b = 1$ if $q < p$.

Observe that the series $\sum_{k \in \omega} (k + 1)^{-1/m} e_k$ converges unconditionally in the Banach space ℓ_M but diverges in ℓ_m . The choice of b ensures that the series

$$\sum_{k \in \omega} x_k = \sum_{k \in \omega} i^b (k + 1)^{-1/m} e_k$$

converges unconditionally in the Banach space $\ell_p \oplus i\ell_q$.

Assuming for a contradiction that the Banach ring $\ell_p \oplus i\ell_q$ is Hirsch, for the neighborhood $U = \{z \in \ell_p \oplus i\ell_q : \|z\| < 1\}$ of zero, find a neighborhood $V \subseteq \ell_p \oplus i\ell_q$ of zero such that for every $n \in \omega$ and elements $(a_k)_{k \in n} \in V^n$ we have $\sum_{k \in n} a_k x_k \in U$. Find $\varepsilon > 0$ such that $\{z \in \ell_p \oplus i\ell_q : \|z\| \leq \varepsilon\} \subseteq V$. Since the series $\sum_{k \in \omega} (k + 1)^{-1}$ diverges, there exists $n \in \omega$ such that

$$\left(\sum_{k \in n} (k + 1)^{-1} \right)^{\frac{1}{m}} > \frac{1}{\varepsilon}.$$

For every $k \in n$ let $a_k = \varepsilon i e_k \in V$. Then

$$\begin{aligned} \left\| \sum_{k \in n} a_k x_k \right\| &= \left\| \sum_{k \in n} (\varepsilon i e_k) (i^b (k + 1)^{-1/m} e_k) \right\| = \left\| \sum_{k \in n} \varepsilon i^{b+1} (k + 1)^{-1/m} e_k \right\| \\ &= \left\| \sum_{k \in n} \varepsilon (k + 1)^{-1/m} e_k \right\|_{\ell_m} = \varepsilon \left(\sum_{k \in n} (k + 1)^{-1} \right)^{\frac{1}{m}} > 1 \end{aligned}$$

and hence $\sum_{k \in n} a_k x_k \notin U$, which contradicts the choice of the neighborhood V . □

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Ми будемо називати топологічне кільце R кільцем Гірша, якщо для будь-яких безумовно збіжного ряду $\sum_{n \in \omega} x_n$ в R і околу U нуля 0 кільця R існує такий окіл $V \subseteq R$ нуля 0 , що $\sum_{n \in F} a_n x_n \in U$ для будь-яких скінченної множини $F \subset \omega$ та послідовності $(a_n)_{n \in F} \in V^F$. Ми розпізнаємо кільця Гірша серед певних відомих класів топологічних кілець. Для цього ми впроваджуємо та розвиваємо техніку напівнорм на актогрупах. Ми доводимо, зокрема, що топологічне кільце R є кільцем Гірша, якщо R є локально компактним або R має базу в нулі, котра складається з відкритих ідеалів або R є замкненим підкільцем банахового кільця $C(K)$, де K є компактним хаусдорфовим простором. З цього випливає, що банахове кільце ℓ_∞ і його підкільця c_0 і c є кільцями Гірша. Використовуючи новий результат Банаха та Кадеця, ми доводимо, що для довільного дійсного числа $p \geq 1$ комутативне банахове кільце ℓ_p є кільцем Гірша тоді і тільки тоді, коли $p \leq 2$. Також для кожного $p \in (1, \infty)$ (некомутативне) банахове кільце $L(\ell_p)$ неперервних ендоморфізмів банахового кільця ℓ_p не є кільцем Гірша.

Ключові слова і фрази: топологічне кільце, безумовна збіжність, локально компактне топологічне кільце, локально компактна абельова топологічна група.