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On the error of the approximate calculation of double integrals with variable upper limits

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In the paper, the estimations of the approximate calculation of double integrals with variable upper limits are suggested. The theorem on estimation for the main error term of approximate integration using the approach of substituting the integrand with an approximating function is formulated and proved, as well as the theorem on the estimation of the main error term in the case of calculating double integrals using the sequential integration approach. Estimations of integration errors are obtained when dividing the variable region of integration into rectangular grids, in particular, four possible options for adapting the grid to the change of the integration region are considered. For each of these options the estimations are found for rectangular and curvilinear elements of the grid as well as total estimation of double integration with variable upper limits.

Key words and phrases: error, approximation, double integral, variable upper limit, Taylor series, integration variable region, rectangular grid.

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Introduction

In practice finding a double integral can cause significant difficulties, since, as a rule, it is impossible to express such an integral per elementary functions and to find its exact value. Therefore, there is a need for the approximate integrating of double integrals of the form $\iint_{(V)} f(x',x'')\,dx''\,dx'$. Here $f(x',x'')$ is the continuous function in variables x' and x'' in the region (V). In order that, as a rule, two basic approaches are applied [2,7].

The first of them is based on the substitution of the integrand by functions approximating it, the integrals of which are calculated in elementary functions. A function $\overline{f}\left(x',x''\right)$ in the region (V) by the mean-value theorem is

$$\overline{f}\left(x',x''\right) = \frac{1}{S}\iint_{(V)} f\left(x',x''\right)\,dx''\,dx', \tag{1}$$

where S is the area of the region (V).

Assuming that the average value of the function is approximately equal to the value of the function in the center of the region (V), i.e. in the centroid (x'_c, x''_c), $\overline{f}\left(x',x''\right) \approx f\left(x'_c,x''_c\right)$, from relation (1) we obtain the approximate formula for numerical integration

$$\iint_{(V)} f\left(x',x''\right)\,dx''\,dx' \approx S\,f\left(x'_c,x''_c\right). \tag{2}$$

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Accepting the function $f\left(x', x''\right)$ to be sufficiently smooth, the error of the formula (2) is sought using the expansion of the integrand into the Taylor series. And the difference between the exact and approximate values of the integral is found with restriction to a given order of accuracy.

The second approach is based on the sequential integration with use of numerical methods of the single integration along the axis Ox' and Ox'' [1]. For example, by change of the variables, the region of integration (V) is reduced to a rectangle and the integral (1) is given in the form

$$I = \iint_{(V)} f\left(x', x''\right) dx''\, dx' = \int_a^b \int_c^d f\left(x', x''\right) |J|\, dx''\, dx', \tag{3}$$

where $|J|$ is Jacobian of the change $\left(|J| \neq 0\right)$.

Expression (3) is written as

$$I = \int_a^b F\left(x'\right) dx', \quad F\left(x'\right) = \int_c^d f\left(x', x''\right) |J|\, dx''. \tag{4}$$

To calculate each of these integrals, well-known methods are used [6, 7, 11], for example, methods of rectangles or trapezoids, Simpson method, etc. In the general case, (4) can be presented as follows

$$F\left(x_i'\right) = h_1 \sum_j q_{1,j} \tilde{f}\left(x_i', x_j''\right),$$

where $\tilde{f}\left(x_i', x_j''\right) = f\left(x', x''\right)|J|$; h_1 is the length of the element along the Ox'-axis, $q_{1,j}$ are the coefficients depending on the integration method along the Ox'-axis.

And then

$$I \approx h_2 \sum_i q_{2,i} F\left(x_i'\right) = h_1 h_2 \sum_{i,j} q_{1,j} q_{2,i} \tilde{f}\left(x_i', x_j''\right),$$

where h_2 is the length of the element along the Ox''-axis, $q_{2,i}$ are the coefficients depending on the integration method along the Ox''-axis.

The integration error is searched similarly to the previous approach.

A more general case is finding double integrals when the limits of integration are variable (that is, functions of some magnitude). Such problems arise, in particular, in engineering practice, for example, when calculating the parameters of a multiphase water filter, modeling the migration of radioactive substances in the soil, etc. In [4], it is proposed the numerical method for finding double integrals, when the upper limit of the inner integral is a linear function of the type $x'' = x'$ (x' and x'' are the variables of integration), and the upper limit of the outer integral depends on the outer variable, then the region of integration is a triangle, the area of which increases with the increase of the outer variable. In addition, at approximate calculating of integrals, an important characteristic is the estimation of the error of the corresponding method. In the case of constant limits of integration, the estimations of integration methods are proposed, in particular, in works [5, 10, 12]. However, in the case of the variable region of integration, similar estimations were not found.

Therefore, in this work, we propose establishing the estimations for the approximate finding of double integrals with variable upper limits using two different approaches, when substituting the integrand with an approximating function and using the sequential integration approach.

1 Determination of the error of the approximate finding of the double integral by the approach of substituting the integrand with an approximating function

Consider the double integral with variable upper limits of integration

$$I(x) = \int_0^x \int_0^{g(x')} f\left(x, x', x''\right) \, dx'' \, dx', \tag{5}$$

where the integrand is $(n+1)$ times continuously differentiable function with respect to the variables x' and x'', the function of the upper limit of the inner integral $g\left(x'\right)$ is continuous with respect to the variable x'. The dependence of the integrand $f\left(x, x', x''\right)$ on the external variable is considered as the dependence on the parameter.

Let us apply the first approach to finding the approximate integral (5) over the variable region $\big(V(x)\big) = \left[0; x\right] \times \left[0; g(x)\right]$, $x \in [0; X]$, $X < \infty$, namely

$$I(x) = \int_0^x \int_0^{g(x')} f\left(x, x', x''\right) \, dx'' \, dx' \approx S(x)\, f\big(x, x_c'(x), x_c''(x)\big),$$

where $S(x)$ is the area of the variable region of integration $\big(V(x)\big)$, $\big(x_c'(x), x_c''(x)\big) \in \big(V(x)\big)$. Note that the point $\big(x_c'(x), x_c''(x)\big)$ is the middle point (centroid) of the region (V), which can shift with change of x.

We can estimate the numerical integration error as the difference between the value of the double integral with variable upper limits and the product of the variable area of the integration region by the value of the integrand at the central point (x, x_c', x_c'') of the integration region

$$R(x) = \int_0^x \int_0^{g(x')} f\left(x, x', x''\right) \, dx'' \, dx' - S(x)\, f\big(x, x_c'(x), x_c''(x)\big). \tag{6}$$

In order to apply the approach to estimating the error by expanding the integrand into a Taylor series, we consider that the function $g\left(x'\right)$ is continuous on the interval $\left[0; g(x)\right]$, and therefore bounded. That is, there exists a number $g_{max} = \max\limits_{x \in [0;X]} \max\limits_{x' \in [0;x]} g\left(x'\right)$ such that the variable region of integration $\big(V(x)\big)$ is always a subregion of $(V_{max}) = [0; x] \times [0; g_{max}]$. Then

$$\left|R(x)\right| \le \left| \int_0^x \int_0^{g_{max}} f\left(x, x', x''\right) \, dx'' \, dx' - S_{max}(x)\, f\big(x, x_c'(x), x_c''(x)\big) \right|, \tag{7}$$

where $S_{max}(x)$ is the area of the region (V_{max}).

Since the integrand $f\left(x, x', x''\right)$ has continuous derivatives up to $(n+1)$th order inclusively, we can expand it into a Taylor series by the variables x' and x'' at the point (x_c', x_c''), namely

$$\begin{aligned} f\left(x, x', x''\right) = f\left(x, x_c', x_c''\right) &+ \frac{\partial f\left(x, x_c', x_c''\right)}{\partial x'}(x' - x_c') + \frac{\partial f\left(x, x_c', x_c''\right)}{\partial x''}\left(x'' - x_c''\right) \\ &+ \frac{1}{2}\frac{\partial^2 f(x, x_c', x_c'')}{\partial x'^2}(x' - x_c')^2 + \frac{\partial^2 f(x, x_c', x_c'')}{\partial x' \partial x''}(x' - x_c')(x'' - x_c'') \\ &+ \frac{1}{2}\frac{\partial^2 f\left(x, x_c', x_c''\right)}{\partial x''^2}\left(x'' - x_c''\right)^2 + \dots . \end{aligned} \tag{8}$$

Substitute the expansion (8) into the formula (7) and obtain the main term of the error

$$\begin{aligned} \left|R(x)\right| &\le \left| \frac{\partial f\left(x, x_c', x_c''\right)}{\partial x'} \frac{g_m}{2}\, x\left(x - 2x_c'\right) + \frac{\partial f\left(x, x_c', x_c''\right)}{\partial x''} \frac{g_m}{2}\, x\left(g_m - 2x_c''\right) \right| \\ &\le \frac{g_m}{2}\, x \left| M_1'(x - 2x_c') + M_1''(g_m - 2x_c'') \right|, \end{aligned} \tag{9}$$

where $M_1' = \max\limits_{x\in[0;X]} \left|\dfrac{\partial f\left(x, x_c', x_c''\right)}{\partial x'}\right|$, $M_1'' = \max\limits_{x\in[0;X]} \left|\dfrac{\partial f\left(x, x_c', x_c''\right)}{\partial x''}\right|$.

If $x = 2x_c'$ and together $g_m = 2x_c''$, for example, we have a rectangular region of integration, then the right-hand side of the inequality (9) is equal to zero, and the main term of the error takes the form

$$\left|R(x)\right| \leq \left|\frac{1}{24}\, g_m x^3\, \frac{\partial^2 f\left(x, x_c', x_c''\right)}{\partial x'^{\,2}} + \frac{1}{24}\, x g_m^3\, \frac{\partial^2 f\left(x, x_c', x_c''\right)}{\partial x''^{\,2}}\right| \leq \frac{1}{24}\, x g_m \left|M_2' x^2 + M_2'' g_m^2\right|, \quad (10)$$

where

$$M_2' = \max_{x\in[0;X]} \left|\frac{\partial^2 f\left(x, x_c', x_c''\right)}{\partial x'^{\,2}}\right|, \quad M_2'' = \max_{x\in[0;X]} \left|\frac{\partial^2 f\left(x, x_c', x_c''\right)}{\partial x''^{\,2}}\right|. \quad (11)$$

Thus, we have proved the following result.

Theorem 1. *If the integrand $f\left(x, x', x''\right)$ is twice continuously differentiable function with respect to the variables x' and x'', $f\left(x, x', x''\right) \in C^2_{(V)}$, the function $g\left(x'\right) \in C_{\left[0;g(x)\right]}$ is continuous and $g_{max} = \max\limits_{x\in[0;X]} \max\limits_{x'\in[0;x]} g\left(x'\right)$, then the estimation for the main term of the error of the approximate integration by the approach of substituting the integrand by the approximating function is $\left|R(x)\right| = O\left(x^2 g_m\right) + O\left(x g_m^2\right)$, and for the symmetric region $\left(V(x)\right)$ it is $\left|R(x)\right| = O\left(x^3 g_m\right) + O\left(x g_m^3\right)$.*

In the general case, the values x and g_m are not small, so the main term of the error (9) or (10) can take large values. To increase the accuracy of calculations, they overlay a grid in the region $\left(V(x)\right)$, which divides the region of integration into elementary elements. Estimating of the total error for approximating finding the double integral with variable upper limits by overlaying different types of variability of the rectangular grid for the variable region of integration is presented in Section 3.

2 Determination of the error of the approximate finding of the double integral by the approach of sequential integration

Let us consider application of the second approach to approximate finding the integral (5) over the variable region (V), i.e. we consider the case of sequential integration.

Let the integrand $f\left(x, x', x''\right)$ and the function $g\left(x'\right)$ of the upper limit of the inner integral are $(n+1)$ times continuously differential.

We can present the desired double integral with the variable upper limits as

$$I(x) = \int_0^x \int_0^{g(x')} f\left(x, x', x''\right)\, dx''\, dx' = \int_0^x F\left(x, x'\right)\, dx', \quad (12)$$

where the function $F\left(x, x'\right)$ in turn is also integral

$$F\left(x, x'\right) = \int_0^{g(x')} f\left(x, x', x''\right)\, dx''.$$

For each fixed value of x, we can consider the function $F\left(x, x'\right)$ as a function of one variable (that is, we can interpret the variable x as a parameter).

Let the function $F(x,x')$ have continuous partial derivatives up to $(n+1)$th order, then it can be expanded [8] into a Taylor series around certain point $x' = x'_c$, namely

$$F\left(x,x'\right) = F\left(x,x'_c\right) + \left.\frac{\partial F\left(x,x'\right)}{\partial x'}\right|_{x'=x'_c}\left(x'-x'_c\right) + \left.\frac{\partial^2 F\left(x,x'\right)}{\partial x'^2}\right|_{x'=x'_c}\left(x'-x'_c\right)^2 + \dots$$
$$= \sum_{k=0}^{n}\frac{1}{k!}\left.\frac{\partial^{(k)} F\left(x,x'\right)}{\partial x'^k}\right|_{x'=x'_c}\left(x'-x'_c\right)^k + r_{n+1}\left(x,x'\right). \tag{13}$$

Here $r_{n+1}(x,x')$ is the remainder term of the Taylor series in Lagrange form [8, 14], presented as

$$r_{n+1}\left(x,x'\right) = \frac{\left(x'-x'_c\right)^{n+1}}{(n+1)!}\left.\left(\frac{\partial^{(n+1)} F\left(x,x'\right)}{\partial x'^{\,n+1}}\right)\right|_{x'=x'_c+\theta(x'-x'_c)},$$

where $0 < \theta < 1$.

To find the partial derivatives of the function $F(x,x')$, we use Leibniz's rule of parameter differentiation [8]. Then we obtain

$$\frac{\partial F(x,x')}{\partial x'} = \frac{\partial}{\partial x'}\left(\int_0^{g(x')} f(x,x',x'')\,dx''\right) = \int_0^{g(x')}\frac{\partial f(x,x',x'')}{\partial x'}\,dx'' + f(x,x',g(x'))\,\frac{dg(x')}{dx'},$$

$$\frac{\partial^2 F\left(x,x'\right)}{\partial x'^2} = \frac{\partial}{\partial x'}\left(\int_0^{g(x')}\frac{\partial f\left(x,x',x''\right)}{\partial x'}\,dx'' + f\left(x,x',g\left(x'\right)\right)\frac{dg\left(x'\right)}{dx'}\right)$$
$$= \int_0^{g(x')}\frac{\partial^2 f\left(x,x',x''\right)}{\partial x'^2}\,dx'' + 2\,\frac{\partial f\left(x,x',x''\right)}{\partial x'}\,\frac{dg\left(x'\right)}{dx'} + f\left(x,x',x''\right)\frac{d^2 g\left(x'\right)}{dx'^2}$$

and so on.

As a result, we present the Taylor series (13) as

$$F(x,x') = \int_0^{g(x'_c)} f(x,x'_c,x'')\,dx''$$
$$+\left(\int_0^{g(x'_c)}\left.\frac{\partial f\left(x,x',x''\right)}{\partial x'}\right|_{x'=x'_c}dx'' + f\left(x,x'_c,g\left(x'_c\right)\right)\left.\frac{dg\left(x'\right)}{dx'}\right|_{x'=x'_c}\right)\left(x'-x'_c\right)$$
$$+\left(\int_0^{g(x'_c)}\left.\frac{\partial^2 f\left(x,x',x''\right)}{\partial x'^2}\right|_{x'=x'_c}dx'' + 2\,\frac{\partial^2 f\left(x,x',x''\right)}{\partial x'^2}\left.\frac{dg\left(x'\right)}{dx'}\right|_{x'=x'_c}\right.$$
$$\left.+\frac{\partial f\left(x,x',x''\right)}{\partial x'}\left.\frac{d^2 g\left(x'\right)}{dx'^2}\right|_{x'=x'_c}\right)\left(x'-x'_c\right)^2 + \dots. \tag{14}$$

Restricting ourselves to the first two terms of the Taylor series, i.e. assuming $n = 1$, we substitute the expression (14) into the formula (12) and integrate the obtained expressions with respect to the variable x'. Then we obtain the following relation

$$I(x) = I_0(x) + I_1(x) + R_T(x), \tag{15}$$

where

$$I_0(x) = \int_0^x F\left(x,x'_c\right)\,dx' = x\int_0^{g(x'_c)} f\left(x,x'_c,x''\right)\,dx'', \tag{16}$$

$$I_1(x) = \int_0^x\left.\frac{\partial F\left(x,x'\right)}{\partial x'}\right|_{x'=x'_c}\left(x'-x'_c\right)\,dx' = \left.\frac{\partial F\left(x,x'\right)}{\partial x'}\right|_{x'=x'_c}\left(\frac{x^2}{2} - x'_c x\right), \tag{17}$$

$$R_T(x) = \int_0^x r_2\left(x, x'\right)\, dx'. \tag{18}$$

Taking into account the formula for the remainder term of the Taylor series, we get

$$\begin{aligned} R_T(x) &= \frac{1}{2}\left.\frac{\partial^2 F\left(x, x'\right)}{\partial x'^{\,2}}\right|_{x'=x'_c+\theta(x'-x'_c)} \int_0^x \left(x' - x'_c\right)^2 dx' \\ &= \frac{1}{2}\Bigg(\int_0^{g(\overline{x}'_c)} \left.\frac{\partial^2 f\left(x, x', x''\right)}{\partial x'^{\,2}}\right|_{x'=\overline{x}'_c} dx'' + 2\,\frac{\partial f\left(x, x', g\left(x'\right)\right)}{\partial x'}\left.\frac{dg\left(x'\right)}{dx'}\right|_{x'=\overline{x}'_c} \\ &\quad + f\left(x, x', g\left(x'\right)\right)\left.\frac{d^2 g\left(x'\right)}{dx'^{\,2}}\right|_{x'=\overline{x}'_c}\Bigg)\left(\frac{x^3}{3} - x^2 x'_c + x x'^2_c\right), \end{aligned} \tag{19}$$

where $\overline{x}'_c = x'_c + \theta\left(x' - x'_c\right)$.

Since the function $f\left(x, x'_c, x''\right)$ has continuous partial derivatives up to $(n+1)$th order, then in the expression (16) we expand the function into the Taylor series at the point $x'' = x''_c$ (there is no such need in expressions (17) and (18), since these expressions have been integrated)

$$\begin{aligned} f\left(x, x'_c, x''\right) &= f\left(x, x'_c, x''_c\right) + \left.\frac{\partial f\left(x, x'_c, x''\right)}{\partial x''}\right|_{x''=x''_c}\left(x'' - x''_c\right) \\ &\qquad + \left.\frac{\partial^2 f\left(x, x'_c, x''\right)}{\partial x''^{\,2}}\right|_{x''=x''_c}\left(x'' - x''_c\right)^2 + \ldots \\ &= \sum_{k=0}^{n}\frac{1}{k!}\left.\frac{\partial^{(k)} f\left(x, x'_c, x''\right)}{\partial x''^{\,k}}\right|_{x''=x''_c}\left(x'' - x''_c\right)^k + \overline{r}_{n+1}\left(x, x'_c, x''\right). \end{aligned} \tag{20}$$

Here the remainder term of the Taylor series in the Lagrangian form [8] is determined as

$$\overline{r}_{n+1}\left(x, x'_c, x''\right) \equiv \overline{r}_{n+1}\left(x, x''\right) = \frac{\left(x'' - x''_c\right)^{n+1}}{(n+1)!}\left.\left(\frac{\partial^{(n+1)} f\left(x, x'_c, x''\right)}{\partial x''^{\,n+1}}\right)\right|_{x''=x''_c+\overline{\theta}(x''-x''_c)}.$$

Substituting the expression (20) into relation (16) we obtain

$$\begin{aligned} I_0(x) &= x\int_0^{g(x'_c)} f\left(x, x'_c, x''\right) dx'' \\ &= x\Bigg[g(x'_c) f(x, x'_c, x''_c) + \left.\frac{\partial f\left(x, x'_c, x''\right)}{\partial x''}\right|_{x''=x''_c}\left(\frac{g(x'_c)^2}{2} - x''_c g(x'_c)\right) \\ &\quad + \int_0^{g(x'_c)} \overline{r}_2\left(x, x''\right) dx''\Bigg]. \end{aligned} \tag{21}$$

Here

$$\int_0^{(x'_c)} \overline{r}_2\left(x, x''\right) dx'' = \frac{1}{2}\left.\frac{\partial^2 f\left(x, x'_c, x''\right)}{\partial x''^2}\right|_{x''=\overline{x}''_c}\left(\frac{g\left(x'_c\right)^3}{3} - g\left(x'_c\right)^2\overline{x}''_c + g\left(x'_c\right)\overline{x}''^2_c\right), \tag{22}$$

where $\overline{x}''_c = x''_c + \overline{\theta}\left(x'' - x''_c\right)$.

The remainder term of the expansion of the function $f\left(x, x'_c, x''\right)$ into the Taylor series (21) at the point $x'' = x''_c$ is referred to the summand $R_T(x)$ of the expression (15), redenoting it as $\overline{R}_T(x)$, namely

$$\overline{R}_T(x) = \int_0^x r_2\left(x, x'\right) dx' + x\int_0^{g(x'_c)} \overline{r}_2\left(x, x''\right)\, dx''.$$

Allowing for the expressions (19) and (22), we have

$$\overline{R}_T(x) = \left(\int_0^{g(\overline{x}_c')} \frac{\partial^2 f\,(x, x', x'')}{\partial x'^{\,2}}\bigg|_{x'=\overline{x}_c'} dx'' + 2\frac{\partial f\big(x, x', g\,(x')\,\big)}{\partial x'}\,\frac{dg\,(x')}{dx'}\bigg|_{x'=\overline{x}_c'} \right.$$
$$\left. + f\big(x, x', g\,(x')\,\big)\,\frac{d^2 g\,(x')}{dx'^{\,2}}\bigg|_{x'=\overline{x}_c'} \right)\left(\frac{x^3}{6} - \frac{x^2}{2}x_c' + \frac{x}{2}x_c'^2\right)$$
$$+ \frac{x}{2}\,\frac{\partial^2 f\,(x, x_c', x'')}{\partial x''^2}\bigg|_{x''=\overline{x}_c''}\left(\frac{g\,(x_c')^3}{3} - g\,(x_c')^2\,\overline{x}_c'' + g\,(x_c')\,\overline{x}_c''^2\right).$$

Represent the expression (6) in the form $R(x) = I(x) - S\,(x, x_c', x_c'')\; f\,(x, x_c', x_c'')$, into which we substitute the value $I(x)$ (see (15)). Then we obtain

$$R(x) = I_0(x) + I_1(x) + \overline{R}_T(x) - S\,\big(x, x_c', x_c''\big)\; f\,\big(x, x_c', x_c''\big)\,. \tag{23}$$

With provision for the expression (21) for $I_0(x)$, relation (23) is written as

$$R(x) = x\,g\,\big(x_c'\big)\,f\,\big(x, x_c', x_c''\big) + x\,\frac{\partial f\,(x, x_c', x'')}{\partial x''}\bigg|_{x''=x_c''}\left(\frac{g(x_c')^2}{2} - x_c'' g(x_c')\right)$$
$$+ I_1(x) + \overline{R}_T(x) - S\,\big(x, x_c', x_c''\big)\; f\,\big(x, x_c', x_c''\big)$$

or

$$R(x) = x\,\frac{\partial f\,(x, x_c', x'')}{\partial x''}\bigg|_{x''=x_c''}\left(\frac{g(x_c')^2}{2} - x_c'' g(x_c')\right) + I_1(x) + \overline{R}_T(x).$$

Then with regard to the denotation (11), we can write the estimation of the error as follows

$$\big|R(x)\big| \le \left|x\,M_1''\left(\frac{g(x_c')^2}{2} - x_c'' g(x_c')\right) + I_1(x) + \overline{R}_T(x)\right|.$$

Determine the upper-bound estimation of the function $I_1(x)$ by

$$\big|I_1(x)\big| \le \left|\Big(M_{1g}'\,g(x_c') + M_f G\Big)\left(\frac{x^2}{2} - x_c' x\right)\right|.$$

As a result, the main term of the error takes the form

$$\big|R(x)\big| \le x\left|M_1''\,g(x_c')\Big(\frac{g(x_c')}{2} - x_c''\Big) + \Big(M_{1g}'\,g(x_c') + M_f G\Big)\,\Big(\frac{x}{2} - x_c'\Big)\right|,$$

where $M_{1g}' = \max\limits_{x\in[0;X]}\;\max\limits_{x''\in[0;g(x_c')]}\left|\dfrac{\partial f(x, x_c', x'')}{\partial x'}\right|$, $M_f' = \max\limits_{x\in[0;X]}\big|f(x, x_c', g(x_c'))\big|$, $G = \dfrac{dg(x')}{dx'}\bigg|_{x'=x_c'}$.

Thus, we have proved the following result.

Theorem 2. *If the integrand $f\,(x, x', x'')$ and the function $g\,(x')$ is twice continuously differentiable functions with respect to variables x' and x'' for f and x' for g, $f\,(x, x', x'') \in C^2_{(V)}$ and $g\,(x') \in C^2_{[0;g(x)]}$, and x_c' is the abscissa of the centroid of the region $\big(V(x)\big)$, then the estimation for the main term of the error of the approximate integration by the approach of sequential integration is $\big|R(x)\big| = O\,\big(x^2 g\,(x_c')\big) + O\left(x\big(g\,(x_c')\,\big)^2\right)$.*

3 The error of the approximate finding of the double integral with variable upper limits when the integration region is divided by a rectangular grid

3.1 Division of the variable integration region by a rectangular grid

In the general case, $S(x)$ with $x \in [0; X]$, $X < \infty$, is the variable region of integration. We consider only methods of numerical integration, which are based on the division of the integration region by a rectangular grid into elementary elements [7] (the methods of cubatures, in particular, of rectangles, Simpson, Gauss, etc.). Let us overlay a rectangular grid on the integration region, which contains N elements along the Ox'-axis and M elements along the Ox''-axis. Figure 1 shows the division of the variable integration region by square grid in the case of linear $g(x') = ax'$, $a > 1$, (Figure 1**a**) and parabolic $g(x') = x'^2$ (Figure 1**b**) functions that determine the upper limit of integration. Here x_1 and x_2, where $x_1 < x_2$, are certain fixed values of the variable x', $g(x_1)$ and $g(x_2)$ are the corresponding values on the Ox''-axis. Green dots mark the centroids $\left(x, x', x''\right) = \left(x, x_c'^{(i)}, x_c''^{(j)}\right)$ of the corresponding elementary elements of the division. Note that in Figure 1, the blue grid highlights the additional region of integration, which was formed as a result of the increase of the variable x' from x_1 to x_2.

So, the approximate value of the double integral with the variable upper limits can be given as [11]

$$S(x)\, f\left(x, x_c', x_c''\right) = \sum_{i=1}^{N}\sum_{j=1}^{M} S_{ij}(x)\, f\left(x, x_c'^{(i)}, x_c''^{(j)}\right),$$

where $S_{ij}(x)$ is the area of the ijth element of the division of integration region by the grid, $\left(x, x_c'^{(i)}, x_c''^{(j)}\right)$ is the corresponding central point (centroid) of this element.

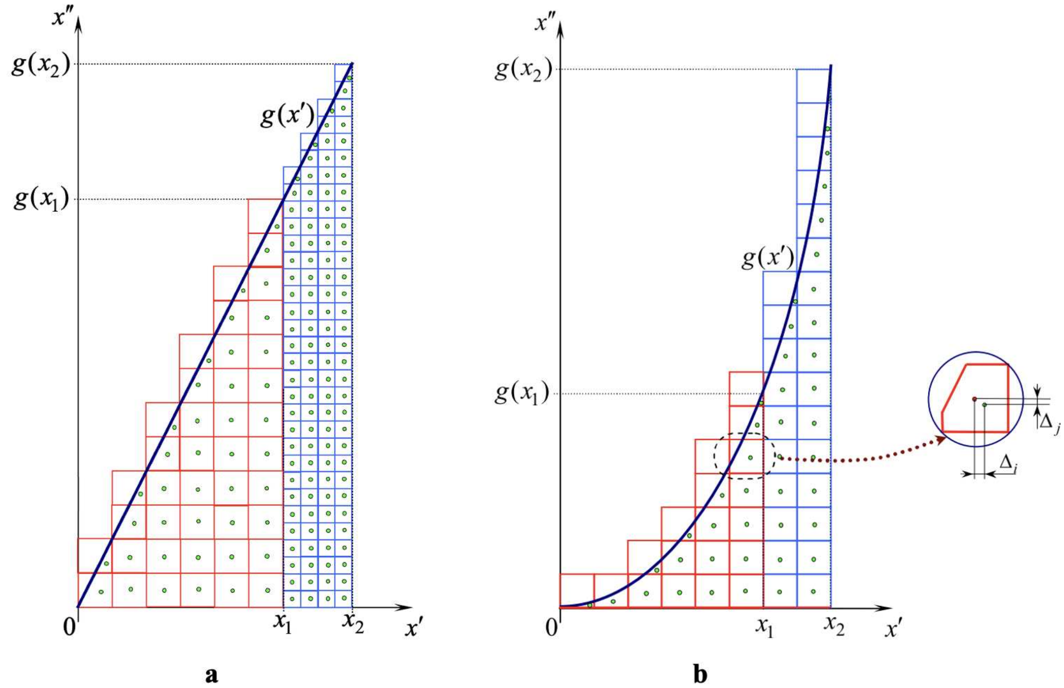

Figure 1. Scheme of the division of integration region by square grid for linear (**a**) and parabolic (**b**) functions $g\left(x'\right)$

As the value $x \in [0; X]$ increases, the integration region $S(x)$ also increases (blue grid over the integration region in Figure 1). Consider the following options for dividing the integration region with a rectangular grid into elementary elements.

1. With the change x, the number of division elements N and M changes, i.e. $N = N(x)$ and $M = M(x)$, and we fix the sizes of the steps $h_{x'}$ and $h_{x''}$ along the Ox' and Ox'' axes (Figure 1**b**). Then the number of elements along the Ox'-axis is determined as $\left[\dfrac{x}{h_{x'}}\right]$, and along the Ox''-axis as $\left[\dfrac{g(x)}{h_{x''}}\right]$, where $[\ \cdot\ ]$ is an integer part of the number. So, in the general case, we can write
$$\int_0^x \int_0^{g(x')} f\left(x, x', x''\right) dx''\, dx' \approx \sum_{el}^{N_{el}(x)} \iint_{(V_{el})} f\left(x, x', x''\right) dx''\, dx', \tag{24}$$
where N_{el} is the number of elementary elements of the grid of the integration region division, (V_{el}) is the elementary element.

2. Change in x leads to changing the length of steps $h_{x'}$ and $h_{x''}$, i.e. $h_{x'} = h_{x'}(x)$ and $h_{x''} = h_{x''}(x)$, and at the same time the number of elements N and M remains unchanged. Note that new values of length and width of the grid are determined as $h_{x'}(x) = \dfrac{x}{N}$, $h_{x''}(x) = \dfrac{g(x)}{M}$. For this case we have
$$\int_0^x \int_0^{g(x')} f\left(x, x', x''\right) dx''\, dx' \approx \sum_{el}^{N_{el}} \iint_{(V_{el}(x))} f\left(x, x', x''\right) dx''\, dx'. \tag{25}$$

3. With the change of x, both the number of division elements and the size of the step may change, but at the same time the ratios $h_{x'}(x) = \dfrac{x}{[N(x)]}$ and $h_{x''}(x) = \dfrac{g(x)}{[M(x)]}$ hold true. That is
$$\int_0^x \int_0^{g(x')} f\left(x, x', x''\right) dx''\, dx' \approx \sum_{el}^{N_{el}(x)} \iint_{(V_{el}(x))} f\left(x, x', x''\right) dx''\, dx'. \tag{26}$$

4. With the increase of the integration region $S(x)$ due to the increase in x, each "additional" kth integration region is divided by its grid with the characteristics $h_{x'}^{[k]}$, N_k along the Ox'-axis and $h_{x''}^{[k]}$, M_k along the Ox''-axis $\left(k = \overline{1; [K(x)]}\right)$ (see Figure 1**a**). In the general case, we obtain
$$\int_0^x \int_0^{g(x')} f\left(x, x', x''\right) dx''\, dx' \approx \sum_{k=1}^{[K(x)]} \sum_{el}^{N_{el}^{[k]}(x)} \iint_{\left(V_{el}^{[k]}(x)\right)} f\left(x, x', x''\right) dx''\, dx'.$$

The region of integration $S(x)$ directly depends on the nature of the function $g\left(x'\right)$, but we can single out typical elements into which it is divided due to the overlaying of the rectangular grid. These are rectangles, triangles, rectangular trapezoids, pentagons (for example, for a linear function $g\left(x'\right)$, Figure 1**a**) and curvilinear trapezoids, which can be approximated by a triangle, rectangular trapezoid or pentagon (for example, for a parabolic function $g(x')$, Figure 1**b**). Note that all such polygons are convex. For simplification, we single out rectangular grid elements $(V_{rect_{el}})$ that are wholly in the integration region, and polygonal elements $(V_{poly_{el}})$ that are formed from rectangular grid elements that intersect with the curve $g(x')$. Then we obtain

$$\sum_{el}^{N_{el}(x)} \iint_{(V_{el}(x))} f\left(x,x',x''\right)\,dx''\,dx' = \sum_{rect_{el}=1}^{N_{rect_{el}}(x)} \iint_{\left(V_{rect_{el}}(x)\right)} f\left(x,x',x''\right)\,dx''\,dx' + \sum_{poly_{el}=1}^{N_{poly_{el}}(x)} \iint_{\left(V_{poly_{el}}(x)\right)} f\left(x,x',x''\right)\,dx''\,dx'. \qquad (27)$$

Here $N_{rect_{el}}(x)$ and $N_{poly_{el}}(x)$ are the numbers of rectangular and polygonal elements that were formed due to overlaying of the rectangular grid over the integration region.

In the case of rectangular elements, there may not be a rectangular element for certain j along the Ox'-axis (for example, the element (V_{11}) for the parabolic function $g(x')$ or for $h_{x''}(x) << h_{x'}(x)$), then we put $S_{ij}(x) = 0$. One element or several rectangular elements can also be formed along the Ox'-axis. So, we have

$$\begin{aligned}\iint_{\bigcup\limits_{rect_{el}}(V_{el})} f\left(x,x',x''\right)\,dx''\,dx' &= \sum_{rect_{el}=1}^{N_{rect_{el}}(x)} \iint_{\left(V_{rect_{el}}(x)\right)} f\left(x,x',x''\right)\,dx''\,dx' \\ &= \sum_{i=1}^{N_{rect}(x)} \sum_{j=1}^{M_{rect}(x)} \iint_{\left(V_{rect_{el}}^{(ij)}(x)\right)} f\left(x,x',x''\right)\,dx''\,dx' \\ &= h_{x'}(x)h_{x''}(x) \sum_{i=1}^{N_{rect}(x)} \sum_{j=1}^{M_{rect}(x)} f\left(x, x_c'^{(i)}, x_c''^{(j)}\right),\end{aligned} \qquad (28)$$

where $N_{rect}(x)$ is the number of rectangular elements along the Ox'-axis bounded by the curve $g(x')$, $M_{rect}(x)$ is the number of rectangular elements along the Ox''-axis.

Herewith, in the option 1 (when the number of division elements $N(x)$ and $M(x)$ changes, and the steps $h_{x'}$ and $h_{x''}$ are constants) with the change of x the number $N_{rect}(x)$ changes stepwise, and the area of the last element is $0 \le S_{ie}(x) \le S_{ij} = const$. The same dependence is observed for the option 3 for the variable grid steps $h_{x'}(x)$ and $h_{x''}(x)$.

In the option 2 we have

$$\iint_{\bigcup\limits_{rect_{el}}(V_{el})} f\left(x,x',x''\right)\,dx''\,dx' = h_{x'}(x)h_{x''}(x) \sum_{i=1}^{N_{rect}} \sum_{j=1}^{M_{rect}} f\left(x, x'_{c(i)}, x''_{c(j)}\right).$$

For the option 4 the formula (28) is specified as follows

$$\begin{aligned}\iint_{\bigcup\limits_{rect_{el}}(V_{el})} f\left(x,x',x''\right)\,dx''\,dx' &= \sum_{k=1}^{[K(x)]} \sum_{i=1}^{N_{rect}^{[k]}} \sum_{j=1}^{M_{rect}^{[k]}} \iint_{\left(V_{rect}^{(ij)[k]}(x)\right)} f\left(x,x',x''\right)\,dx''\,dx' \\ &= \sum_{k=1}^{[K(x)]} h_{x'}^{[k]} h_{x''}^{[k]} \sum_{i=1}^{N_{rect}^{[k]}} \sum_{j=1}^{M_{rect}^{[k]}} f\left(x, x'_{c(i)}, x''_{c(j)}\right).\end{aligned}$$

In the case of polygonal (non-rectangular) elements along the Ox''-axis, there may not be a curvilinear element for certain i, then we put $S_{ij}^{(n)}(x) = 0$, one element or several elements, if $h_{x'}(x) << h_{x''}(x)$. We denote the number of polygonal elements along the Ox'-axis by $N_{poly}(x)$. Then we obtain

$$\begin{aligned}\iint_{\bigcup\limits_{poly_{el}}(V_{el})} f\left(x, x', x''\right)\, dx''\, dx' &= \sum_{poly_{el}=1}^{N_{poly_{el}}(x)} \iint_{\left(V_{poly_{el}}(x)\right)} f\left(x, x', x''\right)\, dx''\, dx' \\ &= \sum_{i=1}^{N_{poly}(x)} \sum_{j=1}^{M_{poly}(x)} \iint_{\left(V_{poly_{el}}^{(ij)}(x)\right)} f\left(x, x', x''\right)\, dx''\, dx' \\ &= \sum_{i=1}^{N_{poly}(x)} \sum_{j=1}^{M_{poly}(x)} S_{ij}^{(n)}(x)\, f\left(x, x'_{c(i)}, x''_{c(j)}\right),\end{aligned} \tag{29}$$

where $S_{ij}^{(n)}(x)$ is the area of the corresponding polygon (triangle, trapezoid, etc.), $\left(x, x'_{c(i)}, x''_{c(j)}\right)$ is the centroid of this figure.

Moreover $N_{rect}(x) + N_{poly}(x) = N(x)$, $M_{rect}(x) + M_{poly}(x) = M(x)$.

We find the area of the convex polygon as

$$S_{ij}^{(n)} = \frac{1}{2}\left|\sum_{l=1}^{n}\left(x_l'^{(i)} + x_{l+1}'^{(i)}\right)\left(x_l''^{(j)} - x_{l+1}''^{(j)}\right)\right|, \tag{30}$$

where n is the number of vertices of the polygon, $\left(x_l'^{(i)}, x_l''^{(j)}\right)$ are the coordinates of the neighboring vertices of the polygon, which are bypassed in turn to avoid self-intersections, and $\left(x_{n+1}'^{(i)}, x_{n+1}''^{(j)}\right) = \left(x_1'^{(i)}, x_1''^{(j)}\right)$. This representation of the area of a polygon is convenient when algorithmically determining which type of polygon was formed. For example, if the length of one of the sides of a polygon equals zero, then the number of its vertices is reduced by one. We can also estimate the area of the convex polygon by the area of the circle circumscribed around this polygon, namely

$$S_{ij}^{(n)}(x) < S_{ij}^{c}(x) = \pi\left(\overline{R}_{ij}(x)\right)^2 = \frac{\pi}{4}\alpha_{ij}^2\left(h_{max}(x)\right)^2, \tag{31}$$

where $\overline{R}_{ij}(x) = \alpha_{ij}h_{max}(x)/2$, $h_{max}(x) = \max\left\{h_{x'}(x), h_{x''}(x)\right\}$, $0 < \alpha_{ij} \le 1$.

Here, we consider the circle of minimum area that wholly contains the polygon (Figure 1**b**) as a circle circumscribed around the polygon. With that not all vertices of a polygon can lie on this circle.

Then we have

$$\begin{aligned}\iint_{\bigcup\limits_{poly_{el}}(V_{el})} f\left(x, x', x''\right)\, dx''\, dx' &= \sum_{poly_{el}=1}^{N_{poly_{el}}(x)} \iint_{\left(V_{poly_{el}}\right)} f\left(x, x', x''\right)\, dx''\, dx' \\ &< \frac{\pi h_{max}^2}{4} \sum_{i=1}^{N_{poly}(x)} \sum_{j=1}^{M_{poly}(x)} \alpha_{ij}^2 f\left(x, x'_{c(i)}, x''_{c(j)}\right).\end{aligned}$$

Note that $N_{poly}(x) = N(x) - N_{rect}(x)$, $M_{poly}(x) = M(x) - M_{rect}(x)$.

3.2 Estimation of the error for rectangular elements of the integration region

We search the error for each elementary element by expanding the function $f\left(x, x', x''\right)$ into the Taylor series at the centroid $\left(x, x_c'^{(i)}, x_c''^{(j)}\right)$ of this element.

If the element ij completely lies in the integration region and it is the rectangle with the area $S_{ij}(x)$, moreover $S_{ij}(x) = h_{x'}(x)h_{x''}(x)$ and $S_{ij}(x) \equiv S_{ij} = h_{x'}h_{x''}$ under constant steps, then

$$\begin{aligned}
\left|R_{ij}(x)\right| &= \left| \iint_{\left(V_{el}^{(ij)}(x)\right)} f\left(x, x', x''\right)\, dx''\, dx' - S_{ij}(x)\, f\left(x, x_c'^{(i)}, x_c''^{(j)}\right) \right| \\
&= \left| \int_{x'^{(i)}}^{x'^{(i)}+h_{x'}(x)} \int_{x''^{(j)}}^{x''^{(j)}+h_{x''}(x)} f\left(x, x', x''\right)\, dx''\, dx' - h_{x'}(x)h_{x''}(x) f\left(x, x_c'^{(i)}, x_c''^{(j)}\right) \right| \\
&= \left| \frac{h_{x'}(x)h_{x''}(x)}{24} \left(\left(h_{x'}(x)\right)^2 \frac{\partial^2 f\left(x, \overline{x}_c'^{(i)}, \overline{x}_c''^{(j)}\right)}{\partial x'^2} + \left(h_{x''}(x)\right)^2 \frac{\partial^2 f\left(x, \overline{x}_c'^{(i)}, \overline{x}_c''^{(j)}\right)}{\partial x''^2} \right) \right|.
\end{aligned}$$

Here $\left(x, \overline{x}_c'^{(i)}, \overline{x}_c''^{(j)}\right) = \left(x, x_c'^{(i)} + \theta\left(x'^{(i)} - x_c'^{(i)}\right), x_c''^{(j)} + \theta\left(x''^{(j)} - x_c''^{(j)}\right)\right)$.

In particular, under constant steps of the division of the integration region we have

$$\begin{aligned}
\left|R_{ij}\right| &= \left| \int_{x'^{(i)}}^{x'^{(i)}+h_{x'}} \int_{x''^{(j)}}^{x''^{(j)}+h_{x''}} f\left(x, x', x''\right)\, dx''\, dx' - h_{x'}h_{x''} f\left(x, x_c'^{(i)}, x_c''^{(j)}\right) \right| \\
&= \left| \frac{h_{x'}h_{x''}}{24} \left(h_{x'}^2 \frac{\partial^2 f\left(x, \overline{x}_c'^{(i)}, \overline{x}_c''^{(j)}\right)}{\partial x'^2} + h_{x''}^2 \frac{\partial^2 f\left(x, \overline{x}_c'^{(i)}, \overline{x}_c''^{(j)}\right)}{\partial x''^2} \right) \right|.
\end{aligned}$$

Denote

$$M_{n+1}(x) = M_2(x) = \max_{(x', x'') \in (V_{el}^{(ij)}(x))} \left\{ \left| \frac{\partial^2 f\left(x, x'^{(i)}, x''^{(j)}\right)}{\partial x'^2} \right|, \left| \frac{\partial^2 f\left(x, x'^{(i)}, x''^{(j)}\right)}{\partial x''^2} \right| \right\},$$

and $M_{max2} = \max\limits_{x \in [0;X]} M_2(x)$. Then

$$\left|R_{ij}\right| \leq \frac{h_{x'}h_{x''}}{24} \left(h_{x'}^2 + h_{x''}^2\right) M_2(x) \leq \frac{h_{x'}h_{x''}}{24} \left(h_{x'}^2 + h_{x''}^2\right) M_{max2}.$$

If together with growth of x the length of the division step changes, then for the integration error on the element ij we have

$$\begin{aligned}
\left|R_{ij}(x)\right| &\leq \frac{h_{x'}(x)h_{x''}(x)}{24} \left(\left(h_{x'}(x)\right)^2 + \left(h_{x''}(x)\right)^2 \right) M_2(x) \\
&\leq \frac{h_{x'}(x)h_{x''}(x)}{24} \left(\left(h_{x'}(x)\right)^2 + \left(h_{x''}(x)\right)^2 \right) M_{max2}.
\end{aligned} \tag{32}$$

Note that the error R_{ij} depends on functional nature of $h_{x'}(x)$ and $h_{x''}(x)$. When x increases, the magnitudes $h_{x'}(x)$ and $h_{x''}(x)$ can both increase and decrease, and these functions are independent of each other. Denote $h_{max}(x) = \max\limits_{x \in [0;X]} \left\{h_{x'}(x); h_{x''}(x)\right\}$.

Then the estimation (32) takes the form

$$\left|R_{ij}(x)\right| \leq \frac{\left(h_{max}(x)\right)^4}{12} M_{max2}.$$

Summarize the estimation over all rectangular elements. Then for constant steps of the division of the integration region (the option 1), using formula (28) we obtain in the general case

$$
\begin{aligned}
|R_{rect}(x)| &\le \frac{h_{x'}h_{x''}}{24}\left(h_{x'}^2+h_{x''}^2\right)M_2(x)N_{rect}(x)M_{rect}(x)\\
&\le \frac{h_{x'}h_{x''}}{24}\left(h_{x'}^2+h_{x''}^2\right)M_{max2}(x)N_{rect}(x)M_{rect}(x)\le \frac{h_{max}^4}{12}M_{max2}N_{rect}(x)M_{rect}(x).
\end{aligned}
$$

For the option 2 of the division, the summation over all rectangular elements leads to the following inequality

$$
\left|R_{rect}(x)\right| \le \frac{h_{max}^4(x)}{12}M_{max2}N_{rect}M_{rect}.
$$

We obtain the summarized estimation of integration over rectangular elements in the option 3 of the division

$$
\left|R_{rect}(x)\right| \le \frac{h_{max}^4(x)}{12}M_{max2}N_{rect}(x)M_{rect}(x).
$$

In the option 4 we obtain the following estimation for the rectangular element

$$
\left|R_{ij}^{[k]}(x)\right| \le \frac{h_{x'}^{[k]}h_{x''}^{[k]}}{24}\left(\left(h_{x'}^{[k]}\right)^2+\left(h_{x''}^{[k]}\right)^2\right)M_{max2}^{[k]} \le \frac{\left(h_{max}^{[k]}(x)\right)^4}{12}M_{max2}^{[k]},
$$

where $M_{max2}^{[k]} = \max\limits_{x\in[0;X]} M_2^{[k]}(x)$, $h_{max}^{[k]} = \max\left\{h_{x'}^{[k]}, h_{x''}^{[k]}\right\}$.

Then the summarized estimation is

$$
\left|R_{rect}(x)\right| \le \sum_{k=1}^{\left[K(x)\right]} \frac{\left(h_{max}^{[k]}\right)^4}{12}M_{max2}^{[k]}N_{rect}^{[k]}M_{rect}^{[k]}(x),
$$

where in the last additional region $k = \left[K(x)\right]$ the number of rectangular elements $N_{rect}^{\left[K(x)\right]}$ and $M_{rect}^{\left[K(x)\right]}$ can depend on x.

3.3 Estimation of the error for curvilinear elements of the integration region

In the case of the curvilinear element, which is approximated by the polygon with the area $S_{ij}(x)$ determined by formula (30), we estimate it by the area $S_{ij}^c(x)$ of the circle circumscribed around the polygon (see (31)). For the option of constant steps of the division we have $S_{ij}^c = \pi R^2 = \frac{\pi}{4}\alpha_{ij}^2 h_{max}^2$. If there is no curvilinear element for certain value ij, we assume that $\alpha_{ij} = 0$. So, we have

$$
\begin{aligned}
\left|R_{ij}(x)\right| &= \left|\iint_{\left(V_{el}^{(ij)}(x)\right)} f\left(x,x',x''\right)\,dx''\,dx' - S_{ij}(x)\,f\left(x,x_c'^{(i)},x_c''^{(j)}\right)\right|\\
&\le \left|\iint_{\left(V_{circle}^{(ij)}(x)\right)} f\left(x,x',x''\right)\,dx''\,dx' - S_{ij}^c(x)\,f\left(x,x_c'^{(i)},x_c''^{(j)}\right)\right|.
\end{aligned} \tag{33}
$$

Let us expand the integrand into the Taylor series at the point $\left(x_c'^{(i)}, x_c''^{(j)}\right)$ as follows

$$
\begin{aligned}
&\iint_{(V_{circle}^{(ij)}(x))} f\left(x, x', x''\right)\, dx''\, dx' \\
&\quad = \iint_{(V_{circle}^{(ij)}(x))} \left(f(x, x_c'^{(i)}, x_c''^{(j)}) + \frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x'} (x' - x'_{c^{(i)}}) \right. \\
&\qquad \left. + \frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x''} (x'' - x''_{c^{(j)}}) + r_{ij}^{(2)}(x) \right) dx''\, dx',
\end{aligned} \tag{34}
$$

and cast out the terms of the series that have at least the second order of smallness with respect to $\left(x' - x'_{c^{(i)}}\right)$ and $\left(x'' - x''_{c^{(j)}}\right)$.

Substituting relation (34) into inequality (33), we get

$$
\begin{aligned}
\left|R_{ij}(x)\right| \leq \Bigg| \iint_{(V_{circle}^{(ij)}(x))} & \left(\frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x'} (x' - x'_{c^{(i)}}) \right. \\
& \left. + \frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x''} (x'' - x''_{c^{(j)}}) \right) dx''\, dx' \Bigg|.
\end{aligned} \tag{35}
$$

We assume that the coordinates of the center of the circle are shifted relative to the centroid of the polygon as follows $\left(x'^{(i)}_{circle}, x''^{(j)}_{circle}\right) = \left(x_c'^{(i)} - \Delta_i(x), x_c''^{(j)} - \Delta_j(x)\right)$.

In Figure 1**b**, it is highlighted separately a curved element approximated by a pentagon around which a circle is circumscribed. Here, the centroid of the pentagon is marked with a green dot, and the center of the corresponding circle of minimum radius, which wholly contains the pentagon, is marked with a red dot.

Taking into account that the center of the circle circumscribed around the polygon may not coincide with its centroid, we displace the origin of the coordinate system to the center of the circle. Then we carry out the change of the variables $\tilde{x}' = x' - x'^{(i)}_{circle} = x' - \left(x_c'^{(i)} - \Delta_i\right)$, $\tilde{x}'' = x'' - x''^{(j)}_{circle} = x'' - \left(x_c''^{(j)} - \Delta_j\right)$ and change to the polar coordinate system with provision for $\tilde{x}' = \rho\cos\phi$, $\tilde{x}'' = \rho\sin\phi$. We obtain

$$
\begin{aligned}
\iint_{(V_0^{(ij)}(x))} \left(\tilde{x}' + \Delta_i\right)\, d\tilde{x}''\, d\tilde{x}' &= \int_0^{2\pi} \int_0^{\overline{R}_{ij}(x)} \rho\left(\rho\cos\phi + \Delta_i\right)\, d\rho\, d\phi \\
&= 2\pi\Delta_i \int_0^{\overline{R}_{ij}(x)} \rho\, d\rho = \pi\Delta_i\left(\overline{R}_{ij}(x)\right)^2 = \frac{\pi\Delta_i\alpha_{ij}^2}{4}\left(h_{max}(x)\right)^2, \\
\iint_{(V_0^{(ij)}(x))} \left(\tilde{x}'' + \Delta_j\right)\, d\tilde{x}''\, d\tilde{x}' &= \int_0^{2\pi} \int_0^{\overline{R}_{ij}(x)} \rho\left(\rho\sin\phi + \Delta_j\right)\, d\rho\, d\phi \\
&= 2\pi\Delta_j \int_0^{\overline{R}_{ij}(x)} \rho\, d\rho = \pi\Delta_j\left(\overline{R}_{ij}(x)\right)^2 = \frac{\pi\Delta_j\alpha_{ij}^2}{4}\left(h_{max}(x)\right)^2.
\end{aligned}
$$

Here $\overline{R}_{ij}(x) = \alpha_{ij}h_{max}(x)/2$, $h_{max}(x) = \max\left\{h_{x'}(x), h_{x''}(x)\right\}$.

Then we transform the expression (35) as follows

$$
\left|R_{ij}(x)\right| \leq \left| \frac{\pi\alpha_{ij}^2\left(h_{max}(x)\right)^2}{4} \left(\Delta_i(x)\frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x'} + \Delta_j(x)\frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x''} \right) \right|.
$$

Let us denote

$$M_{n+1}(x) = M_1(x) = \max_{(x',x'')\in(V_{el}^{(ij)}(x))} \left\{ \left|\frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x'}\right|, \left|\frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x''}\right| \right\}$$
$$= \max_{(x',x'')\in(V_{circle}^{(ij)}(x))} \left\{ \left|\frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x'}\right|, \left|\frac{\partial f(x, x_c'^{(i)}, x_c''^{(j)})}{\partial x''}\right| \right\},$$

$$M_{max1} = \max_{x\in[0;X]} M_1(x), \quad \Delta_{ij} = \max_{x\in[0;X]} \left\{\Delta_i(x), \Delta_j(x)\right\},$$
$$\alpha_{max} = \max_{(ij)=\overline{0;N_{poly_{el}}}} \alpha_{ij}, \quad M_{max} = \max_{x\in[0;X]} M_2(x).$$

Then we obtain the following estimation of integration for the curvilinear element

$$\left|R_{ij}(x)\right| \le \left|\frac{\pi\alpha_{ij}^2 \left(h_{max}(x)\right)^2}{4} M_1(x)\left(\Delta_i(x) + \Delta_j(x)\right)\right| \le \left|\frac{\pi\alpha_{max}^2 \left(h_{max}(x)\right)^2}{2} M_{max1}\Delta_{ij}\right|. \quad (36)$$

Summarize (36) over all curvilinear elements. Then in the general case by formula (29) we obtain the estimation

$$\left|R_{poly}(x)\right| \le \sum_{i=1}^{N_{poly}(x)} \sum_{j=1}^{M_{poly}(x)} \frac{\pi\alpha_{ij}^2 \left(h_{max}(x)\right)^2}{4} M_1(x) \left(\Delta_i(x) + \Delta_j(x)\right)$$
$$\le \frac{\pi\alpha_{ij}^2 \left(h_{max}(x)\right)^2}{2} M_{max1}\Delta_{ij} N_{poly}(x) M_{poly}(x). \quad (37)$$

Estimations (36) and (37) correspond to the option of dividing the integration region by the grid, for which both the number of division elements and the size of the grid step can change together with the change of x (the option 3).

In the case of constant steps of division of the integration region (the option 1), we obtain the following estimation for each curvilinear element and the summarized error estimate for such elements

$$\left|R_{ij}(x)\right| \le \left|\frac{\pi\alpha_{ij}^2 \left(h_{max}\right)^2}{4} M_1(x) \left(\Delta_i(x) + \Delta_j(x)\right)\right| \le \left|\frac{\pi\alpha_{max}^2 \left(h_{max}\right)^2}{2} M_{max1}\Delta_{ij}\right|,$$

$$\left|R_{poly}(x)\right| \le \frac{\pi\alpha_{max}^2 \left(h_{max}\right)^2}{2} M_{max1}\Delta_{ij} N_{poly}(x) M_{poly}(x).$$

For the option 2 of division of the integration region by the rectangular grid, when the size of the steps changes with the change of x, and the number of grid elements along the corresponding axes is constant, the error estimation is as follows

$$\left|R_{ij}(x)\right| \le \left|\frac{\pi\alpha_{ij}^2 \left(h_{max}(x)\right)^2}{4} M_1 \left(\Delta_i(x) + \Delta_j(x)\right)\right| \le \left|\frac{\pi\alpha_{max}^2 \left(h_{max}(x)\right)^2}{2} M_{max1}\Delta_{ij}\right|,$$

$$\left|R_{poly}(x)\right| \le \frac{\pi\alpha_{max}^2 \left(h_{max}(x)\right)^2}{2} M_{max1}\Delta_{ij} N_{poly} M_{poly}.$$

In the option 4, when each additional region of integration has its own grid, we obtain the following estimation for curvilinear elements

$$\left|R_{ij}(x)\right| \leq \left|\frac{\pi\alpha_{ij}^2}{4}\left(h_{max}^{[k]}\right)^2 M_1^{[k]}\left(\Delta_i(x)+\Delta_j(x)\right)\right| \leq \left|\frac{\pi\alpha_{max}^2}{2}\left(h_{max}^{[k]}\right)^2 M_{max1}^{[k]}\Delta_{ij}\right|,$$

$$\left|R_{poly}(x)\right| \leq \frac{\pi\alpha_{max}^2\Delta_{ij}}{2}\sum_{k=1}^{[K(x)]}\left(h_{max}^{[k]}\right)^2 M_{max1}^{[k]} N_{poly}^{[k]} M_{poly}^{[k]}.$$

Here $M_{max1}^{[k]} = \max\limits_{x\in[0;X]} M_1^{[k]}(x)$, $h_{max}^{[k]} = \max\left\{h_{x'}^{[k]}, h_{x''}^{[k]}\right\}$.

3.4 The total error of calculating the double integral when the region of integration is divided by a rectangular grid

We have obtained the estimations of the errors of finding the double integral with variable upper limits for rectangular $R_{rect}(x)$ and curvilinear $R_{poly}(x)$ grid elements of the division of the integration region. So, we can find the estimation of the total error of the calculation of the integral. According to equality (27), the total error is sought as the sum of these two quantities. Here we also highlight four possible options for dividing the grid.

For constant steps of the division (the option 1), we obtain the inequality for estimating the total error of numerically finding the double integral with variable upper limits

$$\left|R(x)\right| \leq \frac{1}{2}\left(\frac{h_{max}^4}{6}M_{max2}N_{rect}(x)M_{rect}(x) + h_{max}^2\pi\alpha_{max}^2\Delta_{ij}M_{max1}N_{poly}(x)M_{poly}(x)\right).$$

For the constant number of grid elements (the option 2), the error estimation is as follows

$$\left|R(x)\right| \leq \frac{1}{2}\left(\frac{h_{max}^4(x)}{6}M_{max2}N_{rect}M_{rect} + h_{max}^2(x)\pi\alpha_{max}^2\Delta_{ij}M_{max1}N_{poly}M_{poly}\right).$$

In the case when the number of both grid elements and step sizes along each axis also changes with the change of x (the option 3), we get

$$\left|R(x)\right| \leq \frac{1}{2}\left(\frac{h_{max}^4(x)}{6}M_{max2}N_{rect}(x)M_{rect}(x) + h_{max}^2(x)\pi\alpha_{max}^2\Delta_{ij}M_{max1}N_{poly}(x)M_{poly}(x)\right).$$

In the option 4, we obtain the following total estimation for the error of numerically finding the double integral

$$\left|R(x)\right| \leq \frac{1}{2}\sum_{k=1}^{[K(x)]}\left(\frac{\left(h_{max}^{[k]}\right)^4}{6}M_{max2}^{[k]}N_{rect}^{[k]}M_{rect}^{[k]} + \left(h_{max}^{[k]}\right)^2\pi\alpha_{max}^2\Delta_{ij}M_{max1}^{[k]}N_{poly}^{[k]}M_{poly}^{[k]}\right).$$

Note that the errors of integration on curvilinear elements is made the greatest contribution to the estimation of the total error $\left|R(x)\right|$. Moreover, the magnitude of this error is affected by how large parts of the elementary curvilinear elements fall into the region of integration, which the coefficient α_{max}^2 determines. The distance between the centroid of the curvilinear element and the center of the circle circumscribed around it also affects the total error.

The obtained estimations depend linearly on the number of elements into which the variable region of integration is divided along the corresponding axes. The estimation of the value of the total error of the numerical integration of the double integral with variable upper limits is directly proportional to the maximum value of the integration step along the Ox'- and Ox''-axes.

We emphasize that the estimations of the integration errors $\left|R(x)\right|$ for the variable regions of integration are functionally dependent on the external variable x. For constant division steps (the option 1), the number of elementary elements increases together with growth x, and the integration error increases accordingly. For the option 2 of overlaying the variable rectangular grid, the length and/or width of the elementary element increases with growth x, which in turn leads also to an increase in $\left|R(x)\right|$. This option is effective for numerical integration over small regions of integration or when initially overlaying the grid with elementary elements of a small area. The option 3 of overlaying the variable grid of the division of the integration region is convenient for specific regions $\left(V(x)\right)$, for example, bounded by a periodic function $g\left(x'\right)$. At the same time, the functional dependence between the number of grid elements and the size of the steps should be chosen, so that the function $h^2_{max}(x)N_{poly}(x)M_{poly}(x)$ is decreasing or limited by a predesigned constant one.

4 Examples

4.1 Integration of simple function

To test the efficiency and reliability of the obtained formulas of the numerical method, we apply it to the integration of sufficiently simple function for which the integration expression can be found analytically.

Let such an integrand be given $f\left(x, x', x''\right) = x'x''$. The surface formed by the function over the integration region is shown in Figure 2.

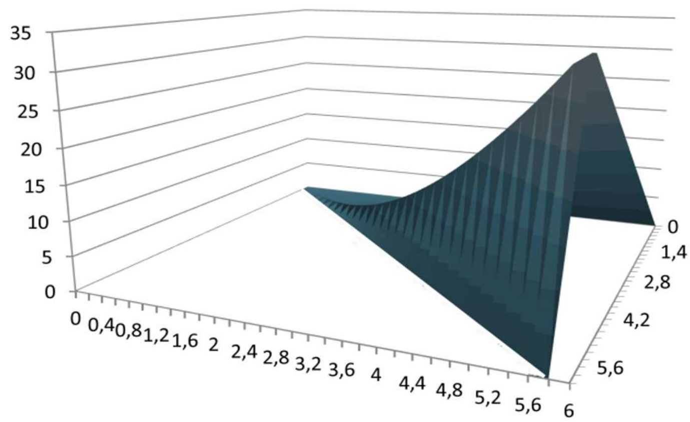

Figure 2. The surface generated by the function $f\left(x, x', x''\right) = x'x''$

We evaluate the integral $I_{num} = \int_0^x \int_0^{x'^2} xx'\,dx''\,dx'$ using formulas (24), (25) and (26). The computation depends on variations in the number of division elements and the grid width. Analytically, the integral is expressed as $I_{analyt} = \dfrac{1}{12}x^6$.

The results of calculations are shown in Tables 1–4 for different numbers of elementary elements of the grid $N_{el}(x)$ and grid width $h(x)$. It is given the values of the difference between the analytical and numerical calculation $|I_{analyt} - I_{num}|$, as well as the relative error $E(x) = |(I_{analyt} - I_{num})/I_{analyt}|$ (see [9, 13]). We choose such basic values of parameters $N_{el}(x) = 10^4$, $h(1) = 10^{-4}$.

Table 1 shows the corresponding values for the case when only the size of the imposed grid $h(x)$ changes with the change of x (option 1). The calculated values for the case when only the number of division elements $N_{el}(x)$ changes with the change of x (option 2) are presented in Table 2. In Tables 3 and 4 it is shown the corresponding values of integration parameters for the case when with the change of x, both the size of the grid and the number of division elements change (option 3). The change in the number of nodes in Table 3 is described by the increasing function $N_{el}(x) = 10^4\sqrt[3]{x}$, then $h(x) = 10^{-4}\sqrt[3]{x^2}$. The change in the number of nodes in Table 4 is described by the decreasing function $N_{el}(x) = 10^4/\sqrt[3]{x}$, and then $h(x) = 10^{-4}/\sqrt[3]{x^2}$.

$N_{el}(x)$	$h(x)$	x	I_{analyt}	I_{num}	$\lvert I_{analyt} - I_{num}\rvert$	$E(x)$
10000	0.0001	1	0.083333333	0.083333334	$3.4300{\cdot}10^{-10}$	$4.1110{\cdot}10^{-9}$
10000	0.000125	1.25	0.317891439	0.317891303	$1.3612{\cdot}10^{-7}$	$4.2821{\cdot}10^{-7}$
10000	0.00015	1.5	0.949218750	0.949219686	$9.3649{\cdot}10^{-7}$	$9.8659{\cdot}10^{-7}$
10000	0.000175	1.75	2.393575033	2.393573986	$1.0469{\cdot}10^{-6}$	$4.3738{\cdot}10^{-7}$
10000	0.0002	2	5.333333333	5.333338492	$5.1585{\cdot}10^{-6}$	$9.6722{\cdot}10^{-7}$
10000	0.000225	2.25	10.81219482	10.81219187	$2.9526{\cdot}10^{-6}$	$2.7308{\cdot}10^{-7}$
10000	0.00025	2.5	20.34505208	20.34506470	$1.2612{\cdot}10^{-5}$	$6.1989{\cdot}10^{-7}$
10000	0.000275	2.75	36.04250081	36.04249159	$9.2213{\cdot}10^{-6}$	$2.5585{\cdot}10^{-7}$
10000	0.0003	3	60.75000000	60.74998992	$1.0081{\cdot}10^{-5}$	$1.6595{\cdot}10^{-7}$
10000	0.000325	3.25	98.20168050	98.20172250	$4.1994{\cdot}10^{-5}$	$4.2763{\cdot}10^{-7}$
10000	0.00035	3.5	153.1888021	153.1888431	$4.0979{\cdot}10^{-5}$	$2.6750{\cdot}10^{-7}$
10000	0.000375	3.75	231.74285889	231.7428215	$3.7385{\cdot}10^{-5}$	$1.6132{\cdot}10^{-7}$
10000	0.0004	4	341.3333333	341.3336936	$3.6029{\cdot}10^{-4}$	$1.0555{\cdot}10^{-6}$
10000	0.000425	4.25	491.0800985	491.0802238	$1.2537{\cdot}10^{-4}$	$2.5529{\cdot}10^{-7}$
10000	0.00045	4.5	691.9804688	691.9806759	$2.0710{\cdot}10^{-4}$	$2.9928{\cdot}10^{-7}$
10000	0.00475	4.75	957.1508993	957.1508567	$4.2583{\cdot}10^{-5}$	$4.4489{\cdot}10^{-8}$
10000	0.0005	5	1302.083333	1302.083944	$6.1080{\cdot}10^{-4}$	$4.6909{\cdot}10^{-7}$

Table 1. Calculation of the integral $\int_0^x \int_0^{x'^2} xx'\,dx''\,dx'$ for the option 1

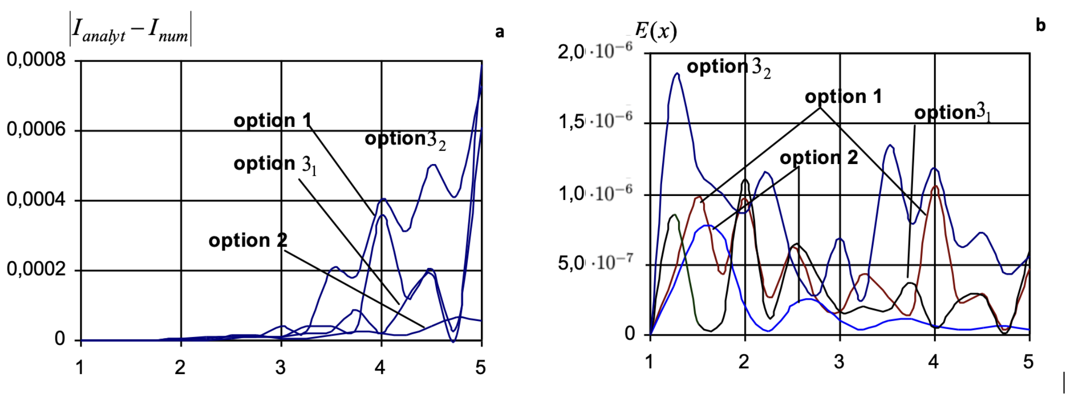

Figure 3. Graphs of absolute (**a**) and relative (**b**) errors for four considered options 1, 2, 3_1 and 3_2 of the relation between the number of nodes and grid width for the integrand $f(x, x', x'') = x'x''$

$N_{el}(x)$	$h(x)$	x	I_{analyt}	I_{num}	$\lvert I_{analyt} - I_{num}\rvert$	$E(x)$
10000	0.0001	1	0.083333333	0.083333334	$3.4258 \cdot 10^{-10}$	$4.1110 \cdot 10^{-9}$
12500	0.0001	1.25	0.317891439	0.317891331	$1.0819 \cdot 10^{-7}$	$3.4033 \cdot 10^{-7}$
15000	0.0001	1.5	0.949218750	0.949219467	$7.1668 \cdot 10^{-7}$	$7.5503 \cdot 10^{-7}$
17500	0.0001	1.75	2.393575033	2.393576733	$1.7002 \cdot 10^{-6}$	$7.1031 \cdot 10^{-7}$
20000	0.0001	2	5.333333333	5.333334425	$1.0914 \cdot 10^{-6}$	$2.0463 \cdot 10^{-7}$
22500	0.0001	2.25	10.81219482	10.81219516	$3.3247 \cdot 10^{-7}$	$3.0750 \cdot 10^{-8}$
25000	0.0001	2.5	20.34505208	20.34505636	$4.2777 \cdot 10^{-6}$	$2.1026 \cdot 10^{-7}$
27500	0.0001	2.75	36.04250081	36.04250948	$8.6631 \cdot 10^{-6}$	$2.4036 \cdot 10^{-7}$
30000	0.0001	3	60.75000000	60.75000654	$6.5431 \cdot 10^{-6}$	$1.0771 \cdot 10^{-7}$
32500	0.0001	3.25	98.20168050	98.20168459	$4.0865 \cdot 10^{-6}$	$4.1614 \cdot 10^{-8}$
35000	0.0001	3.5	153.1888021	153.1888172	$1.5069 \cdot 10^{-5}$	$9.8371 \cdot 10^{-8}$
37500	0.0001	3.75	231.7428589	231.7428858	$2.6914 \cdot 10^{-5}$	$1.1614 \cdot 10^{-7}$
40000	0.0001	4	341.3333333	341.3333551	$2.1806 \cdot 10^{-5}$	$6.3883 \cdot 10^{-8}$
42500	0.0001	4.25	491.0800985	491.0801146	$1.6099 \cdot 10^{-5}$	$3.2783 \cdot 10^{-8}$
45000	0.0001	4.5	691.9804688	691.9805084	$3.9631 \cdot 10^{-5}$	$5.7272 \cdot 10^{-8}$
47500	0.0001	4.75	957.1508993	957.1509638	$6.4589 \cdot 10^{-5}$	$6.7481 \cdot 10^{-8}$
50000	0.0001	5	1302.083333	1302.083388	$5.4509 \cdot 10^{-5}$	$4.1863 \cdot 10^{-8}$

Table 2. Calculation of integral $\int_0^x \int_0^{x'^2} xx'\,dx''\,dx'$ for the option 2

$N_{el}(x)$	$h(x)$	x	I_{analyt}	I_{num}	$\lvert I_{analyt} - I_{num}\rvert$	$E(x)$
10000	0.0001	1	0.083333333	0.083333334	$3.4258 \cdot 10^{-10}$	$4.1110 \cdot 10^{-9}$
10772	0.000116	1.25	0.317891439	0.317891166	$2.7266 \cdot 10^{-7}$	$8.5770 \cdot 10^{-7}$
11447	0.000131	1.5	0.949218750	0.949218818	$6.7514 \cdot 10^{-8}$	$7.1126 \cdot 10^{-8}$
12051	0.0001452	1.75	2.393575033	2.39357526	$2.2753 \cdot 10^{-7}$	$9.5059 \cdot 10^{-8}$
12599	0.0001587	2	5.333333333	5.333339228	$5.8951 \cdot 10^{-6}$	$1.1053 \cdot 10^{-6}$
13104	0.0001717	2.25	10.81219480	10.81219348	$1.3439 \cdot 10^{-6}$	$1.2430 \cdot 10^{-7}$
13572	0.0001842	2.5	20.34505208	20.34506514	$1.3053 \cdot 10^{-5}$	$6.4156 \cdot 10^{-7}$
14010	0.0001963	2.75	36.04250081	36.04251635	$1.5535 \cdot 10^{-5}$	$4.3101 \cdot 10^{-7}$
14422	0.000208	3	60.75000000	60.75000981	$9.8059 \cdot 10^{-6}$	$1.6142 \cdot 10^{-7}$
14812	0.0002194	3.25	98.20168050	98.20166086	$1.9645 \cdot 10^{-5}$	$2.0005 \cdot 10^{-7}$
15183	0.0002305	3.5	153.1888021	153.1887748	$2.7241 \cdot 10^{-5}$	$1.7783 \cdot 10^{-7}$
15536	0.0002414	3.75	231.7428589	231.7429459	$8.7012 \cdot 10^{-5}$	$3.7547 \cdot 10^{-7}$
15874	0.000252	4	341.3333333	341.3333144	$1.8896 \cdot 10^{-5}$	$5.5360 \cdot 10^{-8}$
16198	0.0002624	4.25	491.0800985	491.0799817	$1.1681 \cdot 10^{-4}$	$2.3786 \cdot 10^{-7}$
16510	0.0002726	4.5	691.9804688	691.9806591	$1.9033 \cdot 10^{-4}$	$2.7504 \cdot 10^{-7}$
16810	0.0002826	4.75	957.1508993	957.1509152	$1.5910 \cdot 10^{-5}$	$1.6622 \cdot 10^{-8}$
17100	0.0002924	5	1302.083333	1302.084121	$7.8751 \cdot 10^{-4}$	$6.0481 \cdot 10^{-7}$

Table 3. Calculation of the sought integral for the option 3
for increasing function $N_{el}(x) = 10^4 \sqrt[3]{x}$

$N_{el}(x)$	$h(x)$	x	I_{analyt}	I_{num}	$\lvert I_{analyt} - I_{num}\rvert$	$E(x)$
10000	0.0001	1	0.083333333	0.083333334	$3.4258\cdot10^{-10}$	$4.1110\cdot10^{-9}$
9283	0.0001347	1.25	0.317891439	0.317892018	$5.7896\cdot10^{-7}$	$1.8212\cdot10^{-6}$
8736	0.0001717	1.5	0.949218750	0.949219862	$1.1119\cdot10^{-6}$	$1.1714\cdot10^{-6}$
8298	0.0002109	1.75	2.393575033	2.393577454	$2.4219\cdot10^{-6}$	$1.0118\cdot10^{-6}$
7937	0.000252	2	5.333333333	5.333328689	$4.6447\cdot10^{-6}$	$8.7088\cdot10^{-7}$
7631	0.0002948	2.25	10.81219482	10.81220729	$1.2465\cdot10^{-5}$	$1.1529\cdot10^{-6}$
7368	0.0003393	2.5	20.34505208	20.34506219	1.0111E-05	$4.9699\cdot10^{-7}$
7138	0.0003853	2.75	36.04250081	36.04251081	$9.9928\cdot10^{-5}$	$2.7725\cdot10^{-7}$
6934	0.0004327	3	60.75000000	60.75004159	$4.1594\cdot10^{-5}$	$6.8467\cdot10^{-7}$
6751	0.0004814	3.25	98.20168050	98.20165560	$2.4900\cdot10^{-5}$	$2.5356\cdot10^{-7}$
6586	0.0005314	3.5	153.1888021	153.1890075	$2.0543\cdot10^{-4}$	$1.3410\cdot10^{-6}$
6437	0.0005826	3.75	231.7428589	231.7426760	$1.8290\cdot10^{-4}$	$7.8924\cdot10^{-7}$
6300	0.0006349	4	341.3333333	341.3329299	$4.0344\cdot10^{-4}$	$1.1820\cdot10^{-6}$
6174	0.0006884	4.25	491.0800985	491.0804127	$3.1428\cdot10^{-4}$	$6.3997\cdot10^{-7}$
6057	0.0007429	4.5	691.9804688	691.9799686	$5.0012\cdot10^{-4}$	$7.2273\cdot10^{-7}$
5949	0.0007985	4.75	957.1508993	957.1504815	$4.1775\cdot10^{-4}$	$4.3645\cdot10^{-7}$
5848	0.000855	5	1302.083333	1302.082603	$7.3011\cdot10^{-4}$	$5.6073\cdot10^{-7}$

Table 4. Calculation of the sought integral for the option 3 for decreasing function $N_{el}(x) = 10^4/\sqrt[3]{x}$

Figure 3 demonstrates comparative graphs of absolute (Figure 3**a**) and relative (Figure 3**b**) errors in numerical integration by formulas (24) of the option 1, (25) of the option 2, (26) of the option 3_1 for increasing function $N_{el}(x)$ and the option 3_2 for decreasing $N_{el}(x)$.

Note that the values closest to the analytical values of the integral are obtained in the option of the constant width of the grid and of increase in the number of partition elements together with an increase in the integration (Figure 3**a** and 3**b**). The results calculated by the formula (26) at the imposition of the increasing function of nodes quantity $N_{el}(x)$ (option 3_1, Figure 3) show that the values of absolute and relative errors are quite acceptable.

However, in this case, the number of operations is smaller than in the option 2. Note that only one option 3_2 i.e. imposition of the decreasing function of the number of nodes $N_{el}(x)$, leads to a sharp increase in absolute and relative errors with increasing x and can go beyond a given accuracy of calculations (Figure 3, Table 4). In the options 1, 2 and 3_1 the difference between the analytical and numerical calculations of $\lvert I_{analyt} - I_{num}\rvert$ is within the acceptable deviation.

In this case, at a denser grid overlaid on the variable region of integration both absolute and relative errors decrease. In particular, the difference $\lvert I_{analyt} - I_{num}\rvert$ decreases by two orders of magnitude when $N_{el}(x)$ increases by an order (Tables 1–4).

4.2 Application for integration in problems of water filter operation

The efficiency calculations of multilayered industrial water filters of thickness x_0 are based on solving a nonlinear equation $\sup_{x\in\Omega} c_{sorp}(x, t_*) = N$, where N is the maximum concentration of contaminant particles capable of adsorbing on the filter skeleton (experimentally measurable quantity), $c_{sorp}(x,t)$ is the concentration of adsorbed particles, Ω is the body region, and t_* is the saturation time [3]. The concentration of adsorbed particles under zero initial conditions is determined [3] by the relationship

$$c_{sorp}(x,t) = a\int_0^t c_{sol}\left(x,t'\right)dt',$$

where $c_{sol}(x,t)$ is the concentration of particles in the water solution being filtered, a is the coefficient of sorption intensity.

In multilayered porous systems, the exact solution of the the boundary value problem of convective diffusion of an impurity accompanied by sorption processes can only be found in integral form. Therefore, to determine the concentration of the sorbed impurity, it is necessary to search for integrals of the form

$$I(t) = \int_0^t 3t'e^{-\frac{3}{2}(d_1\pi^2/x_0^2+a)t'^2}\int_0^{\frac{3}{2}t'^2} g\left(t''\right)e^{(d_1\pi^2/x_0^2+a)t''}dt''dt', \tag{38}$$

where d_i is the diffusion coefficient in the solution of the i-th layer, t is time, and the function $g(t)$ are determined as

$$g\left(t'\right) = \frac{A_1c_0e^{\frac{v_1x_l}{2d_1}}\Sigma_n^- + A_2c_*e^{-\frac{v_2\delta x}{2d_2}}\Sigma_m^+}{A_1\lambda^{-1}\Sigma_m^+ + A_2\Sigma_n^-}.$$

Here $A_1 = \frac{2d_1^2}{x_l}$, $A_2 = \frac{2d_2^2}{\delta x}$, $\Sigma_n^\pm = \sum_{n=1}^\infty(\pm)^n y_n^2 e^{-\frac{d_1y_n^2+a}{t-t'}}$, $\Sigma_m^\pm = \sum_{m=1}^\infty(\pm)^m y_m^2 e^{-\frac{d_2y_m^2+a}{t-t'}}$, $y_n = \frac{n\pi}{x_l}$, $y_m = \frac{m\pi}{\delta x}$, x_l and δx are thicknesses of the first and second contacting filtering layers; c_0, c_* are values of the sought function on the upper and lower surfaces of the filtering unit. The results of calculating integral (38) were carried out for option 2 of the integration method, i.e. using formula (25), and are presented in Table 5. The input calculated values are taken as $x_l = 3$, $\delta x = 10$, $c_0 = 1$, $c_* = 0.3$, $d_1 = 1$, $d_2 = 0.2$, $a = 1.1$.

$N_{el}(t)$	$h(t)$	t	$I(t)$
1000	0.0001	0.1	$1.78290466950871 \cdot 10^{-9}$
1500	0.0001	0.15	$5.49042355646216 \cdot 10^{-8}$
2000	0.0001	0.2	$2.13568955720157 \cdot 10^{-7}$
2500	0.0001	0.25	$3.49689244518841 \cdot 10^{-7}$
3000	0.0001	0.3	$2.33823787559044 \cdot 10^{-8}$
3500	0.0001	0.35	$3.06321611479088 \cdot 10^{-6}$
4000	0.0001	0.4	$1.61809743748025 \cdot 10^{-5}$
4500	0.0001	0.45	$5.86758871443194 \cdot 10^{-5}$
5000	0.0001	0.5	0.000171017997979176
5500	0.0001	0.55	0.000426619761144569
6000	0.0001	0.6	0.000944294566160514
6500	0.0001	0.65	0.00189906708090357
7000	0.0001	0.7	0.00353005889551737
7500	0.0001	0.75	0.00614442065807345
8000	0.0001	0.8	0.0101165726659113
8500	0.0001	0.85	0.0158829260525336
9000	0.0001	0.9	0.0239323203253347

Table 5. Calculation of integral $I(t)$ for the option 2

Conclusions

The estimations of the errors of the approximate integration of double integrals with variable upper limits are found. It is shown that in the case of substitution of the integrand with approximating functions, the estimation of the main term of the error depends, in particular, on the maximum of the function of the upper limit of integration. For the case of sequential integration, the error estimation depends on the value of the function of the upper limit of integration at the point defining the abscissa of the centroid of the variable region of integration. The corresponding theorems are formulated and proved. The errors in the numerical determination of the double integral are found in the case of dividing the integration region by a rectangular grid. We have proposed four options for adapting the division grid to a change in the integration region, namely: with a change in the integration region, the number of elements of the grid division changes; the length of the elements changes only; the number of elements and their length changes at the same time; each "additional" region is overlaid by its several grid. Estimations for the rectangular and curvilinear region elements are found, as well as the estimations of the total error for all four options. It is shown that the value of the error depends linearly on the number of rectangular and curvilinear elements approximated by polygons, which for some options of grid adaptation to the change of the integration region are variable values, and they are also directly proportional to the maximum value of the integration steps.

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У роботі запропоновано оцінки наближеного обчислення подвійних інтегралів зі змінними верхніми межами. Сформульовано та доведено теорему про оцінку для головного члена похибки наближеного інтегрування за підходом заміни підінтегральної функції апроксимуючою функцією, а також теорему про оцінку головного члена похибки у випадку обчислення подвійних інтегралів за підходом послідовного інтегрування. Отримано оцінки похибок інтегрування при розбитті змінної області інтегрування прямокутним сітками, зокрема, розглянуто чотири можливі варіанти адаптації сітки до зміни області інтегрування, для кожного з них знайдено оцінки для прямокутних та криволінійних елементів сітки, а також сумарну оцінку подвійного інтегрування зі змінними верхніми межами.

Ключові слова і фрази: похибка, апроксимація, подвійний інтеграл, змінна верхня межа, ряд Тейлора, змінна область інтегрування, прямокутна сітка.