ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2024, **16** (1), 103–113
doi:10.15330/cmp.16.1.103-113

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2024, Т.16, №1, С.103–113

Characterizations of semigroups by their linear Diophantine anti-fuzzy bi-ideals

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The purpose of this paper is to introduce linear Diophantine anti-fuzzification of algebraic structures. In this regard, we define linear Diophantine anti-fuzzy (LDAF) substructures of a semigroup and discuss some of its properties. Moreover, we characterize semigroups in terms of LDAF-ideals and LDAF-bi-ideals. Finally, we apply the linear Diophantine anti-fuzzification to groups and find a relationship between LDAF-subgroups of a group and its LDF-subgroups.

Key words and phrases: group, semigroup, linear Diophantine fuzzy set, LDAF-subgroup, LDAF-subsemigroup, LDAF-ideal, LDAF-bi-ideal, LDAF-simple.

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Introduction

Over the past few decades, there has been a lot of discussion in the literature on associating a fuzzy set to an algebraic object. Generally, an element's membership in a set is determined by whether or not it is a member of the set. Such a concept leaves open many real-life questions. *Fuzzy Set Theory* was first presented by L.A. Zadeh [16] in 1965. The membership of an element in a fuzzy set is a real number in the unit interval $[0,1]$. The sum of an element's degree of membership and degree of non-membership in a fuzzy set is equal one. A. Rosenfeld [15] introduced fuzzy subgroups of a group and R. Biswas [6] introduced anti-fuzzy subgroups of a group. After that, the fuzzification and anti-fuzzification of algebraic structures grew and became areas of research that grabbed many algebraists and set theorists. In 2020, H. Kamaci [9] investigated the linear Diophantine fuzzy subsets of various algebraic structures. Finite linear Diophantine fuzzy substructures of algebraic structures such as groups, rings and fields were of great interest to him.

The linear Diophantine fuzzy subpolygroups of a polygroup and the concept of the linear Diophantine fuzzy n-fold weak subalgebras of a BE-algebra were recently studied by M. Al-Tahan et al. [1,2]. In 2022, G. Muhiuddin et al. [13] implemented the idea of linear Diophantine fuzzy sets in BCK/BCI-algebras. Motivated by the recent work on linear Diophantine fuzzy substructures and the early study on anti-fuzzy algebraic structures, we introduce a new class of linear Diophantine anti-fuzzy algebraic structures.

The following is the structure of our article on linear Diophantine anti-fuzzy subsets of semigroups. In Section 2, we define linear Diophantine anti-fuzzy (LDAF) substructures of a semigroup and discuss some properties. Moreover, we introduce a new relationship between

УДК 517.98
2020 *Mathematics Subject Classification:* 06F05, 06D72, 08A72.

linear Diophantine fuzzy sets and algebraic structures by using LDAF-ideals (bi-ideals) of a semigroup. In Section 3, we apply the linear Diophantine anti-fuzzification to groups and find a connection between LDAF-substructures of a group and its LDF-substructures.

1 Basic concepts

In this section, we introduce some fundamental concepts and results related to linear Diophantine fuzzy sets, as well as to semigroups that are used throughout the paper. For more related details, we refer to [3–8].

Definition 1 ([16]). *Let Ω be a universal set, $I = [0,1]$ and $\mu : \Omega \to I$ be a validity function. Then $A = \{(x, \mu(x)) : x \in \Omega\}$ is a fuzzy set.*

Definition 2 ([14]). *Let Ω be a universal set, $I = [0,1]$ and $U(X), V(x) \in I$ are degrees of belongingness and non-belongingness, respectively. Let $\alpha(x), \beta(x) \in I$ be reference parameters. The degrees satisfy $\alpha(x) + \beta(x) \in I$ and $\alpha(x)U(x) + \beta(x)V(x) \in I$ for all $x \in \Omega$. Then a linear Diophantine fuzzy set (LDFS) D on Ω is described as follows*

$$D = \{(a, < U(a), V(a) >, < \alpha(a), \beta(a) >) : a \in \Omega\}.$$

Remark 1. *A fuzzy set A on a universal set Ω with a validity function μ is a special case of the LDFS. This is easily seen as $A = \{(x, < \mu(x), 0 >, < 1, 0 >) : x \in \Omega\}$ is an LDFS on Ω.*

Definition 3 ([14]). *Let Ω be a universal set and D_1, D_2 be LDFSs on Ω. Then*

(1) the intersection $D_1 \cap D_2$ of D_1 and D_2 is defined as

$$\{(x, < U_1(x) \wedge U_2(x), V_1(x) \vee V_2(x) >, < \alpha_1(x) \wedge \alpha_2(x), \beta_1(x) \vee \beta_2(x) >) : x \in \Omega\};$$

(2) the union $D_1 \cup D_2$ of D_1 and D_2 is defined as

$$\{(x, < U_1(x) \vee U_2(x), V_1(x) \wedge V_2(x) >, < \alpha_1(x) \vee \alpha_2(x), \beta_1(x) \wedge \beta_2(x) >) : x \in \Omega\};$$

(3) the complement D_1^c of D_1 is defined as

$$D_1^c = \{(x, < V_1(x), U_1(x) >, < \beta_1(x), \alpha_1(x) >) : x \in \Omega\}.$$

Here, "$\vee$" represents the maximum and "$\wedge$" represents the minimum.

One can easily see that $(D_1^c)^c = D_1$.

Definition 4. *Let A be a non-void set and*

$$\cdot : A \times A \to A$$

be a map. Then $(A, \cdot)$ is a semigroup if "$\cdot$" is associative on A, i.e. $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ for all $\alpha, \beta, \gamma \in A$.

A non-empty subset M of a semigroup A is called a *subsemigroup* of A if M is a semigroup. A subsemigroup M of A is called a *left ideal* of A if $AM \subseteq M$ and it is called a *right ideal* of A if $MA \subseteq M$. An *ideal* is a left and a right ideal. A subsemigroup M of A is called a *bi-ideal* of A if $MAM \subseteq M$. A semigroup is duo if every its left (right) ideal is an ideal.

2 LDAF-substructures of semigroups

The study of linear Diophantine fuzzy subgroups of a group in [12], and linear Diophantine fuzzy subpolygroups of a polygroup in [1] motivate us to work on linear Diophantine anti-fuzzy (LDAF) substructures of a semigroup.

Let Ω be a universal set and D be an LDFS on Ω given as follows

$$D = \Big\{ \big(x, < U(x), V(x) >, < \alpha(x), \beta(x) > \big) : x \in \Omega \Big\},$$

where $U(X), V(x) \in [0,1]$ are degrees of belongingness and non-belongingness, respectively, and $\alpha(x), \beta(x) \in [0,1]$ are reference parameters. The degrees satisfy $\alpha(x) + \beta(x) \leq 1$ and $\alpha(x)U(x) + \beta(x)V(x) \leq 1$ for all $x \in \Omega$. For $x, y \in \Omega$, we have:

(i) $D(x) \wedge D(y) = \big(< u, v >, < \alpha, \beta > \big)$, where $u = U(x) \wedge U(y)$, $v = V(x) \vee V(y)$, $\alpha = \alpha(x) \wedge \alpha(y)$, $v = \beta(x) \vee \beta(y)$,

(ii) $D(x) \vee D(y) = \big(< u, v >, < \alpha, \beta > \big)$, where $u = U(x) \vee U(y)$, $v = V(x) \wedge V(y)$, $\alpha = \alpha(x) \vee \alpha(y)$, $\beta = \beta(x) \wedge \beta(y)$,

(iii) $D(x) \leq D(y)$ means that $U(x) \leq U(y)$, $V(x) \geq V(y)$, $\alpha(x) \leq \alpha(y)$ and $\beta(x) \geq \beta(y)$.

Definition 5. *Let $(B, \cdot)$ be a semigroup and D be an LDFS of B. Then D is a linear Diophantine anti-fuzzy subsemigroup (LDAF-subsemigroup) of B if*

$$D(b_1 \cdot b_2) \leq D(b_1) \vee D(b_2)$$

for all $b_1, b_2 \in B$.

Definition 6. *Let $(B, \cdot)$ be a semigroup and D be an LDAF-subsemigroup of B. Then D is an LDAF-right ideal of B if*

$$D(b_1 \cdot b_2) \leq D(b_1)$$

for all $b_1, b_2 \in B$.

Definition 7. *Let $(B, \cdot)$ be a semigroup and D be an LDAF-subsemigroup B. Then D is an LDAF-left ideal of B if*

$$D(b_1 \cdot b_2) \leq D(b_2)$$

for all $b_1, b_2 \in B$.

Definition 8. *Let $(B, \cdot)$ be a semigroup and D be an LDAF-subsemigroup of B. Then D is an LDAF-ideal of B if*

$$D(b_1 \cdot b_2) \leq D(b_1) \wedge D(b_2)$$

for all $b_1, b_2 \in B$, i.e. D is an LDAF-left ideal and an LDAF-right ideal of B.

Definition 9. *Let $(B, \cdot)$ be a semigroup and D be an LDAF-subsemigroup of B. Then D is an LDAF-bi-ideal of B if*

$$D(\alpha \cdot b \cdot \beta) \leq D(\alpha) \vee D(\beta)$$

for all $\alpha, b, \beta \in B$.

Proposition 1. *Let $(B, \cdot)$ be a semigroup and D be an LDAF-subsemigroup of B. If D is an LDAF-left ideal (right ideal) of B then D is an LDAF-bi-ideal of B.*

Proof. Let D be an LDAF-left ideal of B and $\alpha, x, \beta \in B$. Since D is an LDAF-subsemigroup of B, it follows that $D(\alpha \cdot x \cdot \beta) \leq D(\alpha) \vee D(x \cdot \beta)$. Having D an LDAF-left ideal of B implies that $D(x \cdot \beta) \leq D(\beta)$ and hence $D(\alpha \cdot x \cdot \beta) \leq D(\alpha) \vee D(\beta)$. Therefore, D is an LDAF-bi-ideal of B. The case D is an LDAF-right ideal of B is done similarly. □

Example 1. *Let $(P, \cdot)$ be the semigroup of positive integers under the standard multiplication of integers and D be the LDFS of P defined by*

$$D(x) = \begin{cases} (<0.5, 0.4>, <0.3, 0.6>), & \text{if } 7|x; \\ (<1, 0.1>, <0.8, 0.1>), & \text{otherwise.} \end{cases}$$

Then D is an LDAF-ideal of P.

We present an example on an LDAF-bi-ideal that is neither an LDAF-right ideal nor an LDAF-left ideal.

Example 2. *Let $B_d = \{d_1, d_2, d_3, d_4\}$ and define "$\cdot_d$" on B_d by Table 1.*

$\cdot_d$	d_1	d_2	d_3	d_4
d_1	d_1	d_2	d_1	d_2
d_2	d_1	d_2	d_1	d_2
d_3	d_3	d_4	d_3	d_4
d_4	d_3	d_4	d_3	d_4

Table 1. The semigroup $(B_d, \cdot_d)$

One can easily see that $(B_d, \cdot_d)$ is a semigroup. Let D be the LDFS of B_d defined as follows:

$$D(d_1) = (<0.8, 0.3>, <0.4, 0.5>),$$
$$D(d_2) = (<0.85, 0.3>, <0.4, 0.5>),$$
$$D(d_3) = D(d_4) = (<0.9, 0.3>, <0.4, 0.5>).$$

One can easily see that D is an LDAF-subsemigroup of B_d. But it is not an LDAF-left ideal of B_d since $D(d_4) = D(d_3 \cdot d_2) \nleq D(d_2)$. Also, D is not an LDAF-right ideal of B_d since $D(d_2) = D(d_1 \cdot d_2) \nleq D(d_1)$.

However, D is an LDAF-bi-ideal of B_d. Let $\alpha, x, \gamma \in B_d$.

1. *If $\alpha \in \{d_3, d_4\}$ or $\gamma \in \{d_3, d_4\}$, then $D(\alpha \cdot x \cdot \gamma) \leq D(\alpha) \vee D(\gamma)$ for all $x \in B_d$.*

2. *If $\alpha, \gamma \in \{d_1, d_2\}$, then $D(\alpha \cdot x \cdot \gamma) \leq D(\alpha) \vee D(\gamma)$ for all $x \in B_d$. The latter follows from considering the following cases.*

Case 1. *If* $\alpha = \gamma = d_1$, *then* $D(d_1 \cdot x \cdot d_1) = d(d_1) \leq D(\alpha) \vee D(\gamma)$.

Case 2. *If* $\alpha = \gamma = d_2$, *then* $D(d_2 \cdot x \cdot d_2) = D(d_2) \leq D(\alpha) \vee D(\gamma)$.

Case 3. *If* $\alpha = d_1$ *and* $\gamma = d_2$, *then* $D(d_1 \cdot x \cdot d_2) = D(d_2) \leq D(\alpha) \vee D(\gamma)$.

Case 4. *If* $\alpha = d_2$ *and* $\gamma = d_1$, *then* $D(d_2 \cdot x \cdot d_1) = D(d_1) \leq D(\alpha) \vee D(\gamma)$.

Now, we present an example on an LDAF-right ideal that is not an LDAF-left ideal.

Example 3. *Let* $(B_d, \cdot_d)$ *be the semigroup in Example 2 and define* D' *on* B_d *as follows:*

$$D'(d_1) = D'(d_2) = (< 0.87, 0.8 >, < 0.3, 0.6 >),$$
$$D'(d_3) = D'(d_4) = (< 0.91, 0.2 >, < 0.3, 0.6 >).$$

We get that D' *is an LDAF right-ideal that is not an LDAF-left ideal since*

$$D'(d_4) = D'(d_3 \cdot d_2) \nleq D'(d_2).$$

Proposition 2. *Let* $(K, \cdot)$ *be a semigroup and* D^*, D^{**} *be LDAF-subsemigroups of* K. *Then* $D^* \cup D^{**}$ *is an LDAF-subsemigroup of* K.

Proof. Let $x, y \in K$ and $D^* = \{(x, < U^*(x), V^*(x) >, < \alpha^*(x), \beta^*(x) >) : x \in K\}$, $D^{**} = \{(x, < U^{**}(x), V^{**}(x) >, < \alpha^{**}(x), \beta^{**}(x) >) : x \in K\}$ be LDAF-subsemigroups of K. Let $D = D^* \cup D^{**} = \{(x, < U(x), V(x) >, < \alpha(x), \beta(x) >) : x \in K\}$.

We have $U^*(x \cdot y) \leq U^*(x) \vee U^*(y)$, $V^*(x \cdot y) \geq V^*(x) \wedge V^*(y)$, $\alpha^*(x \cdot y) \leq \alpha'(x) \vee \alpha^*(y)$, $\beta^*(x \cdot y) \geq \beta^*(x) \wedge \beta^*(y)$, $U^{**}(x \cdot y) \leq U^{**}(x) \vee U^{**}(y)$, $V^{**}(x \cdot y) \geq V^{**}(x) \wedge V^{**}(y)$, $\alpha^{**}(x \cdot y) \leq \alpha^{**}(x) \vee \alpha^{**}(y)$ and $\beta^{**}(x \cdot y) \geq \beta^{**}(x) \wedge \beta^{**}(y)$.

We get $U(x \cdot y) = U^*(x \cdot y) \vee U^{**}(x \cdot y) \leq U^*(x) \vee U^*(y) \vee U^{**}(x) \vee U^{**}(y) = U(x) \vee U(y)$, $V(x \cdot y) = V^*(x \cdot y) \wedge V^{**}(x \cdot y) \geq V^*(x) \wedge V^{**}(y) \wedge V^{**}(x) \wedge V^{**}(y) = V(x) \wedge V(y)$, $\alpha(x \cdot y) = \alpha^*(x \cdot y) \vee \alpha^{**}(x \cdot y) \leq \alpha^*(x) \vee \alpha^*(y) \vee \alpha^{**}(x) \vee \alpha^{**}(y) = \alpha(x) \vee \alpha(y)$ and $\beta(x \cdot y) = \beta^*(x \cdot y) \wedge \beta^{**}(x \cdot y) \geq \beta^*(x) \wedge \beta^{**}(y) \wedge \beta^{**}(x) \wedge \beta^{**}(y) = \beta(x) \wedge \beta(y)$. The latter implies that $D(x \cdot y) \leq D(x) \vee D(y)$, as desired. □

Using a similar proof as that in Proposition 2, we get the following propositions.

Proposition 3. *Let* $(K, \cdot)$ *be a semigroup and* D^*, D^{**} *be LDAF-left ideals of* K. *Then* $D^* \cup D^{**}$ *is an LDAF-left ideal of* K.

Proposition 4. *Let* $(K, \cdot)$ *be a semigroup and* D^*, D^{**} *be LDAF-right ideals of* K. *Then* $D^* \cup D^{**}$ *is an LDAF-right ideal of* K.

Proposition 5. *Let* $(K, \cdot)$ *be a semigroup and* D^*, D^{**} *be LDAF-ideals of* K. *Then* $D^* \cup D^{**}$ *is an LDAF-ideal of* K.

Proposition 6. *Let* $(K, \cdot)$ *be a semigroup and* D^*, D^{**} *be LDAF-bi-ideals of* K. *Then* $D^* \cup D^{**}$ *is an LDAF-bi-ideal of* K.

Remark 2. *Let* $(K, \cdot)$ *be a semigroup and* D^*, D^{**} *be LDAF-subsemigroups of* K. *Then* $D^* \cap D^{**}$ *is not necessary an LDAF-subsemigroup of* K.

We illustrate Remark 2 by Example 4.

Example 4. *Let $(P, +)$ be the semigroup of positive integers under the standard addition of numbers and D_1, D_2 be the LDFSs of P defined as follows:*

$$D_1(x) = \begin{cases} (< 0.34, 0.56 >, < 0.21, 0.67 >), & \text{if } x \in 5P, \\ (< 0.89, 0.21 >, < 0.65, 0.34 >), & \text{otherwise,} \end{cases}$$

$$D_2(x) = \begin{cases} (< 0.3, 0.56 >, < 0.21, 0.67 >), & \text{if } x \in 7P, \\ (< 0.67, 0.1 >, < 0.5, 0.4 >), & \text{otherwise.} \end{cases}$$

One can easily see that D_1 and D_2 are LDAF-subsemigroups of P. Having

$$(D_1 \cap D_2)(5) = (< 0.34, 0.56 >, < 0.21, 0.67 >),$$
$$(D_1 \cap D_2)(7) = (< 0.3, 0.56 >, < 0.21, 0.67 >),$$
$$(D_1 \cap D_2)(12) = (< 0.67, 0.21 >, < 0.5, 0.4 >)$$

and $12 = 5 + 7$ implies that $(D_1 \cap D_2)(12) \nleq (D_1 \cap D_2)(5) \vee (D_1 \cap D_2)(7)$. Thus, $D_1 \cap D_2$ is not an LDAF-subsemigroup of P.

Definition 10. *A semigroup K is an LDAF-duo if every LDAF-left (-right) ideal is an LDAF-ideal.*

Proposition 7. *Let K be a commutative semigroup. Then K is an LDAF-duo.*

Proof. Let D be an LDAF-left ideal of K and $b_1, b_2 \in K$. Then $D(b_1b_2) = D(b_2b_1) \geq D(b_1)$. Note, that D is a left ideal of K. Thus, D is a right ideal of K. Therefore, K is an LDAF-duo. □

Example 5. *Let $(B_d, \cdot_d)$ be the semigroup in Example 2. Then B_d is not an LDAF- duo.*

Definition 11. *Let $(B, \cdot)$ be a semigroup and D be an LDFS of B. Let $u_1, u_2, \alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \in [0, 1]$ and $\alpha u_1 + \beta u_2 \in [0, 1]$. The ceiling set D^t of B corresponding to $t = (< u_1, u_2 >, < \alpha, \beta >)$ is defined as follows*

$$D^t = \{b \in B : D(b) \leq t\}.$$

Theorem 1. *Let $u_1, u_2, \alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \in [0, 1]$ and $\alpha u_1 + \beta u_2 \in [0, 1]$. Let $(K, \cdot)$ be a semigroup and D be an LDFS of K. Then D is an LDAF-subsemigroup of K if and only if for all $t = (< u_1, u_2 >, < \alpha, \beta >)$, $D^t \neq \varnothing$ is a subsemigroup of K.*

Proof. Let D be an LDAF-subsemigroup of K and $D^t \neq \varnothing$. If $x_1, x_2 \in D_t$, then $D(x_1), D(x_2) \leq t$. Since D is an LDAF-subsemigroup of K, it follows that $D(x_1 \cdot x_2) \leq D(x_1) \vee D(x_2) \leq t$. This implies that $x_1 \cdot x_2 \in D^t$. Thus, D^t is a subsemigroup of K.

Conversely, let $x_1, x_2 \in K$ with $D(x_1) = t', D(x_2) = t''$ and $t = t' \vee t''$. Then $x_1, x_2 \in D^t$. Since D^t is a subsemigroup of K, so $x_1 \cdot x_2 \in D^t$. This implies $D(x_1 \cdot x_2) \leq t = D(x_1) \vee D(x_2)$. Thus, D is an LDAF-subsemigroup of K. □

Theorem 2. *Let $u_1, u_2, \alpha, \beta \in [0, 1]$ satisfy $\alpha + \beta \in [0, 1]$ and $\alpha u_1 + \beta u_2 \in [0, 1]$. Let $(K, \cdot)$ be a semigroup and D be an LDFS of K. Then D is an LDAF-left ideal of K if and only if for all $t = (< u_1, u_2 >, < \alpha, \beta >)$, $D^t \neq \varnothing$ is a left ideal of K.*

Proof. Let D be an LDAF-left ideal of K and $D^t \neq \varnothing$. Theorem 1 shows that D^t is a subsemigroup of K. If $x_2 \in D^t$ and $x_1 \in K$, then $D(x_1 \cdot x_2) \leq D(x_2) \leq t$. The latter implies that $x_1 \cdot x_2 \in D^t$ and hence, D^t is a left ideal of K.

Conversely, let $x_1, x_2 \in K$ with $D(x_2) = t'$. Then $x_2 \in D^{t'}$. Since $D^{t'}$ is a left ideal of K, it follows that $x_1 \cdot x_2 \in D_{t'}$. This implies $D(x_1 \cdot x_2) \leq D(x_2)$. □

Theorem 3. *Let $u_1, u_2, \alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \in [0,1]$ and $\alpha u_1 + \beta u_2 \in [0,1]$. Let $(K, \cdot)$ be a semigroup and D be an LDFS of K. Then D is an LDAF-right ideal of K if for all $t = (< u_1, u_2 >, < \alpha, \beta >)$, $D^t \neq \varnothing$ is a right ideal of K.*

Proof. The proof is similar to that of Theorem 2. □

Theorem 4. *Let $u_1, u_2, \alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \in [0,1]$ and $\alpha u_1 + \beta u_2 \in [0,1]$. Let $(K, \cdot)$ be a semigroup and D be an LDFS of K. Then D is an LDAF-ideal of K if and only if for all $t = (< u_1, u_2 >, < \alpha, \beta >)$, $D^t \neq \varnothing$ is an ideal of K.*

Proof. The proof results from Theorems 2 and 3. □

Theorem 5. *Let $u_1, u_2, \alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \in [0,1]$ and $\alpha u_1 + \beta u_2 \in [0,1]$. Let $(K, \cdot)$ be a semigroup and D be an LDFS of K. Then D is an LDAF-bi-ideal of K if and only if for all $t = (< u_1, u_2 >, < \alpha, \beta >)$, $D^t \neq \varnothing$ is a bi-ideal of K.*

Proof. The proof is similar to that of Theorem 2. □

Remark 3. *Let $u, v, \alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \leq 1$ and $\alpha u + \beta v \leq 1$. Let $(K, \cdot)$ be a semigroup and D be an LDFS of K defined as $D(x) = (< u, v >, < \alpha, \beta >)$ for all $a \in A$. Then D is an LDAF-(left ideal/right ideal/ideal/bi-ideal) of K. This LDFS is called the* ***constant LDFS*** *of K.*

Definition 12. *Let $(K, \cdot)$ be a semigroup. Then K is called LDAF-simple if every LDAF-ideal of K is the constant LDFS.*

Theorem 6. *Let $(K, \cdot)$ be a semigroup. Then K is an LDAF-simple if and only if K is a simple semigroup.*

Proof. Let K be an LDAF-simple. Then every LDAF-ideal of K is the constant LDFS. If P is an ideal of K, then P is the ceiling set of K corresponding to $t = (< 0, 1 >, < 0, 1 >)$ for the LDAF-ideal D' of K defined as

$$D'(x) = \begin{cases} (< 0, 1 >, < 0, 1 >), & \text{if } x \in P, \\ (< 1, 0 >, < 1, 0 >), & \text{otherwise.} \end{cases}$$

Since D' is constant, it follows that $P = K$ and hence, K is simple.

Conversely, let K be a simple semigroup and D be an LDAF-ideal of K. Theorem 4 asserts that $D^t \neq \varnothing$ is an ideal of K for all $t = (< u, v >, < \alpha, \beta >)$, where $u, v, \alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \in [0,1]$ and $\alpha u + \beta v \in [0,1]$. Having K a simple semigroup implies that $D^t = K$. Let $x, y \in P$. If $D(x) \leq D(y)$, then $D^t \neq K$ for $t = D(x)$. Similarly, if $D(y) \leq D(x)$, then $D^r \neq K$ for $r = D(y)$. If $D(x) \nleq D(y)$ and $D(y) \nleq D(x)$, then $D^p \neq K$ for $p = D(x)$. This implies that D is a constant LDFS of K. Therefore, K is LDAF-simple. □

Using a similar argument to the proof of Theorem 6, we can prove the following theorem.

Theorem 7. *Let $(K, \cdot)$ be a semigroup. Then K is an LDAF-duo if and only if K is a duo semigroup.*

3 LDAF-substructures of groups

In this section, we define linear Diophantine anti-fuzzy subgroups of a group and find a relationship between its LDAF-subgroups and its LDF-subgroups.

Definition 13 ([12]). *Let $(G,\cdot)$ be a group and D be an LDFS of G. Then D is a linear Diophantine fuzzy subgroup (LDF-subgroup) of G if the following conditions hold:*

(1) $D(\alpha\cdot\beta)\geq D(\alpha)\wedge D(\beta)$ for all $\alpha,\beta\in G$,

(2) $D\left(\alpha^{-1}\right)=D(\alpha)$ for all $\alpha\in G$.

Proposition 8 ([12]). *Let $(G,\cdot)$ be a group with identity "e" and D be a linear Diophantine fuzzy subgroup (LDF-subgroup) of G. Then the following inequalities are true:*

(1) $D(e)\geq D(\alpha)$ for all $\alpha\in G$,

(2) $D(\alpha^k)\geq D(\alpha)$ for all $x\in G$ and $k\in\mathbb{Z}$.

Proposition 9 ([12]). *Let $(G,\cdot)$ be a group and D be an LDFS of G. Then D is LDF-subgroup of G if and only if $D(g_1\cdot g_2^{-1})\geq D(g_1)\wedge D(g_2)$.*

Remark 4. *In [9, Theorem 3.5], H. Kamaci stated that the union of LDF-subgroups of a group is an LDF-subgroup. This result is incorrect. We illustrate it by Example 6.*

Example 6. *Let $(\mathbb{Z},+)$ be the group of integers under the standard addition of numbers, D_3, D_4 be the LDFS on $\mathbb{Z}$ defined as follows:*

$$D_3(x)=\begin{cases}(<0.89,0.21>,<0.65,0.34>), & \text{if } 2|x,\\ (<0.34,0.56>,<0.21,0.67>), & \text{otherwise,}\end{cases}$$

$$D_4(x)=\begin{cases}(<0.67,0.1>,<0.5,0.4>), & \text{if } 3|x,\\ (<0.3,0.56>,<0.21,0.67>), & \text{otherwise.}\end{cases}$$

One can easily see that D_3 and D_4 are LDF-subgroups of $\mathbb{Z}$.

Having

$$\begin{aligned}(D_3\cup D_4)(8)&=(<0.89,0.21>,<0.65,0.34>),\\ (D_3\cup D_4)(27)&=(<0.67,0.1>,<0.5,0.4>),\\ (D_3\cup D_4)(35)&=(<0.34,0.56>,<0.21,0.67>),\end{aligned}$$

and $35=8+27$ implies that $(D_3\cup D_4)(35)\ngeq(D_3\cup D_4)(8)\wedge(D_3\cup D_4)(27)$. Thus, $D_3\cup D_4$ is not an LDF-subgroup of $\mathbb{Z}$.

Definition 14. *Let $(G,\cdot)$ be a group and D be an LDFS of G. Then D is a linear Diophantine anti-fuzzy subgroup (LDAF-subgroup) of G if the following conditions hold:*

(1) $D(\alpha\cdot\beta)\leq D(\alpha)\vee D(\beta)$ for all $\alpha,\beta\in G$,

(2) $D\left(\alpha^{-1}\right)=D(\alpha)$ for all $\alpha\in G$.

Example 7. *Let $(\mathbb{Z},+)$ be the group of integers under standard addition and D_5 be the LDFS of $\mathbb{Z}$ defined as follows:*

$$D_5(\alpha)=\begin{cases}(<0.65,0.45>,<0.3,0.2>), & \text{if } 12|\alpha,\\ (<0.75,0.32>,<0.4,0.2>), & \text{otherwise.}\end{cases}$$

One can easily see that D_5 is an LDAF-subgroup of $\mathbb{Z}$.

Theorem 8. *Let $(G,\cdot)$ be a group and D' be an LDFS of G. Then D' is an LDF-subgroup of G if and only if D'^c is an LDAF-subgroup of G.*

Proof. Let $D'=\{(a,<U(a),V(a)>,<\alpha(a),\beta(a)>):a\in G\}$ be an LDF-subgroup of G. Then $D'^c=\{(a,<V(a),U(a)>,<\beta(a),\alpha(a)>):a\in G\}$.

Now, for all $a,b\in G$, we have $D(a\cdot b)\geq D(a)\wedge D(b)$. Therefore, $U(a\cdot b)\geq U(a)\wedge U(b)$, $V(a\cdot b)\leq V(a)\vee V(b)$, $\alpha(a\cdot b)\geq\alpha(a)\wedge\alpha(b)$ and $\beta(a\cdot b)\leq\beta(a)\vee\beta(b)$. The latter implies that $D'^c(a\cdot b)\leq D'^c(a)\vee D'^c(a)$ for all $a,b\in G$. Moreover, having $D'\left(x^{-1}\right)=D'(x)$, we get $U\left(a^{-1}\right)=U(a)$, $V(a^{-1})=V(a)$, $\alpha\left(a^{-1}\right)=\alpha(a)$ and $\beta\left(a^{-1}\right)=\beta(a)$. This implies that $D'^c\left(a^{-1}\right)=D'^c(a)$. Hence, $D'^c=\{(a,<V(a),U(a)>,<\beta(a),\alpha(a)>):a\in G\}$ is an LDAF-subgroup of G. The other direction is done similarly. □

Corollary 1. *Let $(G,\cdot)$ be a group and D be an LDFS of G. Then D is an LDAF-subgroup of G if and only if D^c is an LDF-subgroup of G.*

Proof. The proof results from Theorem 8 and the fact that $(D^c)^c=D$. □

Proposition 10. *Let $(G,\cdot)$ be a group and D^*,D^{**} be LDAF-subgroups of G. Then $D^*\cup D^{**}$ is an LDAF-subgroup of G.*

Proof. The proof is similar to that of Proposition 2. □

Remark 5. *The intersection of LDAF-subgroups of a group may not be an LDAF-subgroup.*

Proposition 11. *Let $(G,\circ)$ be a group and D be an LDAF-subgroup of G. Then the following statements hold:*

(1) $D(e)\leq D(\alpha)$ for all $\alpha\in G$,

(2) $D\left(\alpha^k\right)\leq D(\alpha)$ for all $\alpha\in G$ and $k\in\mathbb{Z}$.

Proof. Let $D=\{(a,<U(a),V(a)>,<\alpha(a),\beta(a)>):a\in G\}$ be an LDAF-subgroup of G. Then D^c is an LDF-subgroup of G. Therefore, $D^c(e)\geq D^c(\alpha)$ for all $\alpha\in G$. It follows that $D(e)\leq D(\alpha)$ for all $\alpha\in G$. Now, since $D^c\left(\alpha^k\right)\geq D^c(\alpha)$ for all $\alpha\in G$ and $k\in\mathbb{Z}$, therefore $D\left(\alpha^k\right)\leq D(\alpha)$ for all $\alpha\in G$ and $k\in\mathbb{Z}$. □

Using a similar argument to that of Proposition 11, we get the following result.

Proposition 12. *Let $(G,\cdot)$ be a group and D be an LDFS of G. Then D is an LDAF-subgroup of G if and only if $D(g_1\cdot g_2^{-1})\leq D(g_1)\vee D(g_2)$.*

Theorem 9 ([8]). *Let $(B,.)$ be a semigroup. Then B is a group if and only if B is simple.*

Corollary 2. *Let $(B, .)$ be a semigroup. Then B is a group if and only if B is LDAF-simple.*

Proof. Let $(B, .)$ be a semigroup. The proof results from Theorem 6 and Theorem 9. □

Corollary 3. *Every semigroup has either a non-constant LDAF-left ideal or a non-constant LDAF- right ideal.*

Proof. The proof follows from Corollary 2. □

Theorem 10. *Let $u_1, u_2, \alpha, \beta \in [0,1]$ satisfy $\alpha + \beta \in [0,1]$ and $\alpha u_1 + \beta u_2 \in [0,1]$. Let $(G, \cdot)$ be a group and D be an LDFS of G. Then D is an LDAF-subgroup of G if and only if $D^t \neq \varnothing$ is a subgroup of K for all $t = (< u_1, u_2 >, < \alpha, \beta >)$.*

Proof. The proof is similar to that of Theorem 1. □

Proposition 13. *Every non-trivial group has a non-constant LDAF-subgroup.*

Proof. Every non-trivial group with identity "e" has $\{e\}$ as a proper subgroup. One can easily see that the non-constant LDFS $D^{\odot}$ of G defined as

$$D^{\odot}(x) = \begin{cases} (< 0,1 >, < 0,1 >), & \text{if } x = e, \\ (< 1,0 >, < 1,0 >), & \text{otherwise} \end{cases}$$

is an LDAF-subgroup of G. □

Corollary 4. *Every non-trivial group has a non-constant LDF-subgroup.*

Proof. Proposition 13 asserts that every non-trivial group has a non-constant LDAF-subgroup, say D. Corollary 1 asserts that D^c is an LDF-subgroup of G. Having D an non-constant LDFS implies that so D^c. □

4 Conclusion

This paper presented a new link between linear Diophantine fuzzy sets and algebraic structures by introducing LDAF-ideals (bi-ideals) of a semigroup and LDAF-subgroups of a group. The various properties, definitions, and theorems related to the latter concepts have been discussed. Moreover, semigroups and groups were characterized by their LDAF-substructures. The results of the paper can be considered as a generalization of the results known for anti-fuzzy ideals (bi-ideals) of a semigroup and for anti-fuzzy subgroups of a group. For upcoming studies this new approach may be applied to numerous algebraic structures using different methodologies and we hope to apply some applications on the similarity measure like [10, 11]. We expect that the proposed model of LDF-relations and all the ideas in this paper will exist as an establishment for LDFS theory and will lead to new valuable results.

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Received 14.03.2022
Revised 25.10.2022

Аль-Касабех С., Аль Таган М. *Характеризація напівгруп за їхніми лінійними діофантовими антинечіткими бі-ідеалами* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 103–113.

Метою цієї роботи є введення лінійної діофантової антинечіткості алгебраїчних структур. У зв'язку з цим ми визначаємо лінійні діофантові антинечіткі (LDAF) підструктури напівгрупи та обговорюємо деякі її властивості. Крім того, ми характеризуємо напівгрупи в термінах LDAF-ідеалів та LDAF-бі-ідеалів. Насамкінець, ми застосовуємо лінійну діофантову антинечіткість до груп і знаходимо зв'язок між LDAF-підгрупами групи та її LDF-підгрупами.

Ключові слова і фрази: група, напівгрупа, лінійна діофантова нечітка множина, LDAF-підгрупа, LDAF-піднапівгрупа, LDAF-ідеал, LDAF-бі-ідеал.