ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2023, **15** (2), 331–355
doi:10.15330/cmp.15.2.331-355

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2023, Т.15, №2, С.331–355

On the semigroup $B_{\omega}^{\mathscr{F}_n}$, which is generated by the family $\mathscr{F}_n$ of finite bounded intervals of ω

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We study the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, which is introduced in the paper [Visnyk Lviv Univ. Ser. Mech.-Mat. 2020, **90**, 5–19 (in Ukrainian)], in the case when the ω-closed family $\mathscr{F}_n$ generated by the set $\{0,1,\ldots,n\}$. We show that the Green relations $\mathscr{D}$ and $\mathscr{J}$ coincide in $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\operatorname{conv}})$ of partial convex order isomorphisms of $(\omega,\leqslant)$ of the rank $\leqslant n+1$, and $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ admits only Rees congruences. Also, we study shift-continuous topologies on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. In particular, we prove that for any shift-continuous T_1-topology τ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ every non-zero element of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is an isolated point of $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau)$, $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ admits the unique compact shift-continuous T_1-topology, and every $\omega_{\mathfrak{d}}$-compact shift-continuous T_1-topology is compact. We describe the closure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ in a Hausdorff semitopological semigroup and prove the criterium when a topological inverse semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is H-closed in the class of Hausdorff topological semigroups.

Key words and phrases: bicyclic extension, Rees congruence, semitopological semigroup, topological semigroup, bicyclic monoid, inverse semigroup, $\omega_{\mathfrak{d}}$-compact, compact, closure.

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1 Introduction, motivation and main definitions

We shall follow the terminology of [11, 14, 15, 17, 36]. By ω we denote the set of all non-negative integers.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F\in\mathscr{P}(\omega)$ and $n,m\in\omega$ we put $n-m+F=\{n-m+k\colon k\in F\}$ if $F\neq\varnothing$ and $n-m+\varnothing=\varnothing$. A subfamily $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is called *ω-closed* if $F_1\cap(-n+F_2)\in\mathscr{F}$ for all $n\in\omega$ and $F_1,F_2\in\mathscr{F}$.

We denote $[0;0]=\{0\}$ and $[0;k]=\{0,\ldots,k\}$ for any positive integer k. The set $[0;k]$, $k\in\omega$, is called an *initial interval* of ω.

A *partially ordered set* (or shortly a *poset*) $(X,\leqq)$ is the set X with the reflexive, antisymmetric and transitive relation $\leqq$. In this case the relation $\leqq$ is called a partial order on X. A partially ordered set $(X,\leqq)$ is *linearly ordered* or is a *chain* if $x_1\leqq x_2$ or $x_2\leqq x_1$ for any $x_1,x_2\in X$. A map f from a poset $(X,\leqq)$ onto a poset $(Y,\leqslant)$ is said to be an order isomorphism if f is bijective and $x\leqq y$ if and only if $f(x)\leqslant f(y)$. A *partial order isomorphism* f from a poset $(X,\leqq)$ into a poset $(Y,\leqslant)$ is an order isomorphism from a subset A of a poset $(X,\leqq)$ into a subset B of a poset $(Y,\leqslant)$. For any elements x of a poset $(X,\leqq)$ we denote

$$\uparrow_{\leqq}x=\{y\in X\colon x\leqq y\}\qquad\text{and}\qquad\downarrow_{\leqq}x=\{y\in X\colon y\leqq x\}.$$

УДК 512.53
2020 *Mathematics Subject Classification:* 20M18, 20M20, 22A15, 20M12, 20M30, 54D30, 54D40, 54H15.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the mapping $\operatorname{inv}\colon S \to S$ which assigns to every element x of S its inverse element x^{-1} is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$. A *semilattice* is a commutative semigroup of idempotents. By $(\omega, \min)$ or $\omega_{\min}$ we denote the set ω with the semilattice operation $x \cdot y = \min\{x, y\}$.

If S is an inverse semigroup, then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [40].

For semigroups S and T, a map $\mathfrak{h}\colon S \to T$ is called a *homomorphism* if $\mathfrak{h}(s_1 \cdot s_2) = \mathfrak{h}(s_1) \cdot \mathfrak{h}(s_2)$ for all $s_1, s_2 \in S$.

A *congruence* on a semigroup S is an equivalence relation $\mathfrak{C}$ on S such that $(s,t) \in \mathfrak{C}$ implies that $(as, at), (sb, tb) \in \mathfrak{C}$ for all $a, b \in S$. Every congruence $\mathfrak{C}$ on a semigroup S generates the *associated natural homomorphism* $\mathfrak{C}^\natural\colon S \to S/\mathfrak{C}$ which assigns to each element s of S its congruence class $[s]_{\mathfrak{C}}$ in the quotient semigroup $S/\mathfrak{C}$. Also every homomorphism $\mathfrak{h}\colon S \to T$ of semigroups S and T generates the congruence $\mathfrak{C}_{\mathfrak{h}}$ on S: $(s_1, s_2) \in \mathfrak{C}_{\mathfrak{h}}$ if and only if $\mathfrak{h}(s_1) = \mathfrak{h}(s_2)$.

A nonempty subset I of a semigroup S is called a *left ideal* if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$, and a (two-sided) *ideal* if it is both a left and a right ideal. Every ideal I of a semigroup S generates the congruence $\mathfrak{C}_I = (I \times I) \cup \Delta_S$ on S, which is called the *Rees congruence* on S.

Let $\mathscr{I}_\lambda$ denote the set of all partial one-to-one transformations of λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \quad \text{if} \quad x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha\colon y\alpha \in \operatorname{dom}\beta\} \qquad \text{for} \quad \alpha, \beta \in \mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ (see [14]). For any $\alpha \in \mathscr{I}_\lambda$ the cardinality of $\operatorname{dom}\alpha$ is called the *rank* of α and it is denoted by $\operatorname{rank}\alpha$. The symmetric inverse semigroup was introduced by V.V. Wagner [40] and it plays a major role in the theory of semigroups.

Put $\mathscr{I}_\lambda^n = \{\alpha \in \mathscr{I}_\lambda\colon \operatorname{rank}\alpha \leqslant n\}$ for $n \in \{1, 2, 3, \ldots\}$. Obviously, $\mathscr{I}_\lambda^n$ are inverse semigroups, $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}_\lambda$ for each $n \in \{1, 2, 3, \ldots\}$. The semigroup $\mathscr{I}_\lambda^n$ is called the *symmetric inverse semigroup of finite transformations of the rank* $\leqslant n$ [26]. By

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

we denote a partial one-to-one transformation which maps x_1 onto y_1, x_2 onto y_2, ..., and x_n onto y_n. Obviously, in such case we have $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$, $i, j \in \{1, 2, 3, \ldots, n\}$. The empty partial map $\varnothing\colon \lambda \rightharpoonup \lambda$ is denoted by $\mathbf{0}$. It is obvious that $\mathbf{0}$ is zero of the semigroup $\mathscr{I}_\lambda^n$.

For a partially ordered set $(P, \leqq)$, a subset X of P is called *order-convex*, if $x \leqq z \leqq y$ and $\{x, y\} \subset X$ implies that $z \in X$ for all $x, y, z \in P$ [31]. It is obvious that the set of all partial order isomorphisms between convex subsets of $(\omega, \leqslant)$ under the composition of partial self-maps forms an inverse subsemigroup of the symmetric inverse semigroup $\mathscr{I}_\omega$ over the set ω.

We denote this semigroup by $\mathscr{I}_{\omega}(\overrightarrow{\mathrm{conv}})$. We put $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}}) = \mathscr{I}_{\omega}(\overrightarrow{\mathrm{conv}}) \cap \mathscr{I}_{\omega}^{n}$ and it is obvious that $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is closed under the semigroup operation of $\mathscr{I}_{\omega}^{n}$ and the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is called the *inverse semigroup of convex order isomorphisms of* $(\omega, \leqslant)$ *of the rank* $\leqslant n$.

The bicyclic monoid $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined as follows:

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [14].

On the set $\boldsymbol{B}_{\omega} = \omega \times \omega$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 \leqslant i_2; \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 \geqslant i_2. \end{cases}$$

It is well known that the semigroup $\boldsymbol{B}_{\omega}$ is isomorphic to the bicyclic monoid by the mapping $\mathfrak{h}\colon \mathscr{C}(p,q) \to \boldsymbol{B}_{\omega}$, $q^k p^l \mapsto (k,l)$ (see [14, Section 1.12] or [35, Exercise IV.1.11(ii)]).

By $\mathbb{R}$ and $\omega_{\mathfrak{d}}$ we denote the set of real numbers with the usual topology and the infinite countable discrete space, respectively.

Let Y be a topological space. A topological space X is called:

- *compact* if any open cover of X contains a finite subcover;
- *countably compact* if each closed discrete subspace of X is finite;
- *Y-compact* if every continuous image of X in Y is compact.

A *topological* (*semitopological*) *semigroup* is a topological space together with a continuous (separately continuous) semigroup operation. If S is a semigroup and τ is a topology on S such that (S,τ) is a topological semigroup, then we shall call τ a *semigroup topology* on S, and if τ is a topology on S such that (S,τ) is a semitopological semigroup, then we shall call τ a *shift-continuous topology* on S. An inverse topological semigroup with the continuous inversion is called a *topological inverse semigroup*.

Next we shall describe the construction which is introduced in [23].

Let $\boldsymbol{B}_{\omega}$ be the bicyclic monoid and $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_{\omega} \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} \big(i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2\big), & \text{if } j_1 \leqslant i_2; \\ \big(i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)\big), & \text{if } j_1 \geqslant i_2. \end{cases}$$

In [23] it is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed then $(\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$ then the set $\boldsymbol{I} = \{(i,j,\varnothing)\colon i,j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_{\omega}^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}; \\ (\boldsymbol{B}_{\omega} \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [23]. The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ generalizes the bicyclic monoid and the countable semigroup of matrix units. It is proved in [23] that $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is a combinatorial inverse semigroup and Green's relations, the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its set of idempotents are described. The criteria of simplicity, 0-simplicity, bisimplicity, 0-bisimplicity of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and when $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ has the identity, is isomorphic to the bicyclic semigroup or the countable semigroup of matrix units are given. In particular, in [23] it is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the semigrpoup of $\omega\times\omega$-matrix units if and only if $\mathscr{F}$ consists of a singleton set and the empty set.

The semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in the case when the family $\mathscr{F}$ consists of the empty set and some singleton subsets of ω is studied in [21]. It is proved that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the subsemigroup $\mathscr{B}_{\omega}^{\Gamma}(\boldsymbol{F}_{\min})$ of the Brandt ω-extension of the subsemilattice $(\boldsymbol{F},\min)$ of $(\omega,\min)$, where $\boldsymbol{F}=\bigcup\mathscr{F}$. Also topologizations of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ and its closure in semitopological semigroups are studied.

For any $n\in\omega$ we put $\mathscr{F}_n=\{\varnothing,[0;0],[0;1],[0;2],\ldots,[0;n]\}$. It is obvious that $\mathscr{F}_n$ is an ω-closed family of ω.

In this paper, we study the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. We show that the Green relations $\mathscr{D}$ and $\mathscr{J}$ coincide in $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$, and $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ admits only Rees congruences. Also, we study shift-continuous topologizations of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. In particular, we prove that for any shift-continuous T_1-topology τ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ every non-zero element of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is an isolated point of $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau)$, $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ admits the unique compact shift-continuous T_1-topology, and every $\omega_{\mathfrak{d}}$-compact shift-continuous T_1-topology is compact. We describe the closure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ in a Hausdorff semitopological semigroup and prove the criterium when a topological inverse semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is H-closed in the class of Hausdorff topological semigroups.

2 Algebraic properties of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$

An inverse semigroup S with zero is said to be *0-E-unitary* if $0\neq e\preccurlyeq s$, where e is an idempotent in S, implies that s is an idempotent [32]. The class of 0-E-unitary semigroups was first defined by Maria Szendrei [37], although she called them E^*-unitary. The term 0-E-unitary appears to be due to J. Meakin and M. Sapir [33].

In the following proposition we summarise properties which follow from properties of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ in the general case. These properties are corollaries of the results of the paper [23].

Proposition 1. *For any $n\in\omega$ the following statements hold:*

(1) $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *is an inverse semigroup, namely* $\mathbf{0}^{-1}=\mathbf{0}$ *and* $\big(i,j,[0;k]\big)^{-1}=\big(j,i,[0;k]\big)$, *for any* $i,j,k\in\omega$;

(2) $\big(i,j,[0;k]\big)\in\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *is an idempotent if and only if* $i=j$;

(3) $\big(i_1,i_1,[0;k_1]\big)\preccurlyeq\big(i_2,i_2,[0;k_2]\big)$ *in* $E\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ *if and only if* $i_1\geqslant i_2$ *and* $i_1+k_1\leqslant i_2+k_2$ *and this natural partial order on* $E\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ *is presented on Figure 1;*

(4) $\big(i,i,[0;n]\big)$ *is a maximal idempotent of* $E\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ *for any* $i\in\omega$;

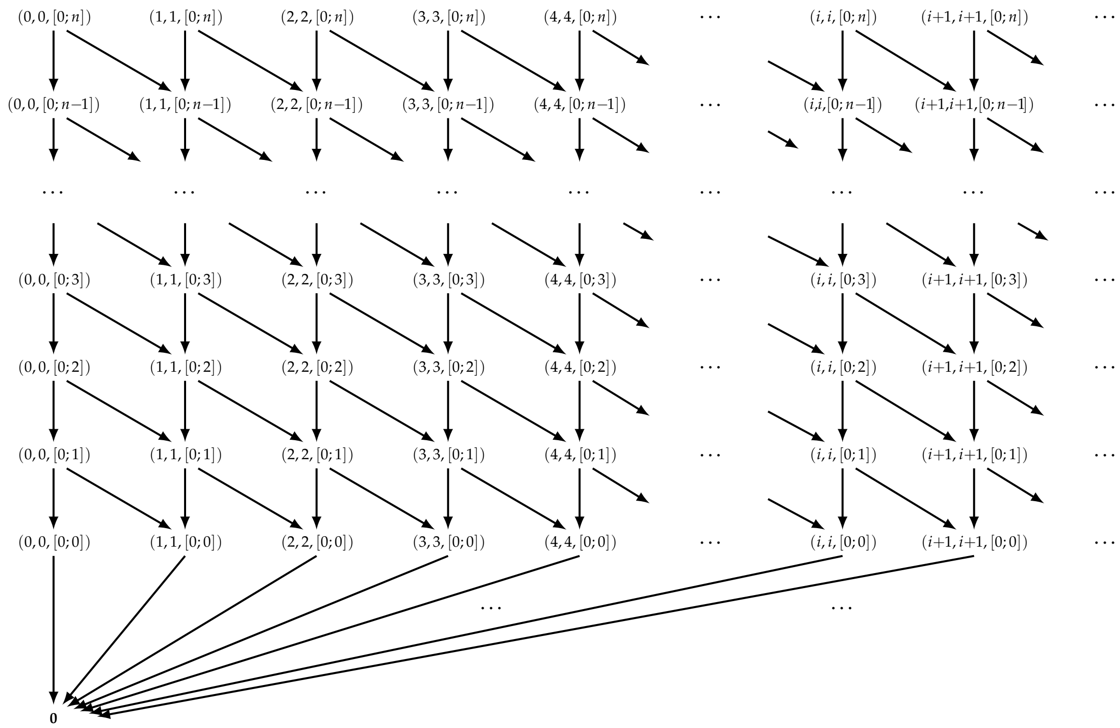

Figure 1. The natural partial order on the band $E\big(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\big)$

(5) $\big(i,i,[0;0]\big)$ *is a primitive idempotent of* $E\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ *for any* $i\in\omega$*;*

(6) $\big(i_1,j_1,[0;k_1]\big)\mathscr{R}\big(i_2,j_2,[0;k_2]\big)$ *in* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *if and only if* $i_1=i_2$ *and* $k_1=k_2$*;*

(7) $\big(i_1,j_1,[0;k_1]\big)\mathscr{L}\big(i_2,j_2,[0;k_2]\big)$ *in* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *if and only if* $j_1=j_2$ *and* $k_1=k_2$*;*

(8) $\big(i_1,j_1,[0;k_1]\big)\mathscr{H}\big(i_2,j_2,[0;k_2]\big)$ *in* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *if and only if* $i_1=i_2$, $j_1=j_2$ *i* $k_1=k_2$*;*

(9) $\big(i_1,j_1,[0;k_1]\big)\mathscr{D}\big(i_2,j_2,[0;k_2]\big)$ *in* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *if and only if* $k_1=k_2$*;*

(10) $\mathscr{D}=\mathscr{J}$ *in* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$*;*

(11) $\big(i_1,j_1,[0;k_1]\big)\preccurlyeq\big(i_2,j_2,[0;k_2]\big)$ *in* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *if and only if* $i_1\geqslant i_2$, $i_1-j_1=i_2-j_2$ *and* $i_1+k_1\leqslant i_2+k_2$*;*

(12) $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *is a* 0*-E-unitary inverse semigroup.*

Proof. Statements (1)–(5) are trivial. Statements (6)–(8) follow from [32, Proposition 3.2.11] and corresponding statements of [23, Theorem 2].

(9) $(\Rightarrow)$ Let $\big(i_1,j_1,[0;k_1]\big)\mathscr{D}\big(i_2,j_2,[0;k_2]\big)$ in $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. Then there exists $\big(i_0,j_0,[0;k_0]\big)\in\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ such that $\big(i_1,j_1,[0;k_1]\big)\mathscr{L}\big(i_0,j_0,[0;k_0]\big)$ and $\big(i_0,j_0,[0;k_0]\big)\mathscr{R}\big(i_2,j_2,[0;k_2]\big)$. By statement (6) we have that $i_0=i_2$ and $k_0=k_2$, and by (7) we get that $j_0=j_1$ and $k_1=k_0$. This implies that $k_1=k_2$.

($\Leftarrow$) Let $(i_1, j_1, [0;k])$ and $(i_2, j_2, [0;k])$ be elements of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. By statements (6) and (7) we have that $(i_1, j_1, [0;k])\mathscr{L}(i_1, j_2, [0;k])\mathscr{R}(i_2, j_2, [0;k])$ and hence $(i_1, j_1, [0;k])\mathscr{D}(i_2, j_2, [0;k])$ in $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$.

(10) It is obvious that the $\mathscr{D}$-class of the zero $\mathbf{0}$ coincides with $\{\mathbf{0}\}$. Also the $\mathscr{J}$-class of the zero $\mathbf{0}$ coincides with $\{\mathbf{0}\}$.

Fix an arbitrary non-zero element $(i_0,j_0,[0;k_0])$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. By (9) the $\mathscr{D}$-class of $(i_0,j_0,[0;k_0])$ is the following set $\mathbf{D} = \{(i, j, [0;k_0]) : i, j \in \omega\}$. By (3) every two distinct idempotents of the set $\mathbf{D}$ are incomparable, and hence every idempotent of the $\mathscr{D}$-class of $(i_0, j_0, [0;k_0])$ is minimal with the respect to the natural partial order on $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. By [32, Proposition 3.2.17], if the $\mathscr{D}$-class D_y has a minimal element then $D_y = J_y$ and hence the $\mathscr{D}$-class of $(i_0, j_0, [0;k_0])$ coincides with its $\mathscr{J}$-class. Therefore we obtain that $\mathscr{D} = \mathscr{J}$ in $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$.

(11) By [23, Proposition 2], the inequality $(i_1, j_1, [0;k_1]) \preccurlyeq (i_2, j_2, [0;k_2])$ is equivalent to the conditions

$$[0;k_1] \subseteq i_2 - i_1 + [0;k_2] = j_2 - j_1 + [0;k_2],$$

which are equivalent to

$$i_2 - i_1 = j_2 - j_1 \leqslant 0 \qquad \text{and} \qquad k_1 \leqslant i_2 - i_1 + k_2.$$

It is obvious that the last conditions are equivalent to

$$i_1 \geqslant i_2, \qquad i_1 - j_1 = i_2 - j_2 \qquad \text{and} \qquad i_1 + k_1 \leqslant i_2 + k_2,$$

which completes the proof of the statement.

Statement (12) follows from (11). □

Lemma 1. *Let $n \in \omega$. Then $\uparrow_{\preccurlyeq}(i_0, j_0, [0;k_0])$ and $\downarrow_{\preccurlyeq}(i_0, j_0, [0;k_0])$ are finite subsets of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ for any its non-zero element $(i_0, j_0, [0;k_0])$, $i_0, j_0 \in \omega$, $k_0 \in \{0, \ldots, n\}$.*

Proof. By Proposition 1(11) there exist finitely many $i, j \in \omega$ and $k \in \{0, \ldots, n\}$ such that $(i, j, [0;k]) \preccurlyeq (i_0, j_0, [0;k_0])$ for some $i, j \in \omega$ and hence the set $\downarrow_{\preccurlyeq}(i_0, j_0, [0;k_0])$ is finite.

The inequality $k \leqslant n$ and Proposition 1(11) imply that there exist finitely many $i, j \in \omega$ and $k \in \{0, \ldots, n\}$ such that $(i_0, j_0, [0;k_0]) \preccurlyeq (i, j, [0;k])$, and hence the set $\uparrow_{\preccurlyeq}(i_0, j_0, [0;k_0])$ is finite, too. □

Lemma 2. *If $n \in \omega$ then for any $\alpha, \beta \in \boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ the set $\alpha \cdot \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \cdot \beta$ is finite.*

Proof. The statement of the lemma is trivial when $\alpha = \mathbf{0}$ or $\beta = \mathbf{0}$.

Fix arbitrary non-zero-elements $\alpha = (i_\alpha, j_\alpha, [0;k_\alpha])$ and $\beta = (i_\beta, j_\beta, [0;k_\beta])$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. If $i \geqslant j_\alpha + n + 1$ or $j \geqslant i_\beta + n + 1$ then for any $k \in \{0, \ldots, n\}$ we have that

$$(i_\alpha, j_\alpha, [0;k_\alpha]) \cdot (i, j, [0;k]) = \Big(i_\alpha - j_\alpha + i, j, (j_\alpha - i + [0;k_\alpha]) \cap [0;k]\Big) = \mathbf{0}$$

and

$$(i, j, [0;k]) \cdot (i_\beta, j_\beta, [0;k_\beta]) = \Big(i, j - i_\beta + j_\beta, [0;k] \cap (i_\beta - j + [0;k_\beta])\Big) = \mathbf{0}.$$

Hence there exist only finitely many $(i, j, [0;k]) \in \boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ such that $\alpha \cdot (i, j, [0;k]) \cdot \beta \neq \mathbf{0}$. This implies the statement of the lemma. □

Lemma 3. *Let $n \in \omega$. Then for any non-zero elements $(i_1, j_1, [0;k_1])$ and $(i_2, j_2, [0;k_2])$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ the sets of solutions of the following equations*

$$(i_1, j_1, [0;k_1]) \cdot \chi = (i_2, j_2, [0;k_2]) \qquad \text{and} \qquad \chi \cdot (i_1, j_1, [0;k_1]) = (i_2, j_2, [0;k_2])$$

in the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ are finite.

Proof. Suppose that χ is a solution of the equation $(i_1, j_1, [0;k_1]) \cdot \chi = (i_2, j_2, [0;k_2])$. The definition of the semigroup operation on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ implies that $\chi \neq \mathbf{0}$ and $k_1 \geqslant k_2$. Assume that $\chi = (i, j, [0;k])$ for some $i, j \in \omega$, $k \in \{0, 1, \ldots, n\}$. Then we have that

$$\begin{aligned}(i_2, j_2, [0;k_2]) &= (i_1, j_1, [0;k_1]) \cdot (i, j, [0;k]) \\ &= \begin{cases} \big(i_1 - j_1 + i, j, (j_1 - i + [0;k_1]) \cap [0;k]\big), & \text{if } j_1 < i; \\ (i_1, j, [0;k_1] \cap [0;k]), & \text{if } j_1 = i; \\ \big(i_1, j_1 - i + j, [0;k_1] \cap (i - j_1 + [0;k])\big), & \text{if } j_1 > i. \end{cases}\end{aligned}$$

We consider the following cases.

1. If $j_1 < i$ then $i = i_2 - i_1 + j_1$, $j = j_2$, $k \geqslant k_2$ and

$$j_1 - i + k_1 = j_1 - i_2 + i_1 - j_1 + k_1 = i_1 - i_2 + k_1 \geqslant k.$$

2. If $j_1 = i$ then $j = j_2$ and $k \geqslant k_2$.

3. If $j_1 > i$ then $i = i_2$, $j = j_2 - j_1 + i = j_2 - j_1 + i_2$ and $i - j_1 + k = i_2 - j_1 + k \geqslant k_2$.

Since $k \leqslant n$ the above considered cases imply that the equation $(i_1, j_1, [0;k_1]) \cdot \chi = (i_2, j_2, [0;k_2])$ has finitely many solutions.

The proof of the statement that the equation $\chi \cdot (i_1, j_1, [0;k_1]) = (i_2, j_2, [0;k_2])$ has finitely many solutions is similar. □

Theorem 1. *For an arbitrary $n \in \omega$ the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to an inverse subsemigroup of $\mathscr{I}_{\omega}^{n+1}$, namely $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$.*

Proof. We define a map $\mathfrak{I} \colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to \mathscr{I}_{\omega}^{n+1}$ by the formulae $\mathfrak{I}(\mathbf{0}) = \mathbf{0}$ and

$$\mathfrak{I}(i, j, [0;k]) = \begin{pmatrix} i & i+1 & \cdots & i+k \\ j & j+1 & \cdots & j+k \end{pmatrix},$$

for all $i, j \in \omega$ and $k \in \{0, 1, \ldots, n\}$.

It is obvious that so defined map $\mathfrak{I}$ is injective.

Next we shall show that $\mathfrak{I} \colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to \mathscr{I}_{\omega}^{n+1}$ is a homomorphism.

It is obvious that

$$\mathfrak{I}(\mathbf{0}\cdot\mathbf{0}) = \mathfrak{I}(\mathbf{0}) = \mathbf{0} = \mathbf{0}\cdot\mathbf{0} = \mathfrak{I}(\mathbf{0})\cdot\mathfrak{I}(\mathbf{0}),$$

$$\mathfrak{I}\Big(\mathbf{0}\cdot\big(i,j,[0;k]\big)\Big) = \mathfrak{I}(\mathbf{0}) = \mathbf{0} = \mathbf{0}\cdot\begin{pmatrix} i & i+1 & \cdots & i+k \\ j & j+1 & \cdots & j+k \end{pmatrix} = \mathfrak{I}(\mathbf{0})\cdot\mathfrak{I}\big(i,j,[0;k]\big),$$

and

$$\mathfrak{I}\Big(\big(i,j,[0;k]\big)\cdot\mathbf{0}\Big) = \mathfrak{I}(\mathbf{0}) = \mathbf{0} = \begin{pmatrix} i & i+1 & \cdots & i+k \\ j & j+1 & \cdots & j+k \end{pmatrix}\cdot\mathbf{0} = \mathfrak{I}(i,j,[0;k])\cdot\mathfrak{I}(\mathbf{0})$$

for any non-zero element $(i,j,[0;k])$ of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$.

Fix arbitrary $i_1,i_2,j_1,j_2\in\omega$ and $k_1,k_2\in\{0,\ldots,n\}$. In the case when $k_1\leqslant k_2$ we have that

$$\mathfrak{I}\Big(\big(i_1,j_1,[0;k_1]\big)\cdot\big(i_2,j_2,[0;k_2]\big)\Big) = \begin{cases} \mathfrak{I}\Big(i_1-j_1+i_2,j_2,\big(j_1-i_2+[0;k_1]\big)\cap[0;k_2]\Big), & \text{if } j_1<i_2;\\ \mathfrak{I}\big(i_1,j_2,[0;k_1]\cap[0;k_2]\big), & \text{if } j_1=i_2;\\ \mathfrak{I}\Big(i_1,j_1-i_2+j_2,[0;k_1]\cap\big(i_2-j_1+[0;k_2]\big)\Big), & \text{if } j_1>i_2 \end{cases}$$

$$= \begin{cases} \mathfrak{I}(\mathbf{0}), & \text{if } j_1<i_2 \quad \text{and} \quad j_1-i_2+k_1<0;\\ \mathfrak{I}\big(i_1-j_1+i_2,j_2,[0;0]\big), & \text{if } j_1<i_2 \quad \text{and} \quad j_1-i_2+k_1=0;\\ \mathfrak{I}\big(i_1-j_1+i_2,j_2,[0;j_1-i_2+k_1]\big), & \text{if } j_1<i_2 \quad \text{and} \quad 1\leqslant j_1-i_2+k_1\leqslant k_2;\\ \mathfrak{I}\big(i_1,j_2,[0;k_1]\big), & \text{if } j_1=i_2;\\ \mathfrak{I}\big(i_1,j_1-i_2+j_2,[0;k_1]\big), & \text{if } j_1>i_2 \quad \text{and} \quad k_1\leqslant i_1-j_1+k_2;\\ \mathfrak{I}\big(i_1,j_1-i_2+j_2,[0;i_2-j_1+k_2]\big), & \text{if } j_1>i_2 \quad \text{and} \quad k_1>i_1-j_1+k_2;\\ \mathfrak{I}\big(i_1,j_1-i_2+j_2,[0;0]\big), & \text{if } j_1>i_2 \quad \text{and} \quad j_1=i_2+k_2;\\ \mathfrak{I}(\mathbf{0}), & \text{if } j_1>i_2 \quad \text{and} \quad j_1>i_2+k_2 \end{cases}$$

$$= \begin{cases} \mathbf{0}, & \text{if } j_1<i_2 \quad \text{and} \quad j_1-i_2+k_1<0;\\ \begin{pmatrix} i_1-j_1+i_2 \\ j_2 \end{pmatrix}, & \text{if } j_1<i_2 \quad \text{and} \quad j_1-i_2+k_1=0;\\ \begin{pmatrix} i_1-j_1+i_2 & \cdots & i_1+k_1 \\ j_2 & \cdots & j_2+j_1-i_2+k_1 \end{pmatrix}, & \text{if } j_1<i_2 \quad \text{and} \quad 1\leqslant j_1-i_2+k_1\leqslant k_2;\\ \begin{pmatrix} i_1 & \cdots & i_1+k_1 \\ j_2 & \cdots & j_2+k_1 \end{pmatrix}, & \text{if } j_1=i_2;\\ \begin{pmatrix} i_1 & \cdots & i_1+k_1 \\ j_1-i_2+j_2 & \cdots & j_1-i_2+j_2+k_1 \end{pmatrix}, & \text{if } j_1>i_2 \quad \text{and} \quad k_1\leqslant i_2-j_1+k_2;\\ \begin{pmatrix} i_1 & \cdots & i_1+i_2-j_1+k_2 \\ j_1-i_2+j_2 & \cdots & j_2+k_2 \end{pmatrix}, & \text{if } j_1>i_2 \quad \text{and} \quad k_1>i_2-j_1+k_2;\\ \begin{pmatrix} i_1 \\ j_1-i_2+j_2 \end{pmatrix}, & \text{if } j_1>i_2 \quad \text{and} \quad j_1=i_2+k_2;\\ \mathbf{0}, & \text{if } j_1>i_2 \quad \text{and} \quad j_1>i_2+k_2 \end{cases}$$

$$= \begin{cases} \mathbf{0}, & \text{if } j_1<i_2 \quad \text{and} \quad j_1-i_2+k_1<0;\\ \begin{pmatrix} i_1+k_1 \\ j_2 \end{pmatrix}, & \text{if } j_1<i_2 \quad \text{and} \quad j_1-i_2+k_1=0;\\ \begin{pmatrix} i_1-j_1+i_2 & \cdots & i_1+k_1 \\ j_2 & \cdots & j_2+j_1-i_2+k_1 \end{pmatrix}, & \text{if } j_1<i_2 \quad \text{and} \quad 1\leqslant j_1-i_2+k_1\leqslant k_2;\\ \begin{pmatrix} i_1 & \cdots & i_1+k_1 \\ j_2 & \cdots & j_2+k_1 \end{pmatrix}, & \text{if } j_1=i_2;\\ \begin{pmatrix} i_1 & \cdots & i_1+k_1 \\ j_1-i_2+j_2 & \cdots & j_1-i_2+j_2+k_1 \end{pmatrix}, & \text{if } j_1>i_2 \quad \text{and} \quad k_1\leqslant i_2-j_1+k_2;\\ \begin{pmatrix} i_1 & \cdots & i_1+i_2-j_1+k_2 \\ j_1-i_2+j_2 & \cdots & j_2+k_2 \end{pmatrix}, & \text{if } j_1>i_2 \quad \text{and} \quad k_1>i_2-j_1+k_2;\\ \begin{pmatrix} i_1 \\ j_2+k_2 \end{pmatrix}, & \text{if } j_1>i_2 \quad \text{and} \quad j_1=i_2+k_2;\\ \mathbf{0}, & \text{if } j_1>i_2 \quad \text{and} \quad j_1>i_2+k_2 \end{cases}$$

and

$$
\begin{aligned}
&\mathfrak{I}(i_1,j_1,[0;k_1])\cdot\mathfrak{I}(i_2,j_2,[0;k_2])\\
&\qquad=\begin{pmatrix} i_1 & \cdots & i_1+k_1\\ j_1 & \cdots & j_1+k_1\end{pmatrix}\cdot\begin{pmatrix} i_2 & \cdots & i_2+k_2\\ j_2 & \cdots & j_2+k_2\end{pmatrix}\\
&\qquad=\begin{cases}
\mathbf{0}, & \text{if } j_1<i_2 \quad\text{and}\quad j_1+k_1<i_2;\\
\begin{pmatrix} i_1+k_1\\ j_2\end{pmatrix}, & \text{if } j_1<i_2 \quad\text{and}\quad j_1+k_1=i_2;\\
\begin{pmatrix} i_1-j_1+i_2 & \cdots & i_1+k_1\\ j_2 & \cdots & j_2+j_1-i_2+k_1\end{pmatrix}, & \text{if } j_1<i_2 \quad\text{and}\quad j_1+k_1\geqslant i_2+1;\\
\begin{pmatrix} i_1 & \cdots & i_1+k_1\\ j_2 & \cdots & j_2+k_1\end{pmatrix}, & \text{if } j_1=i_2;\\
\begin{pmatrix} i_1 & \cdots & i_1+k_1\\ j_1-i_2+j_2 & \cdots & j_1-i_2+j_2+k_1\end{pmatrix}, & \text{if } j_1>i_2 \quad\text{and}\quad j_1+k_1\leqslant i_2+k_2;\\
\begin{pmatrix} i_1 & \cdots & i_1-j_1+i_2+k_2\\ j_1-i_2+j_2 & \cdots & j_2+k_2\end{pmatrix}, & \text{if } j_1>i_2 \quad\text{and}\quad j_1+k_1>i_2+k_2;\\
\begin{pmatrix} i_1\\ j_2+k_2\end{pmatrix}, & \text{if } j_1>i_2 \quad\text{and}\quad j_1=i_2+k_2;\\
\mathbf{0}, & \text{if } j_1>i_2 \quad\text{and}\quad j_1>i_2+k_2.
\end{cases}
\end{aligned}
$$

In the case when $k_1\geqslant k_2$ we have that

$$
\begin{aligned}
&\mathfrak{I}((i_1,j_1,[0;k_1])\cdot(i_2,j_2,[0;k_2]))\\
&\quad=\begin{cases}
\mathfrak{I}(i_1-j_1+i_2,j_2,(j_1-i_2+[0;k_1])\cap[0;k_2]), & \text{if } j_1<i_2;\\
\mathfrak{I}(i_1,j_2,[0;k_1]\cap[0;k_2]), & \text{if } j_1=i_2;\\
\mathfrak{I}(i_1,j_1-i_2+j_2,[0;k_1]\cap(i_2-j_1+[0;k_2])), & \text{if } j_1>i_2
\end{cases}\\
&\quad=\begin{cases}
\mathfrak{I}(\mathbf{0}), & \text{if } j_1<i_2 \quad\text{and}\quad j_1-i_2+k_1<0;\\
\mathfrak{I}(i_1-j_1+i_2,j_2,[0;0]), & \text{if } j_1<i_2 \quad\text{and}\quad j_1-i_2+k_1=0;\\
\mathfrak{I}(i_1-j_1+i_2,j_2,[0;j_1-i_2+k_1]), & \text{if } j_1<i_2 \quad\text{and}\quad 1\leqslant j_1+k_1\leqslant i_2+k_2;\\
\mathfrak{I}(i_1-j_1+i_2,j_2,[0;k_2]), & \text{if } j_1<i_2 \quad\text{and}\quad j_1+k_1>i_2+k_2;\\
\mathfrak{I}(i_1,j_2,[0;k_2]), & \text{if } j_1=i_2;\\
\mathfrak{I}(i_1,j_1-i_2+j_2,[0;i_2-j_1+k_2]), & \text{if } j_1>i_2 \quad\text{and}\quad i_2-j_1+k_2>0;\\
\mathfrak{I}(i_1,j_1-i_2+j_2,[0;0]), & \text{if } j_1>i_2 \quad\text{and}\quad i_2-j_1+k_2=0;\\
\mathfrak{I}(\mathbf{0}), & \text{if } j_1>i_2 \quad\text{and}\quad i_2-j_1+k_2<0
\end{cases}\\
&\quad=\begin{cases}
\mathbf{0}, & \text{if } j_1<i_2 \quad\text{and}\quad j_1-i_2+k_1<0;\\
\begin{pmatrix} i_1-j_1+i_2\\ j_2\end{pmatrix}, & \text{if } j_1<i_2 \quad\text{and}\quad j_1-i_2+k_1=0;\\
\begin{pmatrix} i_1-j_1+i_2 & \cdots & i_1+k_1\\ j_2 & \cdots & j_1-i_2+j_2+k_1\end{pmatrix}, & \text{if } j_1<i_2 \quad\text{and}\quad j_1+k_1\leqslant i_2+k_2;\\
\begin{pmatrix} i_1-j_1+i_2 & \cdots & i_1-j_1+i_2+k_2\\ j_2 & \cdots & j_2+k_2\end{pmatrix}, & \text{if } j_1<i_2 \quad\text{and}\quad j_1+k_1>i_2+k_2;\\
\begin{pmatrix} i_1+k_2 & \cdots & i_1+k_2\\ j_2+k_2 & \cdots & j_2+k_2\end{pmatrix}, & \text{if } j_1=i_2;\\
\begin{pmatrix} i_1 & \cdots & i_1-j_1+i_2+k_2\\ j_1-i_2+j_2 & \cdots & j_2+k_2\end{pmatrix}, & \text{if } j_1>i_2 \quad\text{and}\quad i_2-j_1+k_2>0;\\
\begin{pmatrix} i_1\\ j_1-i_2+j_2\end{pmatrix}, & \text{if } j_1>i_2 \quad\text{and}\quad i_2-j_1+k_2=0;\\
\mathbf{0}, & \text{if } j_1>i_2 \quad\text{and}\quad i_2-j_1+k_2<0
\end{cases}
\end{aligned}
$$

$$
= \begin{cases}
\mathbf{0}, & \text{if } j_1 < i_2 \quad \text{and} \quad j_1 - i_2 + k_1 < 0; \\
\left(\begin{smallmatrix} i_1+k_1 \\ j_2 \end{smallmatrix}\right), & \text{if } j_1 < i_2 \quad \text{and} \quad j_1 - i_2 + k_1 = 0; \\
\left(\begin{smallmatrix} i_1-j_1+i_2 & \cdots & i_1+k_1 \\ j_2 & \cdots & j_1-i_2+j_2+k_1 \end{smallmatrix}\right), & \text{if } j_1 < i_2 \quad \text{and} \quad j_1 + k_1 \leqslant i_2 + k_2; \\
\left(\begin{smallmatrix} i_1-j_1+i_2 & \cdots & i_1-j_1+i_2+k_2 \\ j_2 & \cdots & j_2+k_2 \end{smallmatrix}\right), & \text{if } j_1 < i_2 \quad \text{and} \quad j_1 + k_1 > i_2 + k_2; \\
\left(\begin{smallmatrix} i_1 & \cdots & i_1+k_2 \\ j_2 & \cdots & j_2+k_2 \end{smallmatrix}\right), & \text{if } j_1 = i_2; \\
\left(\begin{smallmatrix} i_1 & \cdots & i_1-j_1+i_2+k_2 \\ j_1-i_2+j_2 & \cdots & j_2+k_2 \end{smallmatrix}\right), & \text{if } j_1 > i_2 \quad \text{and} \quad j_1 < i_2 + k_2; \\
\left(\begin{smallmatrix} i_1 \\ j_2+k_2 \end{smallmatrix}\right), & \text{if } j_1 > i_2 \quad \text{and} \quad j_1 = i_2 + k_2; \\
\mathbf{0}, & \text{if } j_1 > i_2 \quad \text{and} \quad j_1 > i_2 + k_2
\end{cases}
$$

and

$$
\mathfrak{I}(i_1, j_1, [0;k_1]) \cdot \mathfrak{I}(i_2, j_2, [0;k_2]) = \left(\begin{smallmatrix} i_1 & \cdots & i_1+k_1 \\ j_1 & \cdots & j_1+k_1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} i_2 & \cdots & i_2+k_2 \\ j_2 & \cdots & j_2+k_2 \end{smallmatrix}\right)
$$

$$
= \begin{cases}
\mathbf{0}, & \text{if } j_1 < i_2 \quad \text{and} \quad j_1 + k_1 < i_2; \\
\left(\begin{smallmatrix} i_1+k_1 \\ j_2 \end{smallmatrix}\right), & \text{if } j_1 < i_2 \quad \text{and} \quad j_1 + k_1 = i_2; \\
\left(\begin{smallmatrix} i_1-j_1+i_2 & \cdots & i_1+k_1 \\ j_2 & \cdots & j_2+j_1-i_2+k_1 \end{smallmatrix}\right), & \text{if } j_1 < i_2 \quad \text{and} \quad j_1 + k_1 \leqslant i_2 + k_2; \\
\left(\begin{smallmatrix} i_1-j_1+i_2 & \cdots & i_1-j_1+i_2+k_2 \\ j_2 & \cdots & j_2+k_2 \end{smallmatrix}\right), & \text{if } j_1 < i_2 \quad \text{and} \quad j_1 + k_1 > i_2 + k_2; \\
\left(\begin{smallmatrix} i_1 & \cdots & i_1+k_2 \\ j_2 & \cdots & j_2+k_2 \end{smallmatrix}\right), & \text{if } j_1 = i_2; \\
\left(\begin{smallmatrix} i_1 & \cdots & i_1-j_1+i_2+k_2 \\ j_1-i_2+j_2 & \cdots & i_2+k_2 \end{smallmatrix}\right), & \text{if } j_1 > i_2 \quad \text{and} \quad j_1 < i_2 + k_2; \\
\left(\begin{smallmatrix} i_1 \\ j_2+k_2 \end{smallmatrix}\right), & \text{if } j_1 > i_2 \quad \text{and} \quad j_1 = i_2 + k_2; \\
\mathbf{0}, & \text{if } j_1 > i_2 \quad \text{and} \quad j_1 > i_2 + k_2.
\end{cases}
$$

By [35, Lemma II.1.10] the homomorphic image $\mathfrak{I}(\boldsymbol{B}_{\omega}^{\mathscr{F}_n})$ is an inverse subsemigroup of $\mathscr{I}_{\omega}^{n+1}$.

It is obvious that $\mathfrak{I}(\mathbf{0})$ is the empty partial self-map of ω and it is by the assumption is an order convex partial isomorphism of $(\omega, \leqslant)$. Also the image

$$
\mathfrak{I}\big(i, j, [0;k]\big) = \left(\begin{smallmatrix} i & i+1 & \cdots & i+k \\ j & j+1 & \cdots & j+k \end{smallmatrix}\right)
$$

is an order convex partial isomorphism of $(\omega, \leqslant)$ for all $i, j \in \omega$ and $k \in \{0, 1, \ldots, n\}$. The definition of $\mathfrak{I} \colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to \mathscr{I}_{\omega}^{n+1}$ implies that its co-restriction on the image $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ is surjective, and hence $\mathfrak{I} \colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to \mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ is an isomorphism. $\square$

Remark 1. *Observe that the image $\mathfrak{I}(\boldsymbol{B}_{\omega}^{\mathscr{F}_n})$ does not contain all idempotents of the semigroup $\mathscr{I}_{\omega}^{n+1}$, especially $\left(\begin{smallmatrix} 0 & 2 \\ 0 & 2 \end{smallmatrix}\right) \notin \mathfrak{I}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ for any $n \geqslant 1$. But by [23, Proposition 4], the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_0}$ is isomorphic to the semigroup of $\omega \times \omega$-matrix units, and hence $\boldsymbol{B}_{\omega}^{\mathscr{F}_0}$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{1}$.*

A subset D of a semigroup S is said to be *ω-unstable* if D is infinite and for any $a \in D$ and an infinite subset $B \subseteq D$, we have $aB \cup Ba \nsubseteq D$ [20]. A basic example of ω-unstable sets is given in [20]: for an infinite cardinal λ the set $D = \mathscr{I}_{\omega}^{n} \setminus \mathscr{I}_{\omega}^{n-1}$ is an ω-unstable subset of $\mathscr{I}_{\omega}^{n}$.

For any $n\in\omega$ the definition of the semigroup operation on $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ implies that its subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_k}$ is an ideal of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ for any $k\in\{0,\ldots,n\}$. Also, since $\mathscr{I}_{\omega}^{k+1}(\overrightarrow{\mathrm{conv}})\setminus\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$ is an infinite subset of $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ for any $k\in\{0,\ldots,n\}$, the above arguments and Theorem 1 imply the following assertion.

Lemma 4. *For an arbitrary $n\in\omega$ the subsets $\boldsymbol{B}_{\omega}^{\mathscr{F}_0}\setminus\{\boldsymbol{0}\}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}_k}\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}_{k-1}}$ are ω-unstable of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ for any $k\in\{1,\ldots,n\}$.*

Proof. We shall show that the set $\boldsymbol{B}_{\omega}^{\mathscr{F}_k}\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}_{k-1}}$ is ω-unstable, and the proof that the set $\boldsymbol{B}_{\omega}^{\mathscr{F}_0}\setminus\{\boldsymbol{0}\}$ is ω-unstable is similar.

Fix an arbitrary distinct $(i_1,j_1,[0;k]),(i_2,j_2,[0;k])\in\boldsymbol{B}_{\omega}^{\mathscr{F}_k}\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}_{k-1}}$. The definition of the semigroup operation of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ implies that for any $(i,j,[0;k])\in\boldsymbol{B}_{\omega}^{\mathscr{F}_k}\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}_{k-1}}$ we have that

$$(i,j,[0;k])\cdot(i_p,j_p,[0;k])=\begin{cases}\big(i-j+i_p,j_p,(j-i_p+[0;k])\cap[0;k]\big), & \text{if } j<i_p;\\ (i,j_p,[0;k]\cap[0;k]), & \text{if } j=i_p;\\ \big(i,j-i_p+j_p,[0;k]\cap(i_p-j+[0;k])\big), & \text{if } j>i_p\end{cases}$$

for $p\in\{1,2\}$. In the case when $i_1\neq i_2$ we obtain that

$$(i,j,[0;k])\cdot\Big\{(i_1,j_1,[0;k]),(i_2,j_2,[0;k])\Big\}\nsubseteq\boldsymbol{B}_{\omega}^{\mathscr{F}_k}\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}_{k-1}}.$$

In the case when $j_1\neq j_2$ the proof is similar. □

Definition 1 ([20])**.** *An ideal series for a semigroup S is a chain of ideals*

$$I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_m=S.$$

This ideal series is called tight *if I_0 is a finite set and $D_k=I_k\setminus I_{k-1}$ is an ω-unstable subset for each $k\in\{1,\ldots,m\}$.*

Lemma 4 implies the next result.

Proposition 2. *For an arbitrary $n\in\omega$ the following ideal series*

$$\{\boldsymbol{0}\}\subseteq\boldsymbol{B}_{\omega}^{\mathscr{F}_0}\subseteq\boldsymbol{B}_{\omega}^{\mathscr{F}_1}\subseteq\cdots\subseteq\boldsymbol{B}_{\omega}^{\mathscr{F}_{n-1}}\subseteq\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$$

is tight.

Proposition 3. *For any non-negative integer n and arbitrary $p\in\{0,1,\ldots,n-1\}$ the map $\mathfrak{h}_p\colon\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\to\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ defined by the formulae $\mathfrak{h}_p(\boldsymbol{0})=\boldsymbol{0}$ and*

$$\mathfrak{h}_p(i,j,[0;k])=\begin{cases}\boldsymbol{0}, & \text{if } k\in\{0,1,\ldots,p\};\\ (i,j,[0;k-p-1]), & \text{if } k\in\{p+1,\ldots,n\}\end{cases}$$

is a homomorphism which maps the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ onto its subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{n-p-1}}$.

Proof. First we shall show that the map $\mathfrak{h}_0\colon\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\to\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ defined by the formulae $\mathfrak{h}_0(\boldsymbol{0})=\boldsymbol{0}$ and

$$\mathfrak{h}_0(i,j,[0;k])=\begin{cases}\boldsymbol{0}, & \text{if } k=0;\\ (i,j,[0;k-1]), & \text{if } k\in\{1,\ldots,n\}\end{cases}$$

is a homomorphism.

It is obvious that

$$\mathfrak{h}_0(\mathbf{0})\cdot\mathfrak{h}_0\big(i,j,[0]\big)=\mathbf{0}\cdot\mathbf{0}=\mathfrak{h}_0(\mathbf{0})=\mathfrak{h}_0(\mathbf{0}\cdot(i,j,[0]))$$

and

$$\mathfrak{h}_0\big(i,j,[0]\big)\cdot\mathfrak{h}_0(\mathbf{0})=\mathbf{0}\cdot\mathbf{0}=\mathfrak{h}_0(\mathbf{0})=\mathfrak{h}_0\Big(\big(i,j,[0]\big)\cdot\mathbf{0}\Big)$$

for any $i,j\in\omega$.

Fix arbitrary $i_1,i_2,j_1,j_2\in\omega$ and positive integers k_1 and k_2. In the case when $k_1\leqslant k_2$ we have that

$$\begin{aligned}
&\mathfrak{h}_0\big(i_1,j_1,[0;k_1]\big)\cdot\mathfrak{h}_0\big(i_2,j_2,[0;k_2]\big)\\
&\quad=\big(i_1,j_1,[0;k_1-1]\big)\cdot\big(i_2,j_2,[0;k_2-1]\big)\\
&\quad=\begin{cases}
\Big(i_1-j_1+i_2,j_2,\big(j_1-i_2+[0;k_1-1]\big)\cap[0;k_2-1]\Big), & \text{if } j_1<i_2;\\
\big(i_1,j_2,[0;k_1-1]\cap[0;k_2-1]\big), & \text{if } j_1=i_2;\\
\Big(i_1,j_1-i_2+j_2,[0;k_1-1]\cap\big(i_2-j_1+[0;k_2-1]\big)\Big), & \text{if } j_1>i_2
\end{cases}\\
&\quad=\begin{cases}
\mathbf{0}, & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1-1<0;\\
\big(i_1-j_1+i_2,j_2,[0;j_1-i_2+k_1-1]\big), & \text{if } j_1<i_2 \text{ and } 0\leqslant j_1-i_2+k_1-1\leqslant k_2-1;\\
\big(i_1,j_2,[0;k_1-1]\big), & \text{if } j_1=i_2;\\
\big(i_1,j_1-i_2+j_2,[0;k_1-1]\big), & \text{if } j_1>i_2 \text{ and } k_1-1\leqslant i_1-j_1+k_2-1;\\
\big(i_1,j_1-i_2+j_2,[0;i_2-j_1+k_2-1]\big), & \text{if } j_1>i_2 \text{ and } k_1-1>i_1-j_1+k_2-1\geqslant 0;\\
\mathbf{0}, & \text{if } j_1>i_2 \text{ and } k_1-1>i_1-j_1+k_2-1<0
\end{cases}\\
&\quad=\begin{cases}
\mathbf{0}, & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1<1;\\
\big(i_1-j_1+i_2,j_2,[0;j_1-i_2+k_1-1]\big), & \text{if } j_1<i_2 \text{ and } 1\leqslant j_1-i_2+k_1\leqslant k_2;\\
\big(i_1,j_2,[0;k_1-1]\big), & \text{if } j_1=i_2;\\
\big(i_1,j_1-i_2+j_2,[0;k_1-1]\big), & \text{if } j_1>i_2 \text{ and } k_1\leqslant i_2-j_1+k_2;\\
\big(i_1,j_1-i_2+j_2,[0;i_2-j_1+k_2-1]\big), & \text{if } j_1>i_2 \text{ and } k_1>i_2-j_1+k_2\geqslant 1;\\
\mathbf{0}, & \text{if } j_1>i_2 \text{ and } k_1>i_2-j_1+k_2<1,
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{h}_0\Big(\big(i_1,j_1,[0;k_1]\big)\cdot\big(i_2,j_2,[0;k_2]\big)\Big)\\
&\qquad=\begin{cases}
\mathfrak{h}_0\big(i_1-j_1+i_2,j_2,\big(j_1-i_2+[0;k_1]\big)\cap[0;k_2]\big), & \text{if } j_1<i_2;\\
\mathfrak{h}_0\big(i_1,j_2,[0;k_1]\cap[0;k_2]\big), & \text{if } j_1=i_2;\\
\mathfrak{h}_0\Big(i_1,j_1-i_2+j_2,[0;k_1]\cap\big(i_2-j_1+[0;k_2]\big)\Big), & \text{if } j_1>i_2
\end{cases}\\
&\qquad=\begin{cases}
\mathfrak{h}_0(\mathbf{0}), & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1<0;\\
\mathfrak{h}_0\big(i_1-j_1+i_2,j_2,[0;0]\cap[0;k_2]\big), & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1=0;\\
\mathfrak{h}_0\big(i_1-j_1+i_2,j_2,[0;j_1-i_2+k_1]\big), & \text{if } j_1<i_2 \text{ and } 1\leqslant j_1-i_2+k_1\leqslant k_2;\\
\mathfrak{h}_0\big(i_1,j_2,[0;k_1]\big), & \text{if } j_1=i_2;\\
\mathfrak{h}_0\big(i_1,j_1-i_2+j_2,[0;k_1]\big), & \text{if } j_1>i_2 \text{ and } k_1\leqslant i_2-j_1+k_2;\\
\mathfrak{h}_0\big(i_1,j_1-i_2+j_2,[0;k_1]\cap[0;0]\big), & \text{if } j_1>i_2 \text{ and } k_1>i_2-j_1+k_2=0;\\
\mathfrak{h}_0(\mathbf{0}), & \text{if } j_1>i_2 \text{ and } k_1>i_2-j_1+k_2<0
\end{cases}
\end{aligned}$$

$$
= \begin{cases}
\mathfrak{h}_0(\mathbf{0}), & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 < 0;\\
\mathfrak{h}_0\big(i_1 - j_1 + i_2, j_2, [0;0]\big), & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 = 0;\\
\mathfrak{h}_0\big(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1]\big), & \text{if } j_1 < i_2 \ \text{ and } \ 1 \leqslant j_1 - i_2 + k_1 \leqslant k_2;\\
\mathfrak{h}_0\big(i_1, j_2, [0;k_1]\big), & \text{if } j_1 = i_2;\\
\mathfrak{h}_0\big(i_1, j_1 - i_2 + j_2, [0;k_1]\big), & \text{if } j_1 > i_2 \ \text{ and } \ k_1 \leqslant i_2 - j_1 + k_2;\\
\mathfrak{h}_0\big(i_1, j_1 - i_2 + j_2, [0;0]\big), & \text{if } j_1 > i_2 \ \text{ and } \ k_1 > i_2 - j_1 + k_2 = 0;\\
\mathfrak{h}_0(\mathbf{0}), & \text{if } j_1 > i_2 \ \text{ and } \ k_1 > i_2 - j_1 + k_2 < 0
\end{cases}
$$

$$
= \begin{cases}
\mathbf{0}, & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 < 0;\\
\mathbf{0}, & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 = 0;\\
\mathfrak{h}_0\big(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1]\big), & \text{if } j_1 < i_2 \ \text{ and } \ 1 \leqslant j_1 - i_2 + k_1 \leqslant k_2;\\
\mathfrak{h}_0\big(i_1, j_2, [0;k_1]\big), & \text{if } j_1 = i_2;\\
\mathfrak{h}_0\big(i_1, j_1 - i_2 + j_2, [0;k_1]\big), & \text{if } j_1 > i_2 \ \text{ and } \ k_1 \leqslant i_2 - j_1 + k_2;\\
\mathbf{0}, & \text{if } j_1 > i_2 \ \text{ and } \ k_1 > i_2 - j_1 + k_2 = 0;\\
\mathbf{0}, & \text{if } j_1 > i_2 \ \text{ and } \ k_1 > i_2 - j_1 + k_2 < 0
\end{cases}
$$

$$
= \begin{cases}
\mathbf{0}, & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 < 1;\\
\big(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1 - 1]\big), & \text{if } j_1 < i_2 \ \text{ and } \ 1 \leqslant j_1 - i_2 + k_1 \leqslant k_2;\\
\big(i_1, j_2, [0;k_1 - 1]\big), & \text{if } j_1 = i_2;\\
\big(i_1, j_1 - i_2 + j_2, [0;k_1 - 1]\big), & \text{if } j_1 > i_2 \ \text{ and } \ k_1 \leqslant i_2 - j_1 + k_2;\\
\big(i_1, j_1 - i_2 + j_2, [0; i_2 - j_1 + k_2 - 1]\big), & \text{if } j_1 > i_2 \ \text{ and } \ k_1 > i_2 - j_1 + k_2 \geqslant 1;\\
\mathbf{0}, & \text{if } j_1 > i_2 \ \text{ and } \ k_1 > i_2 - j_1 + k_2 < 1.
\end{cases}
$$

In the case when $k_1 \geqslant k_2$ we have that

$$
\mathfrak{h}_0\big(i_1, j_1, [0;k_1]\big) \cdot \mathfrak{h}_0\big(i_2, j_2, [0;k_2]\big) = \big(i_1, j_1, [0;k_1 - 1]\big) \cdot \big(i_2, j_2, [0;k_2 - 1]\big)
$$

$$
= \begin{cases}
\Big(i_1 - j_1 + i_2, j_2, \big(j_1 - i_2 + [0;k_1 - 1]\big) \cap [0;k_2 - 1]\Big), & \text{if } j_1 < i_2;\\
\big(i_1, j_2, [0;k_1 - 1] \cap [0;k_2 - 1]\big), & \text{if } j_1 = i_2;\\
\Big(i_1, j_1 - i_2 + j_2, [0;k_1 - 1] \cap \big(i_2 - j_1 + [0;k_2 - 1]\big)\Big), & \text{if } j_1 > i_2
\end{cases}
$$

$$
= \begin{cases}
\mathbf{0}, & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 - 1 < 0;\\
\big(i_1 - j_1 + i_2, j_2, [0;0] \cap [0;k_2 - 1]\big), & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 - 1 = 0;\\
\big(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1 - 1] \cap [0;k_2 - 1]\big), & \text{if } j_1 < i_2 \ \text{ and } \ 1 \leqslant j_1 - i_2 + k_1 - 1 \leqslant k_2 - 1;\\
\big(i_1 - j_1 + i_2, j_2, [0;k_2 - 1]\big), & \text{if } j_1 < i_2 \ \text{ and } \ k_2 - 1 < j_1 - i_2 + k_1 - 1;\\
\big(i_1, j_2, [0;k_1 - 1] \cap [0;k_2 - 1]\big), & \text{if } j_1 = i_2;\\
\big(i_1, j_1 - i_2 + j_2, [0; i_2 - j_1 + k_2 - 1]\big), & \text{if } j_1 > i_2 \ \text{ and } \ i_2 - j_1 + k_2 - 1 \geqslant 0;\\
\mathbf{0}, & \text{if } j_1 > i_2 \ \text{ and } \ i_2 - j_1 + k_2 - 1 < 0
\end{cases}
$$

$$
= \begin{cases}
\mathbf{0}, & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 < 1;\\
\big(i_1 - j_1 + i_2, j_2, [0;0]\big), & \text{if } j_1 < i_2 \ \text{ and } \ j_1 - i_2 + k_1 = 1;\\
\big(i_1 - j_1 + i_2, j_2, [0; j_1 - i_2 + k_1 - 1]\big), & \text{if } j_1 < i_2 \ \text{ and } \ 1 \leqslant j_1 - i_2 + k_1 - 1 \leqslant k_2 - 1;\\
\big(i_1 - j_1 + i_2, j_2, [0;k_2 - 1]\big), & \text{if } j_1 < i_2 \ \text{ and } \ k_2 < j_1 - i_2 + k_1;\\
\big(i_1, j_2, [0;k_2 - 1]\big), & \text{if } j_1 = i_2;\\
\big(i_1, j_1 - i_2 + j_2, [0; i_2 - j_1 + k_2 - 1]\big), & \text{if } j_1 > i_2 \ \text{ and } \ i_2 - j_1 + k_2 \geqslant 1;\\
\mathbf{0}, & \text{if } j_1 > i_2 \ \text{ and } \ i_2 - j_1 + k_2 < 1
\end{cases}
$$

and

$$\mathfrak{h}_0\Big(\big(i_1,j_1,[0;k_1]\big)\cdot\big(i_2,j_2,[0;k_2]\big)\Big)=\begin{cases}\mathfrak{h}_0\big(i_1-j_1+i_2,j_2,(j_1-i_2+[0;k_1])\cap[0;k_2]\big), & \text{if } j_1<i_2;\\ \mathfrak{h}_0\big(i_1,j_2,[0;k_1]\cap[0;k_2]\big), & \text{if } j_1=i_2;\\ \mathfrak{h}_0\Big(i_1,j_1-i_2+j_2,[0;k_1]\cap\big(i_2-j_1+[0;k_2]\big)\Big), & \text{if } j_1>i_2\end{cases}$$

$$=\begin{cases}\mathfrak{h}_0(\mathbf{0}), & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1<0;\\ \mathfrak{h}_0\big(i_1-j_1+i_2,j_2,[0;0]\big), & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1=0;\\ \mathfrak{h}_0\big(i_1-j_1+i_2,j_2,[0;j_1-i_2+k_1]\big), & \text{if } j_1<i_2 \text{ and } k_2\geqslant j_1-i_2+k_1\geqslant 1;\\ \mathfrak{h}_0\big(i_1-j_1+i_2,j_2,[0;k_2]\big), & \text{if } j_1<i_2 \text{ and } k_2<j_1-i_2+k_1\geqslant 1;\\ \mathfrak{h}_0\big(i_1,j_2,[0;k_2]\big), & \text{if } j_1=i_2;\\ \mathfrak{h}_0\big(i_1,j_1-i_2+j_2,[0;i_2-j_1+k_2]\big), & \text{if } j_1>i_2 \text{ and } 1\leqslant i_2-j_1+k_2;\\ \mathfrak{h}_0\big(i_1,j_1-i_2+j_2,[0;0]\big), & \text{if } j_1>i_2 \text{ and } 0=i_2-j_1+k_2;\\ \mathfrak{h}_0(\mathbf{0}), & \text{if } j_1>i_2 \text{ and } i_2-j_1+k_2<0\end{cases}$$

$$=\begin{cases}\mathfrak{h}_0(\mathbf{0}), & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1<0;\\ \mathfrak{h}_0(\mathbf{0}), & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1=0;\\ \mathfrak{h}_0\big(i_1-j_1+i_2,j_2,[0;j_1-i_2+k_1]\big), & \text{if } j_1<i_2 \text{ and } k_2\geqslant j_1-i_2+k_1\geqslant 1;\\ \mathfrak{h}_0\big(i_1-j_1+i_2,j_2,[0;k_2]\big), & \text{if } j_1<i_2 \text{ and } k_2<j_1-i_2+k_1\geqslant 1;\\ \mathfrak{h}_0\big(i_1,j_2,[0;k_2]\big), & \text{if } j_1=i_2;\\ \mathfrak{h}_0\big(i_1,j_1-i_2+j_2,[0;i_2-j_1+k_2]\big), & \text{if } j_1>i_2 \text{ and } 1\leqslant i_2-j_1+k_2;\\ \mathfrak{h}_0(\mathbf{0}), & \text{if } j_1>i_2 \text{ and } 0=i_2-j_1+k_2;\\ \mathfrak{h}_0(\mathbf{0}), & \text{if } j_1>i_2 \text{ and } i_2-j_1+k_2<0\end{cases}$$

$$=\begin{cases}\mathbf{0}, & \text{if } j_1<i_2 \text{ and } j_1-i_2+k_1\leqslant 0;\\ \big(i_1-j_1+i_2,j_2,[0;j_1-i_2+k_1-1]\big), & \text{if } j_1<i_2 \text{ and } k_2\geqslant j_1-i_2+k_1\geqslant 1;\\ \big(i_1-j_1+i_2,j_2,[0;k_2-1]\big), & \text{if } j_1<i_2 \text{ and } k_2<j_1-i_2+k_1\geqslant 1;\\ \big(i_1,j_2,[0;k_2-1]\big), & \text{if } j_1=i_2;\\ \big(i_1,j_1-i_2+j_2,[0;i_2-j_1+k_2-1]\big), & \text{if } j_1>i_2 \text{ and } 1\leqslant i_2-j_1+k_2;\\ \mathbf{0}, & \text{if } j_1>i_2 \text{ and } i_2-j_1+k_2\leqslant 0.\end{cases}$$

Next observe that by induction we obtain that

$$\mathfrak{h}_p=\underbrace{\mathfrak{h}_0\circ\cdots\circ\mathfrak{h}_0}_{p+1\text{-times}}=\mathfrak{h}_0^{p+1}$$

for any $p\in\{1,\ldots,n-1\}$.

Simple verifications show that the homomorphism $\mathfrak{h}_p\colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n}\to\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ maps the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ onto its subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{n-p-1}}$. □

Proposition 4. *For any positive integer n every congruence on the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is Rees.*

Proof. First we observe that since the semigroup $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ has the zero $\mathbf{0}$ the identity congruence on $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is Rees, and it is obvious that the universal congruence on $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ is Rees, too.

By induction we shall show the following: if $\mathfrak{C}$ is a congruence $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ such that for some $k\leqslant n$ there exist two distinct $\mathfrak{C}$-equivalent elements $\alpha,\beta\in\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$ with $\max\{\operatorname{rank}\alpha,\operatorname{rank}\beta\}=k$, then all elements of subsemigroup $\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$ are equivalent.

In the case when $k=1$ then it is obvious that the semigroup $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$ is isomorphic to the semigroup $\mathscr{I}_{\omega}^{1}$ which is isomorphic to the semigroup $\mathscr{B}_{\omega}$ of $\omega\times\omega$-matrix units. Since the semigroup $\mathscr{B}_{\omega}$ of $\omega\times\omega$-matrix units is congruence-free (see [24, Corollary 3]), the statement that any two distinct elements of the semigroup $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$ are $\mathfrak{C}$-equivalent implies that all elements of $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$ are $\mathfrak{C}$-equivalent. Hence the initial step of induction holds.

Next we shall show the step of induction: if $\mathfrak{C}$ is a congruence $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ such that there exist two distinct $\mathfrak{C}$-equivalent elements $\alpha,\beta\in\mathscr{I}_{\omega}^{k+1}(\overrightarrow{\mathrm{conv}})$ with $\max\{\operatorname{rank}\alpha,\operatorname{rank}\beta\}=k+1$, then the statement that all elements of the subsemigroup $\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$ are $\mathfrak{C}$-equivalent implies that all elements of the subsemigroup $\mathscr{I}_{\omega}^{k+1}(\overrightarrow{\mathrm{conv}})$ are $\mathfrak{C}$-equivalent, as well.

Next we consider all possible cases.

(I) Suppose that $\alpha=\left(\begin{smallmatrix} a & a+1 & \cdots & a+k\\ b & b+1 & \cdots & b+k\end{smallmatrix}\right)$, $\beta=\mathbf{0}$ and $\alpha\mathfrak{C}\beta$. Since C is a congruence on $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$, for any element $\gamma=\left(\begin{smallmatrix} c & c+1 & \cdots & c+k_1\\ d & d+1 & \cdots & d+k_1\end{smallmatrix}\right)$ of the subsemigroup $\mathscr{I}_{\omega}^{k+1}(\overrightarrow{\mathrm{conv}})$, where $k_1\leqslant k+1$, we have that

$$\gamma=\begin{pmatrix} c & c+1 & \cdots & c+k_1\\ a & a+1 & \cdots & a+k_1\end{pmatrix}\cdot\alpha\cdot\begin{pmatrix} b & b+1 & \cdots & b+k_1\\ d & d+1 & \cdots & d+k_1\end{pmatrix}$$

is $\mathfrak{C}$-equivalent to

$$\begin{pmatrix} c & c+1 & \cdots & c+k_1\\ a & a+1 & \cdots & a+k_1\end{pmatrix}\cdot\mathbf{0}\cdot\begin{pmatrix} b & b+1 & \cdots & b+k_1\\ d & d+1 & \cdots & d+k_1\end{pmatrix}=\mathbf{0},$$

and hence $\gamma\mathfrak{C}\mathbf{0}$.

(II) Suppose that $\alpha=\left(\begin{smallmatrix} a & a+1 & \cdots & a+k\\ a & a+1 & \cdots & a+k\end{smallmatrix}\right)$ and $\beta=\left(\begin{smallmatrix} b & b+1 & \cdots & b+k_1\\ b & b+1 & \cdots & b+k_1\end{smallmatrix}\right)$ are non-zero $\mathfrak{C}$-equivalent idempotents of the subsemigroup $\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$ such that $k_1\leqslant k$ and $\beta\preccurlyeq\alpha$. In this case we have that $[b;b+k_1]\subseteq[a;a+k]$. We put

$$\varepsilon=\begin{cases}\begin{pmatrix} a+1 & \cdots & a+k\\ a+1 & \cdots & a+k\end{pmatrix}, & \text{if } a=b;\\ \begin{pmatrix} a & \cdots & a+k-1\\ a & \cdots & a+k-1\end{pmatrix}, & \text{if } a+k=b+k_1\end{cases}$$

and $\gamma=\left(\begin{smallmatrix} a & a+1 & \cdots & a+k\\ a+1 & a+2 & \cdots & a+k+1\end{smallmatrix}\right)$ if $a<b$ and $b+k_1<a+k$.

In the case when either $a=b$ or $a+k=b+k_1$ we obtain that $\varepsilon\alpha$ and $\varepsilon\beta$ are distinct $\mathfrak{C}$-equivalent idempotents of the subsemigroup $\mathscr{I}_{\omega}^{k-1}(\overrightarrow{\mathrm{conv}})$ and hence by the assumption of induction all elements of $\mathscr{I}_{\omega}^{k-1}(\overrightarrow{\mathrm{conv}})$ are $\mathfrak{C}$-equivalent.

In the case when $a<b$ and $b+k_1<a+k$ we obtain that $\gamma\alpha\gamma^{-1}$ and $\gamma\beta\gamma^{-1}$ are distinct $\mathfrak{C}$-equivalent idempotents of the subsemigroup $\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$, because they have distinct rank $\leqslant k$. Hence by the assumption of induction all elements of $\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$ are $\mathfrak{C}$-equivalent.

In both above cases we get that $\alpha\mathfrak{C}\mathbf{0}$, which implies that case **(I)** holds.

(III) Suppose that α and β are distinct incomparable non-zero $\mathfrak{C}$-equivalent idempotents of the subsemigroup $\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$ of $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ such that $\operatorname{rank}\alpha=k+1$. Then $\alpha=\alpha\alpha\mathfrak{C}\alpha\beta$ and $\alpha\beta\preccurlyeq\alpha$ which implies that either case **(II)** or case **(I)** holds.

(IV) Suppose that α and β are distinct non-zero $\mathfrak{C}$-equivalent elements of the subsemigroup $\mathscr{I}_{\omega}^{k}(\overrightarrow{\mathrm{conv}})$ of $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ such that $\operatorname{rank}\alpha=k+1$. Then at least one of the following conditions $\alpha\alpha^{-1}\neq\beta\beta^{-1}$ or $\alpha^{-1}\alpha\neq\beta^{-1}\beta$ holds, because by Proposition 1(8) and Theorem 1 all $\mathscr{H}$-classes in $\mathscr{I}_{\omega}^{n}(\overrightarrow{\mathrm{conv}})$ are singletons. By [32, Proposition 2.3.4(1)], $\alpha\alpha^{-1}\mathfrak{C}\beta\beta^{-1}$ and $\alpha^{-1}\alpha\mathfrak{C}\beta^{-1}\beta$, and hence at least one of cases **(II)** or **(III)** holds. □

Theorem 1 and Proposition 4 imply the description of all congruences on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$.

Theorem 2. *For an arbitrary $n \in \omega$ the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ admits only Rees congruences.*

Theorem 3. *Let n be a non-negative integer and S be a semigroup. For any homomorphism $\mathfrak{h}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to S$ the image $\mathfrak{h}(\boldsymbol{B}_{\omega}^{\mathscr{F}_n})$ is either isomorphic to $\boldsymbol{B}_{\omega}^{\mathscr{F}_k}$ for some $k \in \{0,1,\ldots,n\}$, or is a singleton.*

Proof. By Theorem 2 the homomorphism $\mathfrak{h}$ generates the Rees congruence $\mathfrak{C}_{\mathfrak{h}}$ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. By Proposition 1(9) the following ideal series

$$\{\mathbf{0}\} \subsetneqq \boldsymbol{B}_{\omega}^{\mathscr{F}_0} \subsetneqq \boldsymbol{B}_{\omega}^{\mathscr{F}_1} \subsetneqq \cdots \subsetneqq \boldsymbol{B}_{\omega}^{\mathscr{F}_{n-1}} \subsetneqq \boldsymbol{B}_{\omega}^{\mathscr{F}_n}$$

is maximal in $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, i.e. if $\mathscr{J}$ is an ideal of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ then either $\mathscr{J} = \{\mathbf{0}\}$ or $\mathscr{J} = \boldsymbol{B}_{\omega}^{\mathscr{F}_m}$ for some $m \in \{0,1,\ldots,n\}$.

It is obvious that if $\mathscr{J} = \{\mathbf{0}\}$ then the Rees congruence $\mathfrak{C}_{\mathscr{J}}$ generates the injective homomorphism $\mathfrak{h}_{\mathfrak{C}_{\mathscr{J}}}$, and hence the image $\mathfrak{h}_{\mathfrak{C}_{\mathscr{J}}}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ is isomorphic to the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. Similar in the case when $\mathscr{J} = \boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ we have that the image $\mathfrak{h}_{\mathfrak{C}_{\mathscr{J}}}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ is a singleton.

Suppose that $\mathscr{J} = \boldsymbol{B}_{\omega}^{\mathscr{F}_m}$ for some $m \in \{0,1,\ldots,n-1\}$. Then the Rees congruence $\mathfrak{C}_{\mathscr{J}}$ generates the natural homomorphism $\mathfrak{h}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to \boldsymbol{B}_{\omega}^{\mathscr{F}_n}/\mathscr{J}$. It is obvious that $\alpha\mathfrak{C}_{\mathscr{J}}\beta$ if and only if $\mathfrak{h}_m(\alpha) = \mathfrak{h}_m(\beta)$ for $\alpha,\beta \in \boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, where $\mathfrak{h}_m\colon \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \to \boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is the homomorphism defined in Proposition 3. Then by Proposition 3 the image $\mathfrak{h}\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ is isomorphic to the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_{n-m-1}}$. $\square$

3 On topologizations and closure of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$

In this section, we establish topologizations of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ and its compact-like shift-continuous topologies.

Theorem 4. *Let n be a non-negative integer. Then for any shift-continuous T_1-topology τ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ every non-zero element of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is an isolated point of $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ and hence every subset in $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ which contains zero is closed. Moreover, for any non-zero element $(i,j,[0;k])$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ the set ${\uparrow_{\preccurlyeq}}(i,j,[0;k])$ is open-and-closed in $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$.*

Proof. Fix an arbitrary non-zero element $(i,j,[0;k])$ of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, where $i,j \in \omega$, $k \in \{0,\ldots,n\}$. Proposition 3 and [20, Proposition 7] imply there exists an open neighbourhood $U_{(i,j,[0;k])}$ of the point $(i,j,[0;k])$ in $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ such that

- $U_{(i,j,[0;k])} \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \setminus \boldsymbol{B}_{\omega}^{\mathscr{F}_{k-1}}$ and $(i,j,[0;k])$ is an isolated point in $\boldsymbol{B}_{\omega}^{\mathscr{F}_k}$ if $k \in \{1,\ldots,n\}$;
- $U_{(i,j,[0;k])} \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \setminus \{\mathbf{0}\}$ and $(i,j,[0;k])$ is an isolated point in $\boldsymbol{B}_{\omega}^{\mathscr{F}_0}$ if $k = 0$.

By separate continuity of the semigroup operation in $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ there exists an open neighbourhood $V_{(i,j,[0;k])}$ of $(i,j,[0;k])$ such that $V_{(i,j,[0;k])} \subseteq U_{(i,j,[0;k])}$ and

$$(i,i,[0;k]) \cdot V_{(i,j,[0;k])} \cdot (j,j,[0;k]) \subseteq U_{(i,j,[0;k])}.$$

We claim that $V_{(i,j,[0;k])} \subseteq {\uparrow}_{\preccurlyeq}(i,j,[0;k])$. Suppose to the contrary that there exists $(i_1,j_1,[0;k_1]) \in V_{(i,j,[0;k])} \setminus {\uparrow}_{\preccurlyeq}(i,j,[0;k])$. Then by [32, Lemma 1.4.6(4)] we have that

$$(i,i,[0;k]) \cdot (i_1,j_1,[0;k_1]) \cdot (j,j,[0;k]) \neq (i,j,[0;k]).$$

Since $\boldsymbol{B}_{\omega}^{\mathscr{F}_k}$ is an ideal of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ the above inequality implies that

$$(i,i,[0;k]) \cdot V_{(i,j,[0;k])} \cdot (j,j,[0;k]) \nsubseteq U_{(i,j,[0;k])},$$

a contradiction. Hence $V_{(i,j,[0;k])} \subseteq {\uparrow}_{\preccurlyeq}(i,j,[0;k])$. By Lemma 1 the set ${\uparrow}_{\preccurlyeq}(i,j,[0;k])$ is finite, which implies that $(i,j,[0;k])$ is an isolated point of $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$, because $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ is a T_1-space.

The last statement follows from the equality

$${\uparrow}_{\preccurlyeq}(i,j,[0;k]) = \left\{(a,b,[0;p]) \in \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \colon (i,i,[0;k]) \cdot (a,b,[0;p]) = (i,j,[0;k])\right\}$$

and the assumption that τ is a shift-continuous T_1-topology on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. □

Recall [17], that a topological space X is called:

- *scattered* if X contains no non-empty subset which is dense-in-itself;
- 0-*dimensional* if X has a base which consists of open-and-closed subsets;
- *collectionwise normal* if for every discrete family $\{F_i\}_{i\in\mathscr{S}}$ of closed subsets of X there exists a pairwise disjoint family of open sets $\{U_i\}_{i\in\mathscr{S}}$ such that $F_i \subseteq U_i$ for all $i \in \mathscr{S}$.

Corollary 1. *Let n be a non-negative integer. Then for any shift-continuous T_1-topology τ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ the space $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ is scattered,* 0*-dimensional and collectionwise normal.*

Proof. Theorem 4 implies that $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ is a scattered, 0-dimensional space.

Let $\{F_s\}_{s\in\mathscr{S}}$ be a discrete family of closed subsets of $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$. By Theorem 4 every non-zero element of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is an isolated point of $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$. In the case when every element of the family $\{F_s\}_{s\in\mathscr{S}}$ does not contain the zero $\mathbf{0}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ by [17, Theorem 5.1.17] the space $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ is collectionwise normal. Suppose that $\mathbf{0} \in F_{s_0}$ for some $s_0 \in \mathscr{S}$. Let $U(\mathbf{0})$ be an open neighbourhood of the zero $\mathbf{0}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, which intersects at more one element of the family $\{F_s\}_{s\in\mathscr{S}}$. Put $U_{s_0} = U(0) \cup F_{s_0}$ and $U_s = F_s$ for all $s \in \mathscr{S} \setminus \{s_o\}$. Then $U_s \cap U_t = \varnothing$ for all distinct $s,t \in \mathscr{S}$ and hence by [17, Theorem 5.1.17] the space $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ is collectionwise normal. □

Example 1. *Let n be a non-negative integer. We define a topology τ_{Ac} on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ in the following way. All non-zero elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ are isolated points of $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau_{\mathrm{Ac}}\right)$ and the family $\mathscr{B}_{\mathrm{Ac}}(\mathbf{0}) = \left\{A \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \colon \mathbf{0} \in A \text{ and } \boldsymbol{B}_{\omega}^{\mathscr{F}_n} \setminus A \text{ is finite}\right\}$ determines the base of the topology τ_{Ac} at the point $\mathbf{0}$.*

It is obvious that the topological space $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau_{\mathrm{Ac}}\right)$ is homeomorphic to the Alexandroff one-point compactification of the discrete infinite countable space, and hence $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau_{\mathrm{Ac}}\right)$ is a

Hausdorff compact space. Then the space $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$ *is normal and since it has a countable base, by the Urysohn Metrization Theorem (see [17, Theorem 4.2.9]) the space* $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$ *is metrizable.*

Next we shall show that $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$ *is a semitopological semigroup. Let* α *and* β *be non-zero elements of the semigroup* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$*. Since* α *and* β *are isolated points in* $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$*, it is sufficient to show how to find for a fixed open neighbourhood* $U_{\mathbf{0}}$ *open neighbourhoods* $V_{\mathbf{0}}$ *and* $W_{\mathbf{0}}$ *of the zero* $\mathbf{0}$ *in* $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$ *such that*

$$V_{\mathbf{0}} \cdot \alpha \subseteq U_{\mathbf{0}} \qquad \text{and} \qquad \beta \cdot W_{\mathbf{0}} \subseteq U_{\mathbf{0}}.$$

Since the space $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$ *is compact, any open neighbourhood* $U_{\mathbf{0}}$ *of the zero* $\mathbf{0}$ *is cofinite subset in* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$*. By Lemma 2,*

$$V_{\mathbf{0}} = \{\gamma \in U_{\mathbf{0}} \colon \gamma \cdot \alpha \in U_{\mathbf{0}}\} \qquad \text{and} \qquad W_{\mathbf{0}} = \{\gamma \in U_{\mathbf{0}} \colon \beta \cdot \gamma \in U_{\mathbf{0}}\}$$

are cofinite subsets of $U_{\mathbf{0}}$ *and hence by the definition of the topology* τ_{Ac} *the sets* $V_{\mathbf{0}}$ *and* $W_{\mathbf{0}}$ *are required open neighbourhoods of the zero* $\mathbf{0}$ *in* $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$*.*

Since all non-zero elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *are isolated points in* $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$ *and every open neighbourhood* $U_{\mathbf{0}}$ *of the zero in* $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$ *has the finite complement in* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$*, the inversion is continuous in* $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$*.*

The following theorem describes all compact-like shift-continuous T_1-topologies on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$.

Theorem 5. *Let* n *be a non-negative integer. Then for any shift-continuous* T_1*-topology* τ *on the semigroup* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ *the following conditions are equivalent:*

(1) $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$ *is a compact semitopological semigroup;*

(2) $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$ *is topologically isomorphic to* $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau_{\mathrm{Ac}}\right)$*;*

(3) $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$ *is a compact semitopological semigroup with continuous inversion;*

(4) $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$ *is an* $\omega_{\mathfrak{d}}$*-compact space.*

Proof. Implications (1) $\Rightarrow$ (4), (2) $\Rightarrow$ (1), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. Since by Theorem 4 every non-zero element of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is an isolated point in $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$, statement (1) implies (2).

(4) $\Rightarrow$ (1) Suppose there exists a shift-continuous T_1-topology τ on the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ such that $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$ is an $\omega_{\mathfrak{d}}$-compact non-compact space. Then there exists an open cover $\mathscr{U} = \{U_s\}$ of $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$, which has no a finite subcover. Let $U_{s_0} \in \mathscr{U}$ be such that $U_{s_0} \ni \mathbf{0}$. Then $\boldsymbol{B}_{\omega}^{\mathscr{F}_n} \setminus U_{s_0}$ is an infinite countable subset of isolated points of $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$. We enumerate the set $\boldsymbol{B}_{\omega}^{\mathscr{F}_n} \setminus U_{s_0}$ by positive integers, i.e. $\boldsymbol{B}_{\omega}^{\mathscr{F}_n} \setminus U_{s_0} = \{\alpha_i \colon i \in \mathbb{N}\}$. Next we define a map $f \colon (\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau) \to \omega_{\mathfrak{d}}$ by the formula

$$f(\alpha) = \begin{cases} 0, & \text{if } \alpha \in U_{s_0}; \\ i, & \text{if } \alpha = \alpha_i \text{ for some } i \in \mathbb{N}. \end{cases}$$

By Theorem 4 the set U_{s_0} is open-and-closed in $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$, and hence so defined map f is continuous. But the image $f\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)$ is not a compact subset of $\omega_{\mathfrak{d}}$, a contradiction. The obtained contradiction implies the implication $(4)\Rightarrow(1)$. □

The following proposition states that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ has a similar closure in a T_1-semitopological semigoup as the bicyclic monoid (see [10,16]), the λ-polycyclic monoid [9], graph inverse semigroups [7,34], McAlister semigroups [8], locally compact semitopological 0-bisimple inverse ω-semigroups with a compact maximal subgroup [18], and other discrete semigroups of bijective partial transformations [12,13,19,22,25,27–30].

Proposition 5. *Let n be a non-negative integer. If S is a T_1-semitopological semigroup, which contains $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ as a dense proper subsemigroup, then $I=\left(S\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)\cup\{\mathbf{0}\}$ is an ideal of S.*

Proof. Fix an arbitrary element $\nu\in I$. If $\chi\cdot\nu=\zeta\notin I$ for some $\chi\in\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, then there exists an open neighbourhood $U(\nu)$ of the point ν in the space S such that $\{\chi\}\cdot U(\nu)=\{\zeta\}\subset\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\setminus\{\mathbf{0}\}$. By Lemma 3 the open neighbourhood $U(\nu)$ should contain finitely many elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, which contradicts our assumption. Hence $\chi\cdot\nu\in I$ for all $\chi\in\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ and $\nu\in I$. The proof of the statement that $\nu\cdot\chi\in I$ for all $\chi\in\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ and $\nu\in I$ is similar.

Suppose to the contrary that $\chi\cdot\nu=\omega\notin I$ for some $\chi,\nu\in I$. Then $\omega\in\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ and the separate continuity of the semigroup operation in S yields open neighbourhoods $U(\chi)$ and $U(\nu)$ of the points χ and ν in the space S, respectively, such that $\{\chi\}\cdot U(\nu)=\{\omega\}$ and $U(\chi)\cdot\{\nu\}=\{\omega\}$. Since both neighbourhoods $U(\chi)$ and $U(\nu)$ contain infinitely many elements of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, equalities $\{\chi\}\cdot U(\nu)=\{\omega\}$ and $U(\chi)\cdot\{\nu\}=\{\omega\}$ do not hold, because $\{\chi\}\cdot\left(U(\nu)\cap\boldsymbol{B}_{\omega}^{\mathscr{F}_n}\right)\subseteq I$. The obtained contradiction implies that $\chi\cdot\nu\in I$. □

For any $k\in\{0,1,\ldots,n+1\}$ we denote

$$D_k=\left\{\alpha\in\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})\colon\ \operatorname{rank}\alpha=k\right\}.$$

We observe that by Proposition 1(9) and Theorem 1, $\boldsymbol{D}=\{D_k\colon k=0,1,\ldots,n+1\}$ is the family of all $\mathscr{D}$-classed of the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$.

The following proposition describes the remainder of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ in a semitopological semigroup.

Proposition 6. *Let n be a non-negative integer. If S is a T_1-semitopological semigroup, which contains $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ as a dense proper subsemigroup, then $\chi\cdot\chi=\mathbf{0}$ for all $\chi\in S\setminus\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$.*

Proof. We observe that $\mathbf{0}$ is zero of the semigroup S by [18, Lemma 4.4].

We shall prove the statement of the proposition for the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$, which by Theorem 1 is isomorphic to the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$.

Fix an arbitrary $\chi\in S\setminus\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ and any open neighbourhood $U(\chi)$ of the point χ in S. Since $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ is a dense proper subsemigroup of S the set $U(\chi)\cap\left(\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})\setminus\{\mathbf{0}\}\right)$ is infinite. Since the family $\boldsymbol{D}$ is finite, there exists $i\in\{1,\ldots,n+1\}$ such that the set $U(\chi)\cap D_i$ is infinite. This and the definition of the semigroup $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ imply that at least one of the families

$$\mathfrak{dom}\,D_iU(\chi)=\left\{\operatorname{dom}\alpha\colon\alpha\in U(\chi)\cap D_i\right\}\qquad\text{or}\qquad\mathfrak{ran}\,D_iU(\chi)=\left\{\operatorname{ran}\alpha\colon\alpha\in U(\chi)\cap D_i\right\}$$

has infinitely many members. Assume that the family $\mathfrak{dom}\, D_i U(\chi)$ is infinite. Then the definition of the semigroup operation on $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ implies that there exist infinitely many $\beta \in U(\chi) \cap \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ such that $\mathbf{0} \in \beta \cdot U(\chi)$, and since S is a T_1-space we have that $\beta \cdot \chi = \mathbf{0}$ for such elements β. Also, the infiniteness of $\mathfrak{dom}\, D_i U(\chi)$ and the semigroup operation of $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ imply the existence infinitely many $\gamma \in U(\chi) \cap \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ such that $\mathbf{0} \in U(\chi) \cdot \gamma$, and since S is a T_1-space we have that $\chi \cdot \gamma = \mathbf{0}$ for such elements γ. In the case when the family $\mathfrak{ran}\, D_i U(\chi)$ is infinite similarly we obtain that there exist infinitely many $\beta, \gamma \in U(\chi) \cap \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ such that $\beta \cdot \chi = \mathbf{0}$ and $\chi \cdot \gamma = \mathbf{0}$.

Thus we show that $\mathbf{0} \in V(\chi) \cdot \chi$ and $\mathbf{0} \in \chi \cdot V(\chi)$ for any open neighbourhood $V(\chi)$ of the point χ in S. Since S is a T_1-space, this implies the required equality $\chi \cdot \chi = \mathbf{0}$ for all $\chi \in S \setminus \boldsymbol{B}_\omega^{\mathscr{F}_n}$. □

Let $\mathfrak{STSG}$ be a class of semitopological semigroups. A semigroup $S \in \mathfrak{STSG}$ is called *H-closed in* $\mathfrak{STSG}$, if S is a closed subsemigroup of any topological semigroup $T \in \mathfrak{STSG}$, which contains S both as a subsemigroup and as a topological space. H-closed topological semigroups were introduced by J.W. Stepp in [38], and there they were called *maximal semigroups*. A semitopological semigroup $S \in \mathfrak{STSG}$ is called *absolutely H-closed in the class* $\mathfrak{STSG}$, if any continuous homomorphic image of S into $T \in \mathfrak{STSG}$ is H-closed in $\mathfrak{STSG}$. An algebraic semigroup S is called:

- *algebraically complete in* $\mathfrak{STSG}$, if S with any Hausdorff topology τ such that $(S, \tau) \in \mathfrak{STSG}$ is H-closed in $\mathfrak{STSG}_0$;
- *algebraically h-complete in* $\mathfrak{STSG}$, if S with discrete topology $\tau_{\mathfrak{d}}$ is absolutely H-closed in $\mathfrak{STSG}$ and $(S, \tau_{\mathfrak{d}}) \in \mathfrak{STSG}$.

Absolutely H-closed topological semigroups and algebraically h-complete semigroups were introduced by J.W. Stepp in [39], and there they were called *absolutely maximal* and *algebraic maximal*, respectively. Other distinct types of completeness of (semi)topological semigroups were studied by T. Banakh and S. Bardyla (see [1–6]).

Proposition 3 and [20, Proposition 10] imply the following theorem.

Theorem 6. *For any $n \in \omega$ the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}_n}$ is algebraically complete in the class of Hausdorff semitopological inverse semigroups with continuous inversion, and hence in the class of Hausdorff topological inverse semigroups.*

Theorem 7. *Let n be a non-negative integer. If $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}, \tau\right)$ is a Hausdorff topological semigroup with the compact band then $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}, \tau\right)$ is H-closed in the class of Hausdorff topological semigroups.*

Proof. Suppose to the contrary that there exists a Hausdorff topological semigroup T, which contains $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}, \tau\right)$ as a non-closed subsemigroup. Since the closure of a subsemigroup of a topological semigroup S is a subsemigroup of S (see [11, p. 9]), without loss of generality we can assume that $\boldsymbol{B}_\omega^{\mathscr{F}_n}$ is a dense subsemigroup of T and $T \setminus \boldsymbol{B}_\omega^{\mathscr{F}_n} \neq \varnothing$. Let $\chi \in T \setminus \boldsymbol{B}_\omega^{\mathscr{F}_n}$. Then $\mathbf{0}$ is the zero of the semigroup T by [18, Lemma 4.4], and $\chi \cdot \chi = \mathbf{0}$ by Proposition 6.

Since $\mathbf{0} \cdot \chi = \chi \cdot \mathbf{0} = \mathbf{0}$ and T is a Hausdorff topological semigroup, for any disjoint open neighbourhoods $U(\chi)$ and $U(\mathbf{0})$ of χ and $\mathbf{0}$ in T, respectively, there exist open neighbourhoods $V(\chi) \subseteq U(\chi)$ and $V(\mathbf{0}) \subseteq U(\mathbf{0})$ of χ and $\mathbf{0}$ in T, respectively, such that $V(\mathbf{0}) \cdot V(\chi) \subseteq$

$U(\mathbf{0})$ and $V(\chi)\cdot V(\mathbf{0}) \subseteq U(\mathbf{0})$. By Theorem 4 every non-zero element of $\boldsymbol{B}_\omega^{\mathscr{F}_n}$ is an isolated point in $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n},\tau\right)$ and by [17, Corollary 3.3.11] it is an isolated point of T, and hence the set $E\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}\right)\setminus V(\mathbf{0})$ is finite. Also Hausdorffness and compactness of $E\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}\right)$ imply that without loss of generality we may assume that $V(\chi)\cap E\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}\right) = \varnothing$. Since the neighbourhood $V(\chi)$ contains infinitely many elements of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}_n}$ and the set $E\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}\right)\setminus V(\mathbf{0})$ is finite, there exists $(i,j,[0;k]) \in V(\chi)$ such that either $(i,i,[0;k]) \in V(\mathbf{0})$ or $(j,j,[0;k]) \in V(\mathbf{0})$. Therefore, we have that at least one of the following conditions holds:

$$(V(\mathbf{0})\cdot V(\chi))\cap V(\chi)\neq\varnothing \qquad \text{and} \qquad (V(\chi)\cdot V(\mathbf{0}))\cap V(\chi)\neq\varnothing.$$

Every of the above conditions contradicts the assumption that $U(\chi)$ and $U(\mathbf{0})$ are disjoint open neighbourhoods of χ and $\mathbf{0}$ in T. The obtained contradiction completes the proof. □

Since compactness preserves by continuous maps Theorems 3 and 7 imply the following assertion.

Corollary 2. *Let n be a non-negative integer. If $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n},\tau\right)$ is a Hausdorff topological semigroup with the compact band then $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n},\tau\right)$ is absolutely H-closed in the class of Hausdorff topological semigroups.*

Theorem 8. *Let n be a non-negative integer and $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n},\tau\right)$ be a Hausdorff topological inverse semigroup. If $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n},\tau\right)$ is H-closed in the class of Hausdorff topological semigroups then its band $E\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}\right)$ is compact.*

Proof. We shall prove the statement of the proposition for the semigroup $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$, which by Theorem 1 is isomorphic to the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}_n}$.

Suppose to the contrary that there exists a Hausdorff topological inverse semigroup $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}),\tau\right)$ with the non-compact band such that $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}),\tau\right)$ is H-closed in the class of Hausdorff topological semigroups. By Theorem 4 every non-zero element of the semigroup $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ is an isolated point in $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}),\tau\right)$. Hence there exists an open neighbourhood $U(\mathbf{0})$ of the zero $\mathbf{0}$ in $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}),\tau\right)$ such that the set $A = E\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})\right)\setminus U(\mathbf{0})$ is infinite and closed in $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}),\tau\right)$. Let k be the smallest positive integer $\leqslant n+1$ such that the set $A_k = A\cap\mathscr{I}_\omega^{k}(\overrightarrow{\mathrm{conv}})$ is infinite for the subsemigroup $\mathscr{I}_\omega^{k}(\overrightarrow{\mathrm{conv}})$ of $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$. Without loss of generality we may assume that there exists an increasing sequence of non-negative integers $\{a_j\}_{j\in\omega}$ such that $a_0 \geqslant n+1$ and

$$\widetilde{A}_k = \left\{\begin{pmatrix} a_j & \cdots & a_j+k-1\\ a_j & \cdots & a_j+k-1\end{pmatrix} : j\in\omega\right\}\subseteq A_k.$$

The continuity of the semigroup operation in $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}),\tau\right)$ implies that there exists an open neighbourhood $V(\mathbf{0})\subseteq U(\mathbf{0})$ of the zero $\mathbf{0}$ in $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}),\tau\right)$ such that $V(\mathbf{0})\cdot V(\mathbf{0})\subseteq U(\mathbf{0})$. By the definition of the semigroup operation on $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ we have that the neighbourhood $V(\mathbf{0})$ does not contain at least one of the points

$$\begin{pmatrix} a_j & \cdots & a_j+n\\ a_j & \cdots & a_j+n\end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} a_j-n+k-2 & \cdots & a_j+k-1\\ a_j-n+k-2 & \cdots & a_j+k-1\end{pmatrix}.$$

Since the both above points belong to $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}) \setminus \mathscr{I}_\omega^{n}(\overrightarrow{\mathrm{conv}})$, without loss of generality we may assume that there exists an increasing sequence of non-negative integers $\{b_j\}_{j\in\omega}$ such that $b_j + n + 1 < b_{j+1}$ for all $i \in \omega$ and

$$\widetilde{B}_{n+1} = \left\{ \begin{pmatrix} b_j & \cdots & b_j+n \\ b_j & \cdots & b_j+n \end{pmatrix} : j \in \omega \right\} \nsubseteq V(\mathbf{0}).$$

Since $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}), \tau\right)$ is a Hausdorff topological inverse semigroup, we conclude that the maps $\mathfrak{f}_1 \colon \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}) \to E\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})\right)$, $\alpha\mapsto\alpha\alpha^{-1}$ and $\mathfrak{f}_2 \colon \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}) \to E\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})\right)$, $\alpha \mapsto \alpha^{-1}\alpha$ are continuous, and hence the set $S_{\widetilde{B}_{n+1}} = \mathfrak{f}_1^{-1}(\widetilde{B}_{n+1}) \cup \mathfrak{f}_2^{-1}(\widetilde{B}_{n+1})$ is infinite and open in $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}), \tau\right)$.

Let $\chi \notin \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$. Put $S = \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}) \cup \{\chi\}$. We extend the semigroup operation from $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ onto S in the following way:

$$\chi \cdot \chi = \chi \cdot \alpha = \alpha \cdot \chi = \mathbf{0} \qquad \text{for all} \quad \alpha \in \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}).$$

Simple verifications show that such defined binary operation is associative.

For any $p \in \omega$ we denote

$$\Gamma_p = \left\{ \begin{pmatrix} b_{2j} & \cdots & b_{2j}+n \\ b_{2j+1} & \cdots & b_{2j+1}+n \end{pmatrix} : j \geqslant p \right\}.$$

We determine a topology τ_S on the semigroup S in the following way:

(1) for every $\gamma \in \mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ the bases of topologies τ and τ_S at γ coincide;

(2) $\mathscr{B}(\chi) = \left\{U_p(\chi) = \{\chi\} \cup \Gamma_p \colon p \in \omega\right\}$ is the base of the topology τ_S at the point χ.

Simple verifications show that τ_S is a Hausdorff topology on the semigroup $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$.

For any $p \in \omega$ and any open neighbourhood $V(\mathbf{0}) \subseteq U(\mathbf{0})$ of the zero $\mathbf{0}$ in $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}), \tau\right)$ we have that

$$V(\mathbf{0}) \cdot U_p(\chi) = U_p(\chi) \cdot V(\mathbf{0}) = U_p(\chi) \cdot U_p(\chi) = \{\mathbf{0}\} \subseteq V(\mathbf{0}).$$

We observe that the definition of the set Γ_p implies that for any non-zero element $\gamma = \begin{pmatrix} c & \cdots & c+l \\ d & \cdots & d+l \end{pmatrix}$ of the semigroup $\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}})$ there exists the smallest positive integer j_γ such that $c + l < b_{2j_\gamma}$ and $d + l < b_{2j_\gamma+1}$. Then we have that $\gamma \cdot U_{j_\gamma}(\chi) = U_{j_\gamma}(\chi) \cdot \gamma = \{\mathbf{0}\} \subseteq V(\mathbf{0})$.

Therefore (S, τ_S) is a topological semigroup, which contains $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}), \tau\right)$ as a dense proper subsemigroup. The obtained contradiction implies that $E\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}\right)$ is a compact subset of $\left(\mathscr{I}_\omega^{n+1}(\overrightarrow{\mathrm{conv}}), \tau\right)$. □

Theorem 9. *Let n be a non-negative integer and $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}, \tau\right)$ be a Hausdorff topological inverse semigroup. Then the following conditions are equivalent:*

(1) $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}, \tau\right)$ *is H-closed in the class of Hausdorff topological semigroups;*

(2) $\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}, \tau\right)$ *is absolutely H-closed in the class of Hausdorff topological semigroups;*

(3) *the band $E\left(\boldsymbol{B}_\omega^{\mathscr{F}_n}\right)$ is compact.*

Proof. Implication (2) $\Rightarrow$ (1) is obvious. Implications (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) follow from Theorems 8 and 7, respectively.

Since a continuous image of a compact set is compact, Theorem 3 implies that (3) $\Rightarrow$ (2). □

The following example shows that a counterpart of the statement of Theorem 8 does not hold, when $\left(\boldsymbol{B}_{\omega}^{\mathscr{F}_n},\tau\right)$ is a Hausdorff topological semigroup.

Example 2. *On the semigroup $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$ we define a topology $\tau_{\dagger}$ in the following way. All non-zero elements of the semigroup $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$ are isolated points of $\left(\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}}),\tau_{\dagger}\right)$ and the family $\mathscr{B}_{\dagger}(\mathbf{0})=\left\{U_k(\mathbf{0})\colon k\in\omega\right\}$, where $U_k(\mathbf{0})=\{\mathbf{0}\}\cup\left\{\left(\begin{smallmatrix}2i\\2i+1\end{smallmatrix}\right)\colon i\geqslant k\right\}$, determines the base of the topology $\tau_{\dagger}$ at the point $\mathbf{0}$. It is obvious that $\tau_{\dagger}$ is a Hausdorff topology on $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$. Since $U_k(\mathbf{0})\cdot U_k(\mathbf{0})=\{\mathbf{0}\}$ for any $k\in\omega$ and $U_q(\mathbf{0})\cdot\left\{\left(\begin{smallmatrix}p\\q\end{smallmatrix}\right)\right\}=\left\{\left(\begin{smallmatrix}p\\q\end{smallmatrix}\right)\right\}\cdot U_p(\mathbf{0})=\{\mathbf{0}\}$ for any $p,q\in\omega$, $\left(\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}}),\tau_{\dagger}\right)$ is a topological semigroup.*

Proposition 7. *$\left(\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}}),\tau_{\dagger}\right)$ is H-closed in the class of Hausdorff topological semigroups.*

Proof. Suppose to the contrary that there exists a Hausdorff topological semigroup T, which contains $\left(\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}}),\tau_{\dagger}\right)$ as a non-closed subsemigroup. Since the closure of a subsemigroup of a topological semigroup S is a subsemigroup of S (see [11, p. 9]), without loss of generality we can assume that $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$ is a dense proper subsemigroup of T. Let $\chi\in T\setminus\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$. Then $\mathbf{0}$ is the zero of the semigroup T by [18, Lemma 4.4], and $\chi\cdot\chi=\mathbf{0}$ by Proposition 6.

Fix disjoint open neighbourhoods $U(\chi)$ and $U_p(\mathbf{0})$ of χ and $\mathbf{0}$ in T. By Proposition 6, $E(T)=E\big(\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})\big)$. By [11, Theorem 1.5], $E\big(\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})\big)$ is a closed subset of T and hence without loss of generality we can assume that $U(\chi)\cap E\left(\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})\right)=\varnothing$. Then for any open neighbourhoods $V(\chi)\subseteq U(\chi)$ and $U_q(\mathbf{0})\subseteq U_p(\mathbf{0})$ the infiniteness of $V(\chi)$ and the definition of the semigroup operation on $\mathscr{I}_{\omega}^{1}(\overrightarrow{\mathrm{conv}})$ that imply that

$$V(\chi)\cdot U_q(\mathbf{0})\nsubseteq U_p(\mathbf{0})\qquad\text{or}\qquad U_q(\mathbf{0})\cdot V(\chi)\nsubseteq U_p(\mathbf{0}),$$

which contradicts the continuity of the semigroup operation on T. □

Acknowledgements

The authors acknowledge Serhii Bardyla and Alex Ravsky for their comments and suggestions.

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Received 22.08.2022
Revised 11.04.2023

Гутік О.В., Попадюк О.Б. *Про напівгрупу* $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$*, породжену сім'єю* $\mathscr{F}_n$ *скінченних обмежених інтервалів у* ω // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 331–355.

Ми вивчаємо напівгрупу $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, яка представлена в статті [Вісник Львів. ун-ту. Сер. мех.-мат. 2020, **90**, 5–19], у випадку коли ω-замкнена сім'я $\mathscr{F}_n$ породжена множиною $\{0, 1, \ldots, n\}$. Ми доводимо, що відношення Ґріна $\mathscr{D}$ і $\mathscr{J}$ співпадають в $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$, напівгрупа $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ ізоморфна напівгрупі $\mathscr{I}_{\omega}^{n+1}(\overrightarrow{\mathrm{conv}})$ часткових порядково-опуклих ізоморфізмів множини $(\omega, \leqslant)$ рангу $\leqslant n+1$, і на $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ існують лише конгруенції Ріса. Також вивчаються трансляційно неперервні топології на напівгрупі $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$. Зокрема, доведено, що для довільної трансляційно неперервної T_1-топології τ на $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ кожен ненульовий елемент напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ є ізольованою точкою в $(\boldsymbol{B}_{\omega}^{\mathscr{F}_n}, \tau)$, на $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ існує єдина компактна трансляційно неперервна T_1-топологія, і кожна $\omega_{\mathfrak{d}}$-компактна трансляційно неперервна T_1-топологія компактна. Описано замикання напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ в гаусдорфовій напівтопологічній напівгрупі та доведено критерій H-замкнености топологічної інверсної напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}_n}$ в класі гаусдорфових топологічних напівгруп.

Ключові слова і фрази: біциклічне розширення, конгруенція Ріса, напівтопологічна напівгрупа, топологічна напівгрупа, біциклічний моноїд, інверсна напівгрупа, $\omega_{\mathfrak{d}}$-компактний, компактний, замикання.