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# On approximation of functions from the class $L^{\psi}_{\beta,1}$ by the Abel-Poisson integrals in the integral metric

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In the paper, we investigate an asymptotic behavior of the sharp upper bounds in the integral metric of deviations of the Abel-Poisson integrals from functions from the class $L^{\psi}_{\beta,1}$. The Abel-Poisson integrals are solutions of the partial differential equations of elliptic type with corresponding boundary conditions, and they play an important role in applied problems. The approximative properties of the Abel-Poisson integrals on different classes of differentiable functions were studied in a number of papers. Nevertheless, a problem on the respective approximation on the classes $L^{\psi}_{\beta,1}$ in the metric of the space $L$ remained unsolved. We managed to obtain the estimates for the values of approximation of $(\psi,\beta)$-differentiable functions from the unit ball of the space $L$ by the Abel-Poisson integrals. In some cases, we also write down asymptotic equalities for these quantities, that is we solve the Kolmogorov-Nikol'skii problem for the the Abel-Poisson integrals on the classes $L^{\psi}_{\beta,1}$ in the integral metric.

*Key words and phrases:* Kolmogorov-Nikol'skii problem, Abel-Poisson integral, $(\psi,\beta)$-differentiable function, asymptotic equality, integral metric.

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## 1 Introduction

A.I. Stepanets introduced the classification of periodic functions based on the notion of $(\psi,\beta)$-derivative (see, e.g., [10, Ch. III]). As a result, the following sets appeared: $L^{\psi}_{\beta}$ and $C^{\psi}_{\beta}$, that denote, respectively, the set of summable and continuous $2\pi$-periodic functions $f$, that could be given by the convolution

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x+t)\sum_{k=1}^{\infty}\psi(k)\cos\left(kt+\frac{\beta\pi}{2}\right)dt,$$

where $\int_{-\pi}^{\pi}\varphi(t)dt = 0$, $\psi$ is arbitrary function of a natural argument, such that the series $\sum_{k=1}^{\infty}\psi(k)\cos\left(kt+\frac{\beta\pi}{2}\right)$ is the Fourier series of some function, and $\beta$ is a real number. Such function $\varphi$ is denoted by $f^{\psi}_{\beta}$ and is called $(\psi,\beta)$-derivative of the function $f$. Based on these sets, the classes

$$L^{\psi}_{\beta,1} = \left\{ f\in L^{\psi}_{\beta} : \|\varphi\|_L = \|f^{\psi}_{\beta}\|_1 = \int_{-\pi}^{\pi}|\varphi(x)|dx \le 1 \right\}$$

and

$$C^{\psi}_{\beta,\infty} = \left\{ f\in C^{\psi}_{\beta} : \|\varphi\|_{\infty} = \operatorname*{ess\,sup}_{x\in[-\pi,\pi]}|\varphi(x)| \le 1 \right\}$$

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were introduced.

Note that in the case $\psi(k) = k^{-r}$, $r > 0$, we have $C^{\psi}_{\beta,\infty} = W^r_{\beta}$ and $f^{\psi}_{\beta}(x) = f^r_{\beta}(x)$ is $(r, \beta)$-derivative in the Weyl-Nagy sense; if $r \in N$ and $r = \beta$, then $W^r_{\beta} = W^r$; if $r \in N$ and $r = \beta + 1$, then $W^r_{\beta} = \overline{W}^r$.

Let (see [10, p. 159, 160]) $\mathfrak{M}$ be the set of convex downward, continuous on $[1; \infty)$ and decreasing to zero functions $\psi$,

$$\eta(t) := \psi^{-1}\left(\frac{\psi(t)}{2}\right), \quad \mu(t) = \mu(\psi; t) := \frac{t}{\eta(t) - t},$$

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M} : \quad 0 < \mu(\psi; t) \leq K \quad \forall t \geq 1\}$$

and

$$\mathfrak{M}_C = \{\psi \in \mathfrak{M} : \quad 0 < K_1 \leq \mu(\psi; t) \leq K_2 \quad \forall t \geq 1\},$$

where $K, K_i$, $i = 1, 2$, are the constants that can depend on $\psi$.

For $2\pi$-periodic summable on the period function $f$, by

$$A_{\delta}(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} e^{-\frac{k}{\delta}} \cos kt \right\} dt, \quad \delta > 0, \tag{1}$$

we denote the Abel-Poisson integral (see, e.g., [8]).

The aim of this paper is to study the asymptotic behavior of the quantity

$$\mathcal{E}(L^{\psi}_{\beta,1}; A_{\delta})_L = \sup_{f \in L^{\psi}_{\beta,1}} \|f(\cdot) - A_{\delta}(f; \cdot)\|_L \tag{2}$$

as $\delta \to \infty$ and $\psi \in \mathfrak{M}_0$.

If the function $\phi(\delta) = \phi(L^{\psi}_{\beta,1}; \delta)$ is written in an explicit form such that $\mathcal{E}(L^{\psi}_{\beta,1}; A_{\delta})_L = \phi(\delta) + O(\phi(\delta))$ as $\delta \to \infty$, then following A.I. Stepanets [10, p. 198] we say that the Kolmogorov-Nikol'skii problem is solved on the class $L^{\psi}_{\beta,1}$ in the integral metric.

We should note, that the Kolmogorov-Nikol'skii problem for the Abel-Poisson integrals was studied a lot on different classes of differentiable functions in the uniform metric (see the papers [2–7, 11, 12]). This problem in the integral metric for the classes of $(\psi, \beta)$-differentiable functions remained unsolved. Therefore, in this paper we study the approximative properties of the Abel-Poisson integrals on the classes $L^{\psi}_{\beta,1}$ for arbitrary real $\beta$ and $\psi \in \mathfrak{M}_0$.

## 2 Approximation of $(\psi, \beta)$-differentiable functions by the Abel-Poisson integrals in the integral metric

For the Abel-Poisson integral defined by the relation (1), analogically to that in [14], let us write the so-called summing function

$$\tau_{\delta}(u) = \tau_{\delta}(u; \psi) = \begin{cases} (1 - e^{-u}) \frac{\psi(1)}{\psi(\delta)}, & 0 \leq u \leq \frac{1}{\delta}, \\ (1 - e^{-u}) \frac{\psi(\delta u)}{\psi(\delta)}, & u \geq \frac{1}{\delta}. \end{cases} \tag{3}$$

In the introduced above notation, the following statement holds.

**Theorem 1.** *Let $\psi \in \mathfrak{M}_0$ is such that $\int_1^\infty \frac{\psi(u)}{u}du < \infty$, and the function $g(u) = u\psi(u)$ is convex downward or upward. Then the equalities hold*

$$\mathcal{E}(L^{\psi}_{\beta,1}; A_\delta)_L = \frac{\psi(\delta)}{\pi}\int_{-\infty}^{\infty}\left|\int_0^\infty \tau_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt + O\left(\frac{1}{\delta}\right) \tag{4}$$

*as $\delta \to \infty$, and*

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\int_0^\infty \tau_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt \\ = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\left(\frac{1}{\delta\psi(\delta)}\int_1^\delta \frac{g(u)}{u}du + \frac{1}{\psi(\delta)}\int_\delta^\infty \frac{g(u)}{u^2}du\right) + O\left(1 + \frac{1}{\delta\psi(\delta)}\right).\end{aligned} \tag{5}$$

*Proof.* In view of the fact that the function $\tau_\delta(u)$ defined by the relation (3) is continuous on the whole domain, and its Fourier transform

$$\frac{1}{\pi}\int_0^\infty \tau_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du$$

for all $\psi \in \mathfrak{M}_0$ is summable on the whole axis [13, Lemma 1], then any $f \in L^{\psi}_{\beta,1}$ has the integral representation

$$f(x) - A_\delta(f;x) = \frac{\psi(\delta)}{\pi}\int_{-\infty}^{\infty} f^{\psi}_{\beta}\left(x + \frac{t}{\delta}\right)\int_0^\infty \tau_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\,dt. \tag{6}$$

Respectively, from (2) and (6), analogically to that in the paper [9], we can show that

$$\begin{aligned}\mathcal{E}(L^{\psi}_{\beta,1}; A_\delta)_L = \frac{\psi(\delta)}{\pi}\int_{-\infty}^{\infty}\left|\int_0^\infty \tau_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt \\ + O(\psi(\delta))\int\limits_{|t|\geq\frac{\delta\pi}{2}}\left|\int_0^\infty \tau_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt.\end{aligned} \tag{7}$$

Now, let us estimate the second term from the right-hand side of (7). For this reason we integrate twice by parts and obtain that $\forall\psi \in \mathfrak{M}_0$ we have

$$\begin{aligned}\int_0^\infty \tau_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du = \frac{1}{t}\tau_\delta(u)\sin\left(ut + \frac{\beta\pi}{2}\right)\Big|_0^\infty \\ + \frac{1}{t^2}\tau'_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)\Big|_0^\infty - \frac{1}{t^2}\int_0^\infty \tau''_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du.\end{aligned} \tag{8}$$

According to (3), we get $\tau_\delta(0) = 0$, $\tau_\delta(\infty) = 0$, $\tau'_\delta(\infty) = 0$, $\tau'_\delta(0) = \frac{\psi(1)}{\psi(\delta)}$, hence

$$\left|\int_0^\infty \tau_\delta(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right| \leq \frac{1}{t^2}\int_0^\infty |\tau''_\delta(u)|\,du + \frac{K}{\psi(\delta)t^2}. \tag{9}$$

Now, we estimate the integral from the right-hand side of (9). Analogically to that in the paper [1], we break an integration segment $[0;\infty)$ into three parts: $\left[0;\frac{1}{\delta}\right]$, $\left[\frac{1}{\delta};1\right]$ and $[1;\infty)$.

Taking into account that the function $\tau_\delta(u)$ is convex upward on the first integration segment $\left[0;\frac{1}{\delta}\right]$, we obtain

$$\int_0^{\frac{1}{\delta}} |\tau''_\delta(u)|\,du = -\int_0^{\frac{1}{\delta}} \tau''_\delta(u)\,du = e^{-u}\frac{\psi(1)}{\psi(\delta)}\Big|_0^{\frac{1}{\delta}} = O\left(\frac{1}{\psi(\delta)}\right), \quad \delta \to \infty. \tag{10}$$

Further, from (3) we get

$$\begin{aligned}\int_{\frac{1}{\delta}}^{1}|\tau''_{\delta}(u)|du &\le \int_{\frac{1}{\delta}}^{1}\frac{\left|((1-e^{-u}-u)\psi(\delta u))''\right|}{\psi(\delta)}du+\int_{\frac{1}{\delta}}^{1}\frac{|(u\psi(\delta u))''|}{\psi(\delta)}du\\ &\le \int_{\frac{1}{\delta}}^{1}\frac{u^2\delta^2\psi''(\delta u)}{2\psi(\delta)}du+\int_{\frac{1}{\delta}}^{1}\frac{2u\delta|\psi'(\delta u)|}{\psi(\delta)}du+\int_{\frac{1}{\delta}}^{1}\frac{\psi(\delta u)}{\psi(\delta)}du \\ &\quad+\int_{\frac{1}{\delta}}^{1}\frac{|(u\psi(\delta u))''|}{\psi(\delta)}du := I_1(\delta)+I_2(\delta)+I_3(\delta)+I_4(\delta).\end{aligned} \tag{11}$$

By [10, Theorem 12.1, p. 161], the function $\psi(u)\in\mathfrak{M}$ belongs to the set $\mathfrak{M}_0$ if and only if $\frac{\psi(t)}{t|\psi'(t)|}\ge K>0$, $\psi'(t)=\psi'(t+0)$, $\forall t\ge 1$.

Hence, integrating by parts, for all $\psi\in\mathfrak{M}_0$ we obtain

$$\int_{\frac{1}{\delta}}^{1}\frac{u^2\delta^2\psi''(\delta u)}{2\psi(\delta)}\,du\le K_1+\frac{\psi'(1)}{2\delta\psi(\delta)}+\int_{\frac{1}{\delta}}^{1}\frac{u\delta|\psi'(\delta u)|}{\psi(\delta)}\,du. \tag{12}$$

Using the same Theorem 12.1 from [10], we show that

$$\int_{\frac{1}{\delta}}^{1}\frac{2u\delta|\psi'(\delta u)|}{\psi(\delta)}du\le K_2\int_{\frac{1}{\delta}}^{1}\frac{\psi(\delta u)}{\psi(\delta)}du=K_2\int_{1}^{\delta}\frac{\psi(u)}{\delta\psi(\delta)}du\le K_2\left(1-\frac{1}{\delta}\right)\frac{\psi(1)}{\psi(\delta)}\le\frac{K_3}{\psi(\delta)}. \tag{13}$$

Taking into account the notations for $I_1(\delta)$, $I_2(\delta)$, $I_3(\delta)$ from (11) and the relations (12), (13), we derive to

$$I_1(\delta)+I_2(\delta)+I_3(\delta)\le K_1+\frac{K_4}{\delta\psi(\delta)}+\frac{K_5}{\psi(\delta)}. \tag{14}$$

In view of $(u\psi(\delta u))''=2\delta\psi'(\delta u)+\delta^2u\psi''(\delta u)$ and a convexity of the function $g(u)=u\psi(u)$, we get

$$I_4(\delta)=\int_{\frac{1}{\delta}}^{1}\frac{\left|(u\psi(\delta u))''\right|}{\psi(\delta)}\,du=O\left(\frac{1}{\psi(\delta)}\right). \tag{15}$$

Therefore, combining the relations (11), (14) and (15), we obtain

$$\int_{\frac{1}{\delta}}^{1}\left|\tau''_{\delta}(u)\right|du=O\left(\frac{1}{\psi(\delta)}\right). \tag{16}$$

By (3), for all $u\in[\frac{1}{\delta};\infty)$ the following equality holds

$$\tau''_{\delta}(u)=\frac{(1-e^{-u})\delta^2\psi''(\delta u)}{\psi(\delta)}+\frac{2e^{-u}\delta\psi'(\delta u)}{\psi(\delta)}-\frac{e^{-u}\psi(\delta u)}{\psi(\delta)}. \tag{17}$$

It is known [10, p. 175] that a necessary and sufficient condition for the function $\psi\in\mathfrak{M}$ to belong to $\mathfrak{M}_0$ is the existence for an arbitrary fixed number $c>1$ of such a constant $K$ that for all $t\ge1$ the inequality $\frac{\psi(t)}{\psi(ct)}\le K$ is true. Then, in view of (17), [10, Theorem 12.1, p. 161] and the obvious inequalities $1-e^{-u}\le u$, $e^{-u}\le 1$, $\psi(\delta u)\le\psi(\delta)$, we get

$$\int_{1}^{\infty}|\tau''_{\delta}(u)|du=O(1). \tag{18}$$

Therefore, (10), (16) and (18) yield the estimate

$$\int_0^\infty |\tau''_\delta(u)|\, du = O\left(\frac{1}{\psi(\delta)}\right). \tag{19}$$

In view of the inequality (9), we obtain

$$\int_{|t|\geq\frac{\delta\pi}{2}} \left|\int_0^\infty \tau_\delta(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right| dt = O\left(\frac{1}{\delta\psi(\delta)}\right), \quad \delta\to\infty. \tag{20}$$

Taking into account the estimate (60) from [13], we derive to (5). Hence, (7), (20), (4) and (5) prove the statement of the theorem. □

We should note that for the classes $W^r_\beta$ an analogical theorem in the uniform metric was proved in the paper [3, p. 31].

**Corollary 1.** *If the conditions of Theorem 1 holds, and*

$$\lim_{x\to\infty}\frac{\psi(x)}{x|\psi'(x)|} = \infty, \tag{21}$$

*then for* $\sin\frac{\beta\pi}{2}\neq 0$ *the following asymptotic equality holds*

$$\mathcal{E}(L^{\psi}_{\beta,1}; A_\delta)_L = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right|\int_\delta^\infty \frac{g(u)}{u^2}du + O\left(\psi(\delta)\right) \quad \textit{as}\ \delta\to\infty. \tag{22}$$

*Proof.* First, let us prove that the function $\psi(x)x^{\alpha_0}$ increases beginning at some number $x_0\geq 1$ for all $\alpha_0\in(0,1)$. For this reason, we find

$$(\psi(x)x^{\alpha_0})' = \psi(x)\alpha_0 x^{\alpha_0-1} - x^{\alpha_0}|\psi'(x)| = |\psi'(x)|x^{\alpha_0}\left(\frac{\alpha_0\psi(x)}{x|\psi'(x)|}-1\right).$$

The condition (21) yields that there exists such $x_0 = x_0(\alpha_0)$ that for all $x > x_0$ the inequality $(\psi(x)x^{\alpha_0})' > 0$ holds. Then for all $\alpha\in(\alpha_0,1)$ we have

$$\int_1^\delta \frac{\psi(u)}{\delta\psi(\delta)}du = \int_1^\delta \frac{\psi(u)u^\alpha}{\delta\psi(\delta)}\frac{du}{u^\alpha} \leq \frac{\psi(\delta)\delta^\alpha}{\delta\psi(\delta)}\int_1^\delta \frac{du}{u^\alpha} = O(1) \quad \text{as}\ \delta\to\infty. \tag{23}$$

Further, using the condition (21) and the L'Hôpital's rule, for all $\psi\in\mathfrak{M}_0$ we get

$$\lim_{x\to\infty}\frac{1}{\psi(x)}\int_x^\infty \frac{g(u)}{u^2}du = \lim_{x\to\infty}\frac{1}{\psi(x)}\int_x^\infty \frac{\psi(u)}{u}du = \lim_{x\to\infty}\frac{\psi(x)}{x|\psi'(x)|} = \infty. \tag{24}$$

Hence,

$$\psi(\delta) = o\left(\int_0^\infty \frac{g(u)}{u^2}du\right). \tag{25}$$

Combining the relations (23), (25) with (4) and (5) we derive to (22). Corollary 1 is proved. □

Note that an example of such a function $\psi(u)$ that satisfies the conditions of Corollary 1 is $(\ln(u+K))^{-\gamma}$, where $\gamma>1$ and $K>0$.

**Corollary 2.** *Let $\psi \in \mathfrak{M}_0$ is such that for all $u \geq 1$ the function $g(u) = u\psi(u)$ is convex upward or downward and*

$$\lim_{u\to\infty} g(u) = \infty, \tag{26}$$

$$\lim_{\delta\to\infty} \frac{1}{\delta\psi(\delta)} \int_1^\delta \frac{g(u)}{u} du = \infty. \tag{27}$$

*Then for $\sin\frac{\beta\pi}{2} \neq 0$ the following asymptotic equality holds*

$$\mathcal{E}(L^\psi_{\beta,1}; A_\delta)_L = \frac{2}{\pi}\left|\sin\frac{\beta\pi}{2}\right| \frac{1}{\delta}\int_1^\delta \frac{g(u)}{u} du + O\left(\psi(\delta)\right) \quad \text{as } \delta \to \infty. \tag{28}$$

*Proof.* In view of the fact, that the function $\psi \in \mathfrak{M}_0$ satisfies the conditions (26) and (27), by the L'Hôpital's rule and the properties of the function $g(u) = u\psi(u)$, we get

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x\psi(x)} \int_1^x \frac{g(u)}{u}\, du &= \lim_{x\to\infty} \frac{1}{x\psi(x)} \int_1^x \psi(u)\, du \\ &= \lim_{x\to\infty} \frac{\psi(x)}{\psi(x) + x\psi'(x)} = \frac{1}{1 - \lim\limits_{x\to\infty} \frac{x|\psi'(x)|}{\psi(x)}} = \infty.\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} \frac{\psi(x)}{x|\psi'(x)|} = 1. \tag{29}$$

From the relations (24), (29) and the definition of the function $g(u) = u\psi(u)$ we obtain

$$\int_\delta^\infty \frac{\psi(u)}{u} du = \int_\delta^\infty \frac{g(u)}{u^2} du = O(\psi(\delta)). \tag{30}$$

Combining the relations (4), (5), (26), (27) and (30), we derive to the equality (28). Corollary 2 is proved. □

Note that the functions of the form

$$\psi(u) = \frac{(\ln(u+K))^\gamma}{u}, \quad \gamma > 0, \quad K > 0$$

satisfy the conditions of Corollary 2.

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Жигалло Т.В., Харкевич Ю.І. *Про наближення функцій класу $L^{\psi}_{\beta,1}$ інтегралами Абеля-Пуассона в інтегральній метриці* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 223–229.

Робота присвячена дослідженню асимптотичної поведінки точних верхніх меж відхилень інтегралів Абеля-Пуассона від функцій з класу $L^{\psi}_{\beta,1}$ в інтегральній метриці. Інтеграли Абеля-Пуассона є розв'язками диференціальних рівнянь в частинних похідних еліптичного типу з відповідними крайовими умовами та відіграють важливу роль в задачах прикладного характеру. Вивченню апроксимативних властивостей інтегралів Абеля-Пуассона на різних класах диференційовних функцій присвячено цілий ряд робіт, проте питання про наближення даними інтегралами на класах $L^{\psi}_{\beta,1}$ в метриці простору $L$ залишалось відкритим. В результаті проведених досліджень вдалось знайти оцінки для величин наближення $(\psi, \beta)$-диференційовних функцій з одиничної кулі простору $L$ інтегралами Абеля-Пуассона, а в деяких випадках вдалось записати асимптотичні рівності для цих величин, тобто знайти розв'язки задачі Колмогорова-Нікольського для інтегралів Абеля-Пуаасона на класах $L^{\psi}_{\beta,1}$ в інтегральній метриці.

*Ключові слова і фрази:* задача Колмогорова-Нікольського, інтеграл Абеля-Пуассона, $(\psi, \beta)$-диференційовні функції, асимптотична рівність, інтегральна метрика.