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On compressed zero divisor graphs associated to the ring of integers modulo n

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Let R be a commutative ring with unity $1 \neq 0$. In this paper, we completely describe the vertex and the edge chromatic number of the compressed zero divisor graph of the ring of integers modulo n. We find the clique number of the compressed zero divisor graph $\Gamma_E(\mathbb{Z}_n)$ of $\mathbb{Z}_n$ and show that $\Gamma_E(\mathbb{Z}_n)$ is weakly perfect. We also show that the edge chromatic number of $\Gamma_E(\mathbb{Z}_n)$ is equal to the largest degree proving that $\Gamma_E(\mathbb{Z}_n)$ resides in class 1 family of graphs.

Key words and phrases: ring, compressed zero divisor graph, coloring, clique number.

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1 Introduction

A graph G with vertex set $V(G) \neq \varnothing$ and edge set $E(G)$ of unordered pairs of distinct vertices is called a simple graph. The cardinality of $V(G)$ is called the *order* of G and the cardinality of $E(G)$ is called its *size*. A graph G is connected if and only if there exists a path between every pair of vertices u and v. A graph on n vertices such that any pair of distinct vertices is joined by an edge is called a *complete* graph, denoted by K_n. A complete subgraph of G of largest order is called a maximal clique of G and its order is called the *clique number* of G, denoted by $cl(G)$. The number of edges incident on a vertex is called its degree and a vertex of degree 1 is called a pendent vertex. In a connected graph G, the distance between two vertices u and v is the length of the shortest path between u and v. The *diameter* of a graph G is defined as $diam(G) = \sup\{(d(u,v) : u,v \in V(G))\}$, where $d(u,v)$ denotes the distance between vertices u and v of G.

For any set X, let $|X|$ denote the cardinality of X and X^* denote the set of non-zero elements of X. We denote an empty set by $\varnothing$ and the complement of X shall be denoted by X^c. We denote a ring of integers by $\mathbb{Z}$, a ring of integer modulo n by $\mathbb{Z}_n$. For more definitions and terminology of graphs theory, we refer to [3,8,12] and for ring theory, we refer to [1,7].

Throughout the paper, R shall denote a commutative ring with unity $1 \neq 0$, unless otherwise stated. The concept of zero divisor graph of a commutative ring was first introduced by I. Beck [2]. The zero divisor graph $\Gamma(R)$ has its vertices as the set of non-zero zero divisors $Z^*(R) = Z(R)\backslash\{0\}$ of R and two vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. This graph turns out to best exhibit the properties of the set of zero divisors and other related properties of a commutative ring. The zero divisor graph translates the algebraic properties

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of a ring to graph theoretical tools, thus helps in exploring interesting results in both graph theory and abstract algebra.

For a commutative ring R with $1 \neq 0$, the compressed zero divisor graph $\Gamma_E(R)$ is a simple graph with vertex set $Z(R_E) - \{[0]\} = R_E - \{[0],[1]\}$ defined by $R_E = \{[x] : x \in R\}$, where $[x] = \{y \in R : ann(x) = ann(y)\}$ and two distinct vertices $[x]$ and $[y]$ are adjacent if and only if $[x][y] = [0] = [xy]$. This graph was first defined by S. Spiroff and C. Wickham [11].

The compressed zero divisor graph, denoted by $\Gamma_E(R)$, has some advantages over the earlier zero divisor graph $\Gamma(R)$, or subsequent ideal based zero divisor graph of R. In many cases $\Gamma_E(R)$ is finite when $\Gamma(R)$ is infinite. For example, if $R = \mathbb{Z}[x,y]/(x^3, xy)$, then $\Gamma(R)$ is an infinite graph, while $\Gamma_E(R)$ has only four vertices. In addition, there are no complete compressed zero divisor graphs with three or more vertices since the graph would collapse to a single point. These are two ways in which $\Gamma_E(R)$ represents a more succinct description of the zero divisor activity in R. More information on the zero divisor graphs can also be found in [4–6,9] and the study of edge ideals exposed to the zero divisor graphs can be found in [10].

In this paper, we investigate the vertex and the edge coloring of the compressed zero divisor graphs of the ring of integers modulo n. We give a formula for computing the clique number of $\Gamma_E(\mathbb{Z}_n)$ (Theorem 3). We show that $\Gamma_E(\mathbb{Z}_n)$ is weakly perfect for all $n \in \mathbb{N}$ (Theorem 4). We also give a complete description for the edge chromatic number of $\Gamma_E(\mathbb{Z}_n)$ (Theorem 7). Thus, we completely determine the values for vertex and edge coloring of the compressed zero divisor graphs of ring of integers modulo n.

2 Coloring of the compressed zero divisor graph of $\mathbb{Z}_n$

We begin with the following example.

Example 1. *Consider the ring* $\mathbb{Z}_{20}$.

The zero divisor graph $\Gamma(\mathbb{Z}_{20})$, *with* $V(\Gamma(R)) = \{2,4,5,6,8,10,12,14,15,16,18\}$, *and the compressed zero divisor graph* $\Gamma_E(\mathbb{Z}_{20})$, *with vertex set* $\{[2],[4],[5],[10]\}$, *is given in Figure 1.*

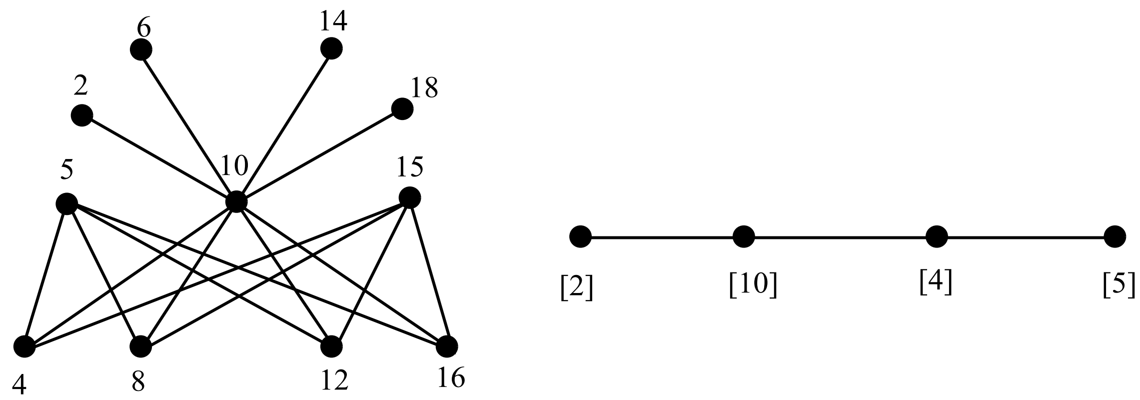

Figure 1. $\Gamma(\mathbb{Z}_{20})$ and $\Gamma_E(\mathbb{Z}_{20})$

Here, $ann(2) = ann(6) = ann(14) = ann(18) = \{10\}$, $ann(5) = ann(15) = \{4,8,12,16\}$, $ann(10) = \{2,4,6,8,12,14,16,18\}$ and $ann(4) = ann(8) = ann(12) = ann(16) = \{5,10,15\}$. Thus, there are four different annihilators so that $V(\Gamma_E(\mathbb{Z}_{20})) = \{[2],[4],[5],[10]\}$ and that $\Gamma_E(\mathbb{Z}_{20}) = P_4$.

The distance $d(u,v)$ between a pair of vertices u and v is the length of the shortest path between u and v; if no path exists, then the distance is infinite. The diameter of a graph is the largest distance between any two distinct vertices.

Theorem 1 ([11]). *The graph $\Gamma_E(R)$ is connected and $diam(\Gamma_E(R)) \leq 3$.*

We divide the vertex set of the graph $\Gamma\left(\mathbb{Z}_{p^n}\right)$ into disjoint subsets $V_1, V_2, \ldots, V_{n-1}$, where $V_i = \left\{u_i p^i : p \nmid u_i\right\}$, $1 \leq i \leq n-1$. Then it is not difficult to see that $|V_i| = (p-1)p^{n-i-1}$, $1 \leq i \leq n-1$, and therefore $\left|\Gamma\left(\mathbb{Z}_{p^n}\right)\right| = \sum_{i=1}^{n-1}(p-1)p^{n-i-1} = p^{n-1}-1$. One can check that $ann(x) = ann(y)$ if and only if $x, y \in V_i$ for some fixed i, $1 \leq i \leq n-1$, and all vertices in a given V_i have the same annihilator. Therefore, we identify the vertex set of $\Gamma_E\left(\mathbb{Z}_{p^n}\right)$ with $\{V_1, V_2, \ldots, V_{n-1}\}$. Thus, the order of $\Gamma_E\left(\mathbb{Z}_{p^n}\right)$ is equal to $n-1$. Further, the adjacency relations are given in the following way: V_i is adjacent to V_j for all $j \geq n-i$, $i = 1, 2, \ldots, n-1$, and V_i is not adjacent to V_{n-j} for all $j < i$.

Let G be a connected graph. The labelling of vertices of G such that the adjacent vertices have distinct labels is known as proper vertex coloring of G. The minimum possible labels being used in a proper coloring of G is known as chromatic number of G. In the following theorem, we determine the chromatic number of $\Gamma_E\left(\mathbb{Z}_{p^n}\right)$.

Theorem 2. *For $n \geq 2$ we have* $\chi(\Gamma_E(\mathbb{Z}_{p^n})) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$

Proof. For $n = 2, 3$, we have $\Gamma_E(\mathbb{Z}_{p^2}) = K_1$ and $\Gamma_E(\mathbb{Z}_{p^3}) = K_2$. So, we assume $n \geq 4$, then $Z^*(\Gamma_E(\mathbb{Z}_{p^n})) = \{V_1, V_2, \ldots, V_{n-1}\}$. Observe that $V_i \leftrightarrow V_j$ for each $\left[\frac{n}{2}\right] \leq i, j \leq n-1$, so the set $\mathcal{C} = \left\{V_{\left[\frac{n}{2}\right]}, V_{\left[\frac{n}{2}\right]+1}, \ldots, V_{n-1}\right\}$ forms a clique. We color all vertices in $\mathcal{C}$ with different colors. Now, the set $\mathcal{C}' = V(\Gamma_E(\mathbb{Z}_{p^n})) \setminus \mathcal{C}$ is an independent set, so each vertex of $\mathcal{C}'$ can be given the same color. Observe that no vertex of $\mathcal{C}'$ is adjacent to $V_{\left[\frac{n}{2}\right]}$, we give the color used on $V_{\left[\frac{n}{2}\right]}$ to all vertices of $\mathcal{C}'$. □

Lemma 1. *Let p be a prime and $n \in \mathbb{N}$. Then* $cl(\Gamma_E(\mathbb{Z}_{p^n})) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$

Proof. We see that $V_i \leftrightarrow V_j$ for each $\left[\frac{n}{2}\right] \leq i, j \leq n-1$. Thus, the clique number of $\Gamma_E\left(\mathbb{Z}_{p^n}\right)$ is equal to

$$(n-1) - \left(\left[\frac{n}{2}\right] - 1\right) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ \dfrac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

□

Corollary 1. *The graph $\Gamma_E\left(\mathbb{Z}_{p^n}\right)$ is weakly perfect.*

The following theorem gives the clique number of the compressed zero divisor graph of the ring of intergers modulo n.

Theorem 3. *For $m = p_1^{\alpha_1} \ldots p_r^{\alpha_r} q_1^{\beta_1} \cdots q_s^{\beta_s} \in \mathbb{N}$, the clique number of $\Gamma_E\left(\mathbb{Z}_m\right)$ is*

$$\left(\prod_{i=1}^{r}\left(\frac{\alpha_i}{2}+1\right)\prod_{j=1}^{s}\frac{\beta_j+1}{2}\right) + s - 1,$$

where α_i's are even integers and β_j's are odd integers.

Proof. Let $m = p_1^{\alpha_1} \dots p_r^{\alpha_r} q_1^{\beta_1} \dots q_s^{\beta_s}$, where α_i's are even and β_j's are odd. For $1 \leq i \leq r$, let $P_i = \left\{1, V_1^{(p_i)}, V_2^{(p_i)}, \dots, V_{\alpha_i-1}^{(p_i)}\right\}$. For $1 \leq j \leq s$, define $Q_j = \left\{1, V_1^{(q_j)}, V_2^{(q_j)}, \dots, V_{\beta_j-1}^{(q_j)}\right\}$. Then the sets P_i and Q_j define the vertex sets of $\Gamma_E\left(\mathbb{Z}_{p_i^{\alpha_i}}\right)$ and $\Gamma_E\left(\mathbb{Z}_{q_j^{\beta_j}}\right)$, respectively. Let $X, Y \subset \mathbb{N}$. Define $XY = \{xy : x \in X,\ y \in Y\}$, where xy denotes the usual multiplication of real numbers. With this construction $V\left(\Gamma_E\left(\mathbb{Z}_m\right)\right) = P_1P_2 \dots P_rQ_1Q_2 \cdots Q_s$. Take $\Delta = \{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\}$. Then the order of $\Gamma_E\left(\mathbb{Z}_{p^m}\right)$ is given by $\prod\limits_{\delta \in \Delta} (\delta + 1) - 2$, where the sum runs over all $\delta \in \Delta$.

Let $v_i' = m/q_i^{(\beta_i+1)/2}$. Define $V_i' = \left\{uv_i' : u \in \mathbb{Z}^{\times}_{p_1 \dots p_r q_1 \dots q_s}\right\}$. Let $\mathcal{P}_i = \left\{V_{\alpha_i/2}, \dots, V_{\alpha_i-1}\right\}$, $1 \leq i \leq r$, $\mathcal{Q}_j = \left\{V_{(\beta_j+1)/2}, \cdots, V_{\beta_j}\right\}$, $1 \leq j \leq s$ and $\mathcal{K} = \{V_1', V_2', \dots, V_s'\}$. Define $\mathcal{P} = \mathcal{P}_1\mathcal{P}_2 \dots \mathcal{P}_r$, $\mathcal{Q} = \mathcal{Q}_1\mathcal{Q}_2 \cdots \mathcal{Q}_s$ and $\mathcal{C} = \mathcal{P}\mathcal{Q} \cup \mathcal{K}$. Clearly, each $\mathcal{P}_i$ and $\mathcal{Q}_j$ form a clique in $\Gamma_E\left(\mathbb{Z}_{p_i^{\alpha_i}}\right)$ and $\Gamma_E\left(\mathbb{Z}_{q_j^{\beta_j}}\right)$, respectively (note that here 0 is also taken as vertex of $\Gamma_E\left(\mathbb{Z}_{p_i^{\alpha_i}}\right)$ and $\Gamma_E\left(\mathbb{Z}_{q_j^{\beta_j}}\right)$). We claim that $\mathcal{C}$ forms a clique in $\Gamma_E\left(\mathbb{Z}_m\right)$. Let $z_a, z_b \in \mathcal{P}\mathcal{Q}$. Then $a, b \geq \lceil m/2 \rceil$ and so z_a and z_b are adjacent. Now, let $z_c \in \mathcal{P}\mathcal{Q}$ and $z_d \in \mathcal{K}$. Then $c \geq \lceil m/2 \rceil$ and $d \geq (\beta_j - 1)/2$, $1 \leq j \leq s$. As β_j is odd, thus $z_dV_\beta = 0$ for all $\beta \geq (\beta_j - 1)/2$ and so $z_cz_d = 0$. Without loss of generality, for the case when $z_a, z_b \in \mathcal{K}$, we have $z_az_b = 0$. Hence, $\mathcal{C}$ forms a clique. We show $\mathcal{C}$ is a maximal clique in $\Gamma_E\left(\mathbb{Z}_m\right)$. For this, let $V = p_1^{\alpha_1'} \dots p_r^{\alpha_r'} q_1^{\beta_1'} \dots q_s^{\beta_s'}$, where $\alpha_i' \leq \alpha_i$, and $\beta_j' \leq \beta_j$. If V is in the clique, then V is adjacent to every vertex in $\mathcal{P}_i$ and $\mathcal{K}$ implying that $\alpha_i' \geq \alpha_i/2$ and $\beta_j' \geq (\beta_j + 1)/2$, $1 \leq i \leq r$, $1 \leq j \leq s$. Hence, V is a vertex in $\mathcal{C}$. Therefore, we have $|\mathcal{P}\mathcal{Q}| = \prod\limits_{i=1}^{r} \left(\frac{\alpha_i}{2} + 1\right) \prod\limits_{j=1}^{s} \frac{\beta_j+1}{2}$. Thus, the size contributed to clique by $\mathcal{P}$ and $\mathcal{Q}$ is $|\mathcal{P}\mathcal{Q}| - 1$ (zero excluded). So the clique number is $\left(\prod\limits_{i=1}^{r} \left(\frac{\alpha_i}{2} + 1\right) \prod\limits_{j=1}^{s} \frac{\beta_j+1}{2}\right) + s - 1$. ☐

Corollary 2. *Let $\mathbb{F}_{p_i}$ be a field of prime order p_i for each $1 \leq i \leq n$, then clique number of $\Gamma_E\left(\mathbb{F}_1 \times \cdots \times \mathbb{F}_n\right)$ is equal to n.*

Proof. Using the notations of Theorem 3, we see that the set $\mathcal{P}\mathcal{Q}$ only contains 0 and so if zero is taken as a vertex (this is known as Beck's zero divisor graph), then clique number contributed by $\mathcal{P}\mathcal{Q}$ is 1. Therefore, size of clique is $(1)^n + n - 1 = n$. ☐

Example 2. *Consider the compressed zero divisor graphs $\Gamma_E\left(\mathbb{Z}_{36}\right)$ and $\Gamma_E\left(\mathbb{Z}_{72}\right)$ (see Figure 2). The examples of cliques are the sets of vertices $\{[6], [12], [18]\}$ and $\{[6], [12], [24], [36]\}$ from $V\left(\Gamma_E\left(\mathbb{Z}_{36}\right)\right)$ and $V\left(\Gamma_E\left(\mathbb{Z}_{72}\right)\right)$. The clique numbers of these graphs can be easily computed using Theorem 3 as follows.*

For $m = 72 = 2^3 3^2$, we have $\alpha_1 = 3, \beta_1 = 2$ and $s = 1$, thus, clique number is

$$\left(\frac{\alpha_1}{2} + 1\right)\left(\frac{\beta_1 + 1}{2}\right) + s - 1 = \left(\frac{2}{2} + 1\right)\left(\frac{3+1}{2}\right) + 1 - 1 = 4.$$

Similiarly for $m = 36 = 2^2 3^2$, we have $\alpha_1 = 2$, $\alpha_2 = 2$ and $s = 0$. Thus, clique number is $\left(\frac{\alpha_1}{2} + 1\right)\left(\frac{\alpha_2}{2} + 1\right) + s - 1 = \left(\frac{2}{2} + 1\right)\left(\frac{2}{2} + 1\right) + 0 - 1 = 3$. It is also trivial to see that the chromatic number of $\Gamma_E\left(\mathbb{Z}_{36}\right)$ is 3 and chromatic number for $\Gamma_E\left(\mathbb{Z}_{72}\right)$ is 4, which agrees with the conclusion of the Theorem 4 below.

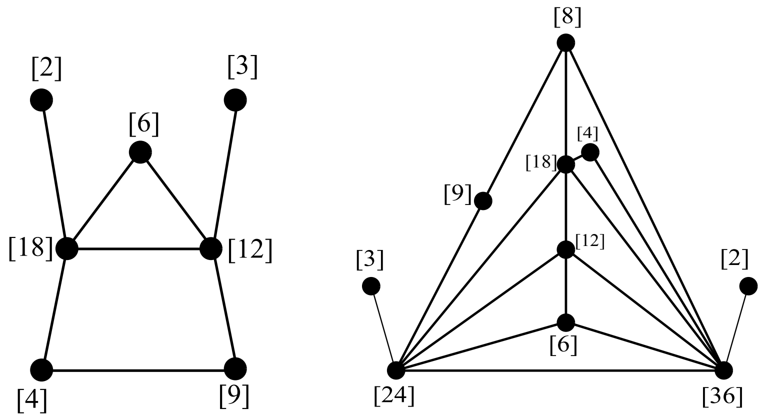

Figure 2. $\Gamma_E(\mathbb{Z}_{36})$ and $\Gamma_E(\mathbb{Z}_{72})$

Theorem 4. *The compressed zero divisor graph of $\mathbb{Z}_n$ is weakly perfect.*

Proof. We use the notations of Theorem 3. Let ω denote the clique number of $\Gamma_E\left(\mathbb{Z}_m\right)$. It is sufficient to show that $\Gamma_E\left(Z_m\right)$ has an ω-coloring. Color the vertices of the clique $\mathcal{C}$ with ω different colors and let v be a vertex not contained in the clique.

Let $P = P_1 P_2 \dots P_r$ and $Q = Q_1 Q_2 \dots Q_s$ and let $V = p_1^{\alpha_1'} \dots p_r^{\alpha_r'} q_1^{\beta_1'} \dots q_s^{\beta_s'}$, where $\alpha_i' \leq \alpha_i$, and $\beta_j' \leq \beta_j$. If $Vq = 0$ for every $q \in Q$, then there exists $p \in P$ such that $pq \neq 0$, for otherwise, V included in the clique makes clique number exceed ω. Let

$$x = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r} q_1^{b_1} \dots q_s^{b_s} = t_1^{c_1} t_2^{c_2} \dots t_{r+s}^{c_{r+s}},$$

where the sets $\{p_1, \dots p_r, q_1, \dots, q_s\}$ and $\{t_1, t_2, \dots, t_{r+s}\}$ are equal sets, possibly with different arrangement of elements. The integers c_i's are defined in the following manner: if $t_i^{c_i}$ is in the clique induced by the vertices $P_i Q_i \cup V_i'$, then set c_i to be the largest integer such that $t_i^{c_i} \mid m$, i.e. $c_i \in \{\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\}$ and if $t_i^{c_i}$ is not in the clique induced by $P_i Q_i \cup V_i'$, let $T_i = \left\{u \in \mathcal{PQ} \, : \, u t_i^{c_i} \neq 0\right\}$ and set $t_i^{c_i}$ be the smallest element in T_i. Clearly, in both cases, x is in the clique $\mathcal{C}$. If V is in the clique, then V and x are assigned distinct colors, by definition. Assume $V \notin \mathcal{C}$. We show that if x and V have the same color, then they are are not adjacent. For some i, $1 \leq i \leq r_s$, assume c_i is such that $t_i^{c_i+1} \mid m$. Then by definition of T_i, we have $c_i = \frac{\alpha_i}{2} \frac{\beta_i+1}{2}$. Since v is not in the clique, we have $t_i^{c_i} u_i \neq 0$. Hence $xv \neq 0$, so that assigning the vertex V the color of the vertex x keeps the coloring proper.

Now, consider the case that there exists $q' \in Q$ such that $vq' \neq 0$. Let T be the collection of all vertices in $\mathcal{K}$, that are not adjacent to V. Let w be the smallest element of T. Since w has the smallest index among the vertices on the clique in $\Gamma_E\left(\mathbb{Z}_{q_i}^{\beta_i}\right)$ and that no two vertices of $V_{\frac{\beta_i - 1}{2}}$ are adjacent, therefore $wV \neq 0$, so we use the color of w to color V. □

Definition. *For a graph G, we define the edge-chromatic number to be the least number of colors required to color the edges of G in such a way that no two adjacent edges are assigned the same color. We denote the edge chromatic number (or edge chromatic index) by $\chi'(G)$.*

Theorem 5 (Vizing's Theorem)**.** *The edge chromatic number of a simple connected graph is either equal to the largest degree or exceeds it by 1.*

Vizing's theorem partitions the set of all (connected) graphs into two classes based on their edge chromatic number. We call a graphs to reside in class 1 if its edge chromatic number equals the largest degree and call it to reside in class 2 if its edge chromatic number exceeds the largest degree by 1.

Theorem 6 ([3, Corollary 5.4]). *If G is a simple graph such that for every vertex x with maximum degree there exists a neighbor y such that $\Delta(G) - deg(y) + 2$ is greater than the number of vertices of maximum degree, then the edge chromatic number equals the largest degree in G.*

We now prove that the family of all compressed zero divisor graphs of the ring of integers modulo n reside in class 1.

Theorem 7. *The edge chromatic number of a compressed zero divisor graph of the ring of integer modulo n equals the largest degree.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots, p_k$ are prime numbers and $k \in \mathbb{N}$. First consider the case when $n = p^{\alpha}$. The structure of $\Gamma_E(\mathbb{Z}_{p^\alpha})$ is given in Figure 1. If $\alpha = 1, 2$, there is nothing to prove and for $\alpha = 3$, the result holds trivially. So we assume $\alpha \geq 4$. Then the vertex $x = [p^{\alpha-1}]$ has the largest degree $\Delta = \alpha - 2$ and the vertex $y = [p]$ is the pendent vertex. Consequently, $\Delta - deg(y) + 2 = n - 2 - 1 + 2 = n - 1 > 1$. So the result follows by Theorem 6.

Now, consider the case when n is square free. For $k = 2$, the result is trivial. So assume that $k \geq 3$. The vertices of the form $x_i' = [n/p_i]$, $1 \leq i \leq k$, have the largest degree $\Delta = 2^{k-1} - 1$ and the vertices of the form $y_i' = [p_i]$, $1 \leq i \leq k$, are the pendent vertices. Also, the number of vertices of the form x_i' is equal to k. Clearly, $x_i' y_i' = [0]$ for each i and $\Delta - deg(y_i') + 2 = 2^{k-1} > k$ for all $k \geq 3$. Thus by Theorem 6, the result follows.

Assume $k > 1$, $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$ and $\alpha_k \geq 2$. Let $u = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$, where $\beta_i \leq \alpha_i$, $1 \leq i \leq k$, be a vertex of $\Gamma_E(\mathbb{Z}_n)$. If u is a vertex of $\Gamma_E(\mathbb{Z}_n)$, then $\beta_i < \alpha_i$ for some i. Let $u_i = [n/p_i]$, then $\deg(u_i) = \Delta$ if $\alpha_i = \alpha_{i+1} = \cdots = \alpha_k$ and in this case, the number of vertices of largest degree is equal to $k - i + 1$. In particular,

$$\Delta = deg(u_k) = \left[\prod \left(\left| \Gamma_E \left(\mathbb{Z}_{p_i^{\alpha_i}} \right) \right| + 1 \right) \left| \Gamma_E \left(\mathbb{Z}_{p_1^{\alpha_k}} \right) \right| - 1 \right] - 1.$$

Also the vertex $v_i = [p_i]$, $1 \leq i \leq k$, is a pendent vertex which is only adjacent to u_i for each $1 \leq i \leq k$. If $\alpha_k > \alpha_j$ for each $1 \leq j \leq k-1$, then u_k is the only vertex of the largest degree. In case $\alpha_1 = \alpha_2 = \cdots = \alpha_k$, there are k vertices of largest degree. In both, the conditions of Theorem 6 follows by easy calculations. Therefore, the conclusion. $\square$

Conclusions

This work caters to the weak perfectness of the compressed zero divisor graphs associated to commutative rings. The work can be extended to the zero divisor graphs (and compressed zero divisor graphs) of non-commutative rings also. There is an open scope for working in the same direction for ideal based zero divisor graphs and other types of graphs associated to rings. The perfectness of the zero divisor graphs of rings can also be carried on the similar lines.

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Нехай R — комутативне кільце з одиницею $1 \neq 0$. У цій статті ми повністю описуємо вершинне та реберне хроматичне число стисненого графа дільників нуля кільця цілих чисел за модулем n. Ми знаходимо клікове число стисненого графа $\Gamma_E(\mathbb{Z}_n)$ дільників нуля кільця $\mathbb{Z}_n$ і показуємо, що $\Gamma_E(\mathbb{Z}_n)$ є слабко досконалим. Ми також показуємо, що реберне хроматичне число графа $\Gamma_E(\mathbb{Z}_n)$ дорівнює найбільшому степеню. Це доводить, що $\Gamma_E(\mathbb{Z}_n)$ знаходиться у сім'ї графів класу 1.

Ключові слова і фрази: кільце, стиснутий граф дільників нуля, розфарбовування, клікове число.