ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2024, **16** (2), 606–616
doi:10.15330/cmp.16.2.606-616

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2024, Т.16, №2, С.606–616

Fixed point theorems for generalised contractions in altering JS-metric space with graph

Alshammari I.[1], Özel C.[2], Gnanachandra P.[3], Varunnya X.M.J.[3]

Throughout this article, we propose an extension of the metric space due to M. Jleli and B. Samet (JS-metric space). This extension is made using the altering distance function, hence the name altering JS-metric. The idea of fixed points for the contraction mappings endowed with graph was first introduced by J. Jachymski in 2008. After that many researchers followed this idea and proposed various contraction mappings endowed with a graph. Motivated by these ideas, we have generalised some well-known contraction mappings in the literature that are endowed with a graph. Additionally, we have provided some fixed point results concerning these contractions in an altering JS-metric space.

Key words and phrases: JS-metric, altering JS-metric, generalised G-contraction, generalised G-Kannan mapping, generalised G-Chatterjea mapping.

[1] University of Hafr Al-Batin, 39524, Hafr Al-Batin, Kingdom of Saudi Arabia
[2] Univeristy of King Abdulaziz, 21589, Jeddah, Kingdom of Saudi Arabia
[3] Ayya Nadar Janaki Ammal College, Sivakasi, 626124, Tamil Nadu, India
E-mail: `iealshamri@hotmail.com` (Alshammari I.), `cenap.ozel@gmail.com` (Özel C.), `pgchandra07@gmail.com` (Gnanachandra P.), `jeffinwhite@gmail.com` (Varunnya X.M.J.)

Introduction

Fixed point theory is a prominant branch of non linear analysis having a broad range of applications in wide disciplines. The raise of theory of fixed points dates back to 19th century thereby gaining the attention of many researchers. Metric fixed point theory is a division, which deals with the fixed points of functions in metric spaces. Through the years, many research works were carried out concerning this discipline. S. Banach proposed Banach contraction principle to assure unique fixed point of continuous contraction map in complete metric space. Enormous amount of results have been given by generalising the contraction condition and the underlying metric space. M. Jleli and B. Samet [8] gave a generalised metric space and showed that Banach type contraction has only one invariant point in their generalised metric space. In 2005, J.J. Nieto and R. Rodríguez-López [11, 12] obtained a result that ensured the fixed point for a non decreasing function in a partially ordered complete metric space and A. Petruşel and I.A. Rus [13] upgraded that result. In 2008, J. Jachymski [7] introduced the idea of contraction mappings endowed with a graph. He studied theorems concerning the fixed points in metric spaces endowed with a graph. Following him many researchers worked on the area of various contraction mappings endowed with a graph. For more ideas one may refer to [1, 2, 9, 15]. Motivated by these ideas, in this paper we have introduced the generalised contractions endowed with a graph and have given examples for such contractions. We have also proved fixed point theorems for such contractions.

УДК 517.988.523
2020 *Mathematics Subject Classification:* 47H10, 37C25, 55M20, 05C20.

1 Preliminaries

Generalisation of metric space, made by M. Jleli and B. Samet, is one of the notable extensions prevailing in the litrature. They made the generalisation by introducing a converging sequence in an axiom. The generalised metric due to M. Jleli and B. Samet [8] is given below.

Definition 1 ([8])**.** *Consider a non-void set S and a mapping $J_\phi : S \times S \to [0, \infty]$. For every $s \in S$, let us define the set*

$$C(J_\phi, S, j) = \{\{j_n\} \subset S : \lim_{n\to\infty} J_\phi(j_n, j) = 0\}.$$

A non-void set S with the map $J_\phi : S \times S \to [0, \infty]$ is a JS-metric space if it satisfies the following axioms:

$(D1)$ $J_\phi(j, p) = 0 \Rightarrow j = p$ *for* $(j, p) \in S \times S$,

$(D2)$ $J_\phi(j, p) = J_\phi(p, j)$ *for* $(j, p) \in S \times S$,

$(D3)$ $\{j_n\} \in C(J_\phi, S, j) \Rightarrow J_\phi(j, p) \leq k \limsup_{n\to\infty} J_\phi(j_n, p)$ *for* $(j, p) \in S \times S$ *and some* $k > 0$.

The idea of altering distance function was proposed by D. Delbosco [5] and F. Skof [14]. Succeeding them, M.S. Khan et. al. [10] utilized the idea to prove many results in fixed point theory. Following the definition of altering distance function given by M.S. Khan in 1984, we give a new generalisation of the JS-metric.

Definition 2 ([10])**.** *Let η be a function from non-negative reals to itself. Then η is an altering distance function if*

(1) η is monotonically non-decreasing and continuous,

(2) $\eta(t) = 0 \Leftrightarrow t = 0$,

(3) $h.t^r \leq \eta(t)$ for $t > 0$ and some constants $h, r > 0$.

Let Ψ denote the set of altering distance functions.

Example 1. *Define $\eta : [0, \infty) \to [0, \infty)$ by $\eta(t) = 5^t t$. It is obvious that $\eta(t)$ is contionuous and non-decreasing in the interval $[0, \infty)$. Also, it is observed that the value of η is 0 only at the point $t = 0$. Also, $t \leq \eta(t)$ for every t. This implies that η is an altering distance function.*

Let us now introduce the altering JS-metric space utilising the function $\phi \in \eta$ as follows.

Definition 3. *Let S be a non-void set and $J_\phi : S \times S \to [0, \infty]$ be a map. We say that J_ϕ is an altering JS-metric on S if:*

$(A1)$ $J_\phi(j, p) = 0 \Rightarrow s = t$ *for* $j, p \in S$,

$(A2)$ $J_\phi(j, p) = J_\phi(p, j)$ *for* $j, p \in S$,

$(A3)$ $\{j_n\} \in C(J_\phi, S, j) \Rightarrow \phi(J_\phi(j, p)) \leq \limsup_{n\to\infty} [\phi(k.J_\phi(j_n, p))]$ *for* $j, p \in S$ *and some* $\phi \in \Psi, k > 0$.

The pair (S, J_ϕ) is an altering JS-metric space.

It can be observed that in an altering JS-metric space if $\phi(t) = t$, then it becomes a JS-metric space. Suppose the set $C\left(J_\phi, S, j\right)$ is empty for every element of S, then $\left(S, J_\phi\right)$ becomes a JS-metric space iff the axioms $(A1)$ and $(A2)$ are satisfied.

Example 2. *Let $S = \mathbb{R}^+$ and let $J_\phi : S \times S \to [0, \infty]$ be defined by $J_\phi(j, p) = (j - p)^2$. Then $\left(S, J_\phi\right)$ satisfies the axioms of an altering JS-metric space. Thus $\left(\mathbb{R}^+, J_\phi\right)$ is an altering JS-metric space.*

It can be noted that a metric space is always an altering JS-metric space.

Definition 4. *A sequence $\{j_n\}$ in an altering JS-metric space $\left(S, J_\phi\right)$ is:*

(i) J_ϕ-convergent to $s \in S$ if $\lim\limits_{n\to\infty} J_\phi\left(j_n, j\right) = 0$, i.e. $\{j_n\} \in C(J_\phi, S, j)$,

(ii) J_ϕ-Cauchy if for given $\varepsilon > 0$ there exists $s \in N$ such that $J_\phi\left(j_n, j_m\right) < \varepsilon$ for all $m, n \geq s$, i.e. $\lim\limits_{n,m\to\infty} J_\phi\left(j_n, j_m\right) = 0$.

The altering JS-metric space $\left(S, J_\phi\right)$ is J_ϕ-complete, if every J_ϕ-Cauchy sequence in S is J_ϕ-convergent.

2 Contractive mappings endowed with a graph

In 2008, the concept of showing fixed points of a contractive map in a metric space endowed with a graph was initiated by J. Jachymski [7].

Consider a non-void set S and a mapping $J_\phi : S \times S \to [0, \infty]$, which is an altering JS-metric on S. Let $\Theta = S \times S$, G be a graph, which is directed having no parallel edges determined by the vertex set $V(G)$, that corresponds to S, i.e. $S = V(G)$ and the directed edge set $E(G) \subseteq V(G) \times V(G)$ containing all loops, i.e. $\Theta \subseteq E(G)$. The amount of distance joining the end vertices is allocated as the weight of every edge. Let the directed graph acquired by flipping over the edges of G be indicated by G^{-1} and the graph acquired by neglecting the directions of edges be indicated by $\tilde{G}$.

Definition 5 ([3])**.** *A sequence $\{j_n\}$ of vertices in $V(G)$ is G-increasing if*

$$(j_n, j_{n+1}) \in E(G) \quad \text{for all} \quad n \in \mathbb{N}.$$

Example 3. *Let $S = \mathbb{N}$ be defined with the usual metric and G be a directed graph whose vertex set concur with $\mathbb{N}$ and the edge set is defined to be $E(G) = \left\{(j, p) : j < p\right\}$. Let $\{j_n\}$ be a sequence in $V(G)$ given by $j_n = n^2$. Then $\{j_n\}$ is G-increasing.*

The sequence $\{j_n\}$ is not G-increasing, when $S = \mathbb{Q}$.

Definition 6. *Let S be a non-void set and G be a directed graph such that $S = V(G)$. Then G is transitive provided $(j, r) \in E(G)$ whenever $(j, p), (p, r) \in E(G)$ for $j, p, r \in V(G)$.*

Example 4. *Let the vertex set of the directed graph coincide with the set of natural numbers $\mathbb{N}$ and the edge set is defined to be $E(G) = \left\{(j, p) : j, p \in \mathbb{N} \text{ and } j \mid p\right\}$. Then G is transitive.*

The graph defined in Example 3 is also transitive.

J. Jachymski [7] used the property that was proposed by J.J. Nieto and R. Rodríguez-López [11] for metric space equipped by a graph. This property is also mentioned as the JNRL property by many researchers.

Property 1 (JNRL property)**.** *The directed graph G complies the JNRL property, whenever a G-increasing sequence $\{j_n\}$ is J_ϕ-convergent to s, then $(j_n, j) \in E(G)$.*

The following is the J. Jachymski's definition of graph contraction.

Definition 7 ([7])**.** *Consider the metric space (S, d) and the directed graph having the vertex set S. A map T from S to itself is a G-contraction if:*

(i) the edges of G are preserved by T, i.e. $j, p \in S, (j, p) \in E(G) \Rightarrow (Tj, Tp) \in E(G)$,

(ii) $(j, p) \in E(G) \Rightarrow d(Tj, Tp) \leq rd(j, p)$ for $j, p \in S$ and some $r \in (0, 1)$.

The following definition of Kannan graph contraction was given by F. Bojor [4].

Definition 8 ([4])**.** *Consider the metric space (S, d) and the directed graph G having the vertex set S. A map T from S to itself is a G-Kannan mapping if:*

(i) $(j, p) \in E(G) \Rightarrow (Tj, Tp) \in E(G)$ for all $j, p \in S$,

(ii) there exists $0 \leq r < \frac{1}{2}$ such that

$$d(Tj, Tp) \leq r[d(j, Tj) + d(p, Tp)]$$

for $(j, p) \in E(G)$.

In 2016, K. Fallahi and A. Aghanians [6] introduced the G-Chatterjea mapping using the Jachymski's idea of contraction endowed with a graph for Chatterjea's contraction.

Definition 9 ([6])**.** *Consider the metric space (S, d) and the directed graph G having the vertex set S. A map T from S to itself is a G-Chatterjea mapping if:*

(i) $(j, p) \in E(G) \Rightarrow (Tj, Tp) \in E(G)$ for $j, p \in S$,

(ii) there exists $0 \leq r < \frac{1}{2}$ such that

$$d(Tj, Tp) \leq r\big[d(j, Tp) + d(p, Tj)\big]$$

for $(j, p) \in E(G)$.

3 Generalised graph contractions

The contractions that have been proposed in the literature so far work in the case of existence of edge between the vertices that we consider. Now, here we have replaced the existence of edge with existence of path between the vertices, which gave rise to the idea of generalised contractions endowed with a graph.

Definition 10. *Let us consider the altering JS-metric space (S, J_ϕ) and the directed graph $G = (V(G), E(G))$, whose vertex set corresponds to S. A sequence $\{j_n\}_{n=0}^{m}$ of $m + 1$ dissimilar vertices from $j_0 = j$ to $j_m = p$ is a path of length m in a graph G if $(j_n, j_{n+1}) \in E(G)$ for $n = 0, 1, 2, \ldots, m - 1$.*

The distances of each edge involved in a path adds up to give the distance of the path. For any $j, p \in V(G)$, let $P(j, p)$ denotes the collection of all paths between s and t in the undirected graph $\tilde{G}$.

Definition 11. *Consider the metric space (S, d) and the directed graph G having the vertex set S. A map T from S to itself is a generalised G-contraction if:*

(i) the edges of G are preserved by T, i.e. $(j, p) \in E(G) \Rightarrow (Tj, Tp) \in E(G)$ for $(j, p \in S)$,

(ii) for some $r \in (0, 1)$ and for every $j, p \in S$, such that $P(j, p)$ is non-void, we have

$$d(Tj, Tp) \leq rd(j, p).$$

Example 5. *Consider the non-void set $S = \mathbb{N}$ endowed with the usual metric and the graph G, whose vertex set coincides with S. The edge set is defined as $E(G) = \{(j, p) \colon j, p \in \mathbb{N}$ and $j \mid p\}$. Let the mapping $T : S \to S$ be given by $Tj = \frac{j}{2}$.*

Now, $(j, p) \in E(G) \Rightarrow j \mid p$. Consider $\frac{Tp}{Tj} = \frac{\frac{p}{2}}{\frac{j}{2}} = \frac{p}{j}$. Since $j \mid p$, it follows that $Tj \mid Tp$, which gives $(Tj, Tp) \in E(G)$. Thus the edges of G are preserved by T.

Let $j, p \in E(G)$ be such that $P(j, p)$ is non-void. Thus, there exists a path $\{j_n\}_{n=0}^{m}$ from j to p, where $j_0 = j$ and $j_m = p$. Since the edges of G are preserved by T, there exists a path $\{Tj_n\}_{n=0}^{m}$ from Tj to Tp, where $Tj_0 = Tj$ and $Tj_m = Tp$. We have

$$\begin{aligned}
d(Tj, Tp) = d(Tj_0, Tj_m) &= d(Tj_0, Tj_1) + d(Tj_1, Tj_2) + \cdots + d(Tj_{m-1}, Tj_m) \\
&= |Tj_0 - Tj_1| + |Tj_1 - Tj_2| + \cdots + |Tj_{m-1} - Tj_m| \\
&= \left|\frac{j_0}{2} - \frac{j_1}{2}\right| + \left|\frac{j_1}{2} - \frac{j_2}{2}\right| + \cdots + \left|\frac{j_{m-1}}{2} - \frac{j_m}{2}\right| \\
&= \frac{1}{2}\{\, |j_0 - j_1| + |j_1 - j_2| + \cdots + |j_{m-1} - j_m| \,\} \\
&= \frac{1}{2}\{d(j_0, j_1) + d(j_1, j_2) + \cdots + d(j_{m-1}, j_m)\,\} \\
&= \frac{1}{2}d(j_0, j_m) = \frac{1}{2}d(j, p)
\end{aligned}$$

Hence, T is a generalised G-contraction.

Definition 12. *Consider the metric space (S, d) and the directed graph G having the vertex set S. A map T from S to itself is a generalised G-Kannan mapping if:*

(i) $(j, p) \in E(G) \Rightarrow (Tj, Tp) \in E(G)$ for $j, p \in S$,

(ii) there exists $0 \leq r < \frac{1}{2}$ with $d(Tj, Tp) \leq r\big[d(j, Tj) + d(p, Tp)\big]$ for all j, p such that $P(j, p)$ is non-void.

Example 6. *Let the set $S = \{1, 2, 3, 4\}$ is endowed with the usual metric $d(j, p) = |j - p|$. Let the directed graph G be defined with the vertex set S and the edge set be given by $E(G) = \big\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (3, 4)\big\}$. The self map T is defined by*

$$Tj = \begin{cases} \dfrac{j}{2}, & \text{if } j \text{ is even,} \\ \dfrac{j+1}{2}, & \text{if } j \text{ is odd.} \end{cases}$$

Now, $(T_1, T_1) = (T_2, T_2) = (T_1, T_2) = (1,1)$; $(T_3, T_3) = (T_4, T_4) = (T_3, T_4) = (2,2)$; $(T_1, T_3) = (1,2)$. *Thus the edges of* G *are preserved by* T. *In this graph, all the edges form a path of length 1. Apart from that,* $P(1,4)$ *is non empty. So all these paths satisfy the condition*

$$d(Tj, Tp) \leq r\big[d(s, Ts) + d(t, Tt)\big]$$

with the constant $r = \dfrac{2}{5}$.

Let us check the contraction condition for these cases:

$$\begin{aligned}
d\,(T_1, T_1) = d(1,1) = 0 \quad &\text{and} \quad \frac{2}{5}\big[d\,(1, T_1) + d\,(1, T_1)\,\big] = \frac{2}{5}\big[d(1,1) + d(1,1)\big] = \frac{0}{5},\\
d\,(T_2, T_2) = d(1,1) = 0 \quad &\text{and} \quad \frac{2}{5}\big[d\,(2, T_2) + d\,(2, T_2)\,\big] = \frac{2}{5}\big[d(2,1) + d(2,1)\big] = \frac{4}{5},\\
d\,(T_3, T_3) = d(2,2) = 0 \quad &\text{and} \quad \frac{2}{5}\big[d\,(3, T_3) + d\,(3, T_3)\,\big] = \frac{2}{5}\big[d(3,2) + d(3,2)\big] = \frac{4}{5},\\
d\,(T_4, T_4) = d(2,2) = 0 \quad &\text{and} \quad \frac{2}{5}\big[d\,(4, T_4) + d\,(4, T_4)\,\big] = \frac{2}{5}\big[d(4,2) + d(4,2)\big] = \frac{8}{5},\\
d\,(T_1, T_2) = d(1,1) = 0 \quad &\text{and} \quad \frac{2}{5}\big[d\,(1, T_2) + d\,(2, T_1)\big] = \frac{2}{5}\big[d(1,1) + d(2,1)\big] = \frac{1}{5},\\
d\,(T_1, T_3) = d(1,2) = 1 \quad &\text{and} \quad \frac{2}{5}\big[d\,(1, T_3) + d\,(3, T_1)\,\big] = \frac{2}{5}\big[d(1,2) + d(3,1)\big] = \frac{6}{5},\\
d\,(T_1, T_4) = d(1,2) = 1 \quad &\text{and} \quad \frac{2}{5}\big[d\,(1, T_4) + d\,(4, T_1)\,\big] = \frac{2}{5}\big[d(1,2) + d(4,1)\big] = \frac{8}{5},\\
d\,(T_3, T_4) = d(1,2) = 1 \quad &\text{and} \quad \frac{2}{5}\big[d\,(3, T_4) + d\,(4, T_3)\,\big] = \frac{2}{5}\big[d(3,2) + d(4,2)\big] = \frac{6}{5}.
\end{aligned}$$

Thus, T *is a generalised* G*-Kannan mapping.*

Definition 13. *Consider the metric space* (S, d) *and the directed graph* G *having the vertex set* S. *A map* T *from* S *to itself is a generalised* G*-Chatterjea mapping if:*

(i) $(j, p) \in E(G) \Rightarrow (Tj, Tp) \in E(G)$ *for* $j, p \in S$,

(ii) *there exists* $0 \leq r < \frac{1}{2}$ *with* $d(Tj, Tp) \leq r\big[d(j, Tp) + d(p, Tj)\big]$ *for every* j, p *such that* $P(j, p)$ *is non-void.*

Definition 14. *The map* T *is a* G*-continuous map with respect to the altering JS-metric* J_ϕ *over the non-void set* S, *if for the given* G*-increasing sequence* $\{j_n\}$ *in* S *and* $s \in S$, $J_\phi\,(j_n, j) \to 0$ *implies* $J_\phi\,(Tj_n, Tj) \to 0$.

Proposition 1.

(i) *A* G*-contraction mapping is a generalised* G*-contraction mapping and a* G*-Kannan mapping is a generalised* G*-Kannan mapping. The same holds for* G*-Chatterjea mapping.*

(ii) *If* T *is a generalised* G*-contraction, then* T *is also a generalised* G^{-1}*-contraction and a generalised* $\tilde{G}$*-contraction.*

(iii) *If* T *is a generalised* G*-Kannan mapping, then* T *is also a generalised* G^{-1}*-Kannan mapping and generalised* $\tilde{G}$*-Kannan mapping.*

(iv) *If* T *is a generalised* G*-Chatterjea mapping, then* T *is also a generalised* G^{-1}*-Chatterjea mapping and generalised* $\tilde{G}$*-Chatterjea mapping.*

Proof. *(i)* The fact that an edge is a path of length one makes *(i)* hold true.

(ii) If $(j,p) \in E(G)$, then $(p,j) \in E\left(G^{-1}\right)$. Since the edges of G are preserved by T, $(Tj,Tp) \in E(G)$, which implies $(Tt,Ts) \in E\left(G^{-1}\right)$. Hence the edges of G^{-1} are preserved by T. Again, $\tilde{G}$ is a graph obtained by neglecting the directions of edges, so the edges of $\tilde{G}$ are preserved by T. Further, the symmetric axiom of the metric space ensures that T is a generalised G^{-1}-contraction and a generalised $\tilde{G}$-contraction.

Items *(iii)* and *(iv)* can be proved by similar way. □

Throughout this section, we consider (S, J_ϕ) to be an altering JS-metric space endowed with a directed graph $G = \big(V(G), E(G)\big)$, whose vertex set coincides with the set S, i.e. $S = V(G)$.

Theorem 1. *Let (S, J_ϕ) be a complete altering JS-metric space endowed with G such that G satisfies the JNRL property. Let $T : S \to S$ be a map satisfying:*

(i) $(j, Tj) \in E(G)$ for all $j \in V(G)$,

(ii) T is a generalised G-Kannan mapping.

Then T has a fixed point.

Proof. Consider $s_0 \in S$. Define an iterative sequence $\{j_n\}$ by Picard's iteration defined by $Ts_n = s_{n+1}$ for every $n \in \mathbb{N}$. Suppose for some $n \in \mathbb{N}$ we get $s_n = s_{n+1}$, then $s_n = Ts_n$ and hence the claim. So, let us consider the case $s_n \neq s_{n+1}$ for every $n \in \mathbb{N}$.

By the hypothesis *(i)*, we get $(j_n, Tj_n) \in E(G)$ for every $n \in \mathbb{N}$, which gives $(j_n, j_{n+1}) \in E(G)$ for every $n \in \mathbb{N}$. Therefore $\{j_n\}$ is a G-increasing sequence in S and $P\left(j_n, j_{n+1}\right)$ is non-void for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned} J_\phi\left(j_n, j_{n+1}\right) = J_\phi\left(Tj_{n-1}, Tj_n\right) &\leq r\Big[J_\phi\left(j_{n-1}, Tj_{n-1}\right) + J_\phi\left(j_n, Tj_n\right)\Big] \\ &= r\Big[J_\phi\left(j_{n-1}, j_n\right) + J_\phi\left(j_n, j_{n+1}\right)\Big](1-r)J_\phi\left(j_n, j_{n+1}\right) \\ &\leq rJ_\phi\left(j_{n-1}, j_n\right) J_\phi\left(j_n, j_{n+1}\right) \leq \frac{r}{1-r} J_\phi\left(j_{n-1}, j_n\right). \end{aligned}$$

Continuing in this way we obtain

$$J_\phi\left(j_n, j_{n+1}\right) \leq \left(\frac{r}{1-r}\right)^n J_\phi\left(j_0, j_1\right)$$

for all $n \in \mathbb{N}$. If $n \to \infty$, then $J_\phi\left(j_n, j_{n+1}\right) \to 0$. Let $j_n, j_m \in V(G)$ with $n < m$. Now, we get

$$\left(j_n, j_{n+1}\right), \left(j_{n+1}, j_{n+2}\right), \ldots, \left(j_{m-1}, j_m\right) \in E(G).$$

Hence, there exists a path from j_n to j_m in G. We have

$$\begin{aligned} J_\phi\left(j_n, j_m\right) = J_\phi\left(Tj_{n-1}, Tj_{m-1}\right) &\leq r\Big[J_\phi\left(j_{n-1}, Tj_{n-1}\right) + J_\phi\left(j_{m-1}, Tj_{m-1}\right)\Big] \\ &= r\Big[J_\phi\left(j_{n-1}, j_n\right) + J_\phi\left(j_{m-1}, j_m\right)\Big]. \end{aligned}$$

Letting $n, m \to \infty$, we obtain $J_\phi\left(j_n, j_m\right) \to 0$. Thus, $\{j_n\}$ is a Cauchy sequence in S. Completeness of S gives that $j_n \xrightarrow{J_\phi} j$ for $j \in S$. Since G satisfies the JNRL property, $(j_n, j) \in E(G)$.

Now,

$$J_\phi\left(Tj_n, Tj\right) \leq r\Big[J_\phi\left(j_n, Tj_n\right) + J_\phi(j, Tj)\Big],$$

$$J_\phi\left(j_{n+1}, Tj\right) \leq r\Big[J_\phi\left(j_n, j_{n+1}\right) + J_\phi\left(j, Tj\right)\Big].$$

Letting $n \to \infty$, we get $J_\phi(j, Tj) \leq rJ_\phi(j, Tj)$, which is possible iff $J_\phi(j, Tj) = 0 \Rightarrow j = Tj$. □

Corollary 1. *Suppose $\left(S, J_\phi\right)$ is complete and endowed with a graph G. If $T : S \to S$ is a G-Kannan mapping such that G is either complete or transitive, then T has a fixed point.*

Theorem 2. *Suppose $\left(S, J_\phi\right)$ is a complete altering JS-metric space with a graph G satisfying JNRL property. Let $T : S \to S$ be a map such that:*

(i) $(j, Tj) \in E$ *for all* $j \in S$,

(ii) $J_\phi\left(j_0, j_w\right) < \infty$ *for every G-increasing sequence* $\{j_n\}_{n=0}^{\infty}$ *and for all* $w \geq 0$,

(iii) $J_\phi(Tj, Tp) \leq r.J_\phi(j, p)$, *with* $r \in (0, 1)$, *for every* j, p *such that* $P(j, p)$ *is non-void.*

Then the mapping T has a fixed point in S.

Proof. Let $j_0 \in S$ and the sequence $\{j_n\}$ be defined as $j_{n+1} = Tj_n$, $n \in \mathbb{N}$. Let $j_n \neq j_{n+1}$ for all $n \in \mathbb{N}$. By the hypothesis, $(j_n, Tj_n) \in E(G)$ for all $n \in \mathbb{N}$, which implies $(j_n, j_{n+1}) \in E(G)$, $n \in \mathbb{N}$. Hence, $\{j_n\}$ is a G-increasing sequence in S and $P\left(j_n, j_{n+1}\right)$ is non-void for all $n \in \mathbb{N}$.

Note, that

$$\begin{aligned} J_\phi\left(j_n, j_{n+1}\right) = J_\phi\left(Tj_{n-1}, Tj_n\right) &\leq r.J_\phi\left(j_{n-1}, j_n\right) = r.J_\phi\left(Tj_{n-2}, Tj_{n-1}\right) \\ &\leq r^2 J_\phi\left(j_{n-2}, j_{n-1}\right) \ldots J_\phi\left(j_n, j_{n+1}\right) \leq r^n J_\phi\left(j_0, j_1\right). \end{aligned}$$

As $n \to \infty$, we have $J_\phi\left(j_n, j_{n+1}\right) \to 0$. Consider $n, m \in \mathbb{N}$ such that $n < m$. Now, to ensure that $\{j_n\}$ is a J_ϕ-Cauchy sequence, consider

$$\begin{aligned} J_\phi\left(j_n, j_m\right) = J_\phi\left(Tj_{n-1}, Tj_{m-1}\right) &\leq r.J_\phi\left(j_{n-1}, j_{m-1}\right) = r.J_\phi\left(Tj_{n-2}, Tj_{m-2}\right) \\ &\leq r^2 J_\phi\left(j_{n-2}, j_{m-2}\right) \ldots J_\phi(j_n, j_m) \leq r^n J_\phi\left(j_0, j_{m-n}\right). \end{aligned}$$

Since $\{j_n\}$ is a G-increasing sequence, $J_\phi\left(j_0, j_{m-n}\right) < \infty$. Letting $n \to \infty$, we obtain that $\{j_n\}$ is a J_ϕ-Cauchy sequence. Completeness of S implies that $\{j_n\}$ is J_ϕ-convergent to a point j in S. By the JNRL property of G, $(j_n, j) \in E(G)$.

Hence, $J_\phi\left(Tj_n, Tj\right) \leq r.J_\phi\left(j_n, j\right), J_\phi\left(j_{n+1}, Tj\right) \leq r.J_\phi\left(j_n, j\right)$. As $n \to \infty$, we get $J_\phi(j, Tj) \leq 0$. This is possible iff $j = Tj$. □

Corollary 2. *Suppose (S, J_ϕ) is a complete altering JS-metric space with graph G and $T : S \to S$ is a G-continuous map such that:*

(i) $(j, Tj) \in E$ *for all* $j \in S$,

(ii) $J_\phi\left(j_0, j_w\right) < \infty$ *for every G-increasing sequence* $\{j_n\}_{n=0}^{\infty}$ *and for all* $w \geq 0$,

(iii) $J_\phi(Tj, Tp) \leq r.J_\phi(j, p)$ *with* $r \in (0, 1)$ *for every* j, p *such that* $P(j, p)$ *is non-void.*

Then the map T has a fixed point in S.

Proof. As in the proof of Theorem 2, completeness of the space (S, J_ϕ) yields $j_n \xrightarrow{J_\phi} j$ for $j \in S$, i.e. $J_\phi(j_n, j) \to 0$. Since T is a G-continuous map, we obtain

$$J_\phi(Tj_n, Tj) \to 0,$$

i.e. $J_\phi(j_{n+1}, Tj) \to 0$, so $J_\phi(j, Tj) = 0$.

Hence, $Tj = j$. So, j is the fixed point of T. □

Theorem 3. *Suppose (S, J_ϕ) is an altering JS-metric space with a graph G that satisfies JNRL property. Let $T : S \to S$ be a map such that:*

(i) there exists a G-increasing sequence $\{j_n\}_{n=0}^{\infty}$ with $J_\phi(j_n, j_{m-n}) < \infty$ for every $m, n \geq 0$ such that $n < m$,

(ii) T is a generalised G-Chatterjea mapping.

Then the map T has a fixed point in S.

Proof. Consider the G-increasing sequence $\{j_n\}_{n=0}^{\infty}$ satisfying $J_\phi(s_n, s_{m-n}) < \infty$ for every $m, n \geq 0$ such that $n < m$. First we show that $\{j_n\}_{n=0}^{\infty}$ is a J_ϕ-Cauchy sequence. Since $\{j_n\}_{n=0}^{\infty}$ is a G-increasing sequence, $(j_n, j_{n+1}) \in E(G)$ for every $n \in \mathbb{N}$. Hence, there exists a path from j_n to j_m. So, we have

$$\begin{aligned}
J_\phi(j_n, j_m) &= J_\phi(Tj_{n-1}, Tj_{m-1}) \leq r\Big[J_\phi(j_{n-1}, Tj_{m-1}) + J_\phi(j_{m-1}, Tj_{n-1})\Big] \\
&= r\Big[J_\phi(j_{n-1}, j_m) + J_\phi(j_{m-1}, j_n)\Big] = r\Big[J_\phi(Tj_{n-2}, Tj_{m-1}) + J_\phi(Tj_{m-2}, Tj_{n-1})\Big] \\
&\leq r^2\Big[\{J_\phi(j_{n-2}, Tj_{m-1}) + J_\phi(j_{m-1}, Tj_{n-2})\} + \{J_\phi(j_{m-2}, Tj_{n-1}) + J_\phi(j_{n-1}, Tj_{m-2})\}\Big] \\
&= r^2\Big[\{J_\phi(j_{n-2}, j_m) + J_\phi(j_{m-1}, j_{n-1})\} + \{J_\phi(j_{m-2}, j_n) + J_\phi(j_{n-1}, j_{m-1})\}\Big] \\
&= r^2\Big[J_\phi(j_{n-2}, j_m) + 2.J_\phi(j_{n-1}, j_{m-1}) + J_\phi(j_n, j_{m-2})\Big] \\
&= r^2\Big[J_\phi(Tj_{n-3}, Tj_{m-1}) + 2.J_\phi(Tj_{n-2}, Tj_{m-2}) + J_\phi(Tj_{n-1}, Tj_{m-3})\Big] \\
&\leq r^3\Big[\{J_\phi(j_{n-3}, Tj_{m-1}) + J_\phi(j_{m-1}, Tj_{n-3})\} \\
&\quad + 2.\{J_\phi(j_{n-2}, Tj_{m-2}) + J_\phi(j_{m-2}, Tj_{n-2})\} + \{J_\phi(j_{n-1}, Tj_{m-3}) + J_\phi(j_{m-3}, Tj_{n-1})\}\Big] \\
&= r^3\Big[\{J_\phi(j_{n-3}, j_m) + J_\phi(j_{m-1}, j_{n-2})\} \\
&\quad + 2.\{J_\phi(j_{n-2}, j_{m-1}) + J_\phi(j_{m-2}, j_{n-1})\} + \{J_\phi(j_{n-1}, j_{m-2}) + J_\phi(j_{m-3}, j_n)\}\Big] \\
&= r^3\Big[J_\phi(j_{n-3}, j_m) + 3.J_\phi(j_{n-2}, j_{m-1}) + 3.J_\phi(j_{n-1}, j_{m-2}) + J_\phi(j_n, j_{m-3})\Big], \\
&\dots \\
J_\phi(j_n, j_m) &\leq r^n\Big\{\sum_{i=0}^{n} n\, C_i\, J_\phi(j_i, j_{m-i})\Big\}.
\end{aligned}$$

Letting $n \to \infty$, we obtain $J_\phi(j_n, j_m) \to 0$. Since S is complete, $\{j_n\}$ is J_ϕ-convergent to a point j in S. By the JNRL property of G, we get $(j_n, j) \in E(G)$.

Note, that

$$J_\phi\left(Tj_n, Tj\right) \leq r\Big[J_\phi\left(j_n, Tj\right) + J_\phi\left(j, Tj_n\right)\Big] \Rightarrow J_\phi\left(j_{n+1}, Tj\right) \leq r\Big[J_\phi\left(j_n, Tj\right) + J_\phi\left(j, Tj_n\right)\Big].$$

As $n \to \infty$, we get

$$J_\phi(j, Tj) \leq r\big[J_\phi(j, Tj) + J_\phi(j, Tj)\big] \Rightarrow J_\phi(j, Tj) \leq 2.r\big[J_\phi(j, Tj)\big] \Rightarrow J_\phi(j, Tj) = 0 \Rightarrow j = Tj.$$

Thus, T has a fixed point. □

Example 7. *Consider the complete altering JS-metric space with the set $S = \mathbb{R}^+$ and J_ϕ defined by*

$$J_\phi(j,p) = \begin{cases} 3, & \text{if either } j \text{ or } p \text{ is irrational,} \\ \dfrac{1}{3}, & \text{if either } j \text{ or } p \text{ is an integer,} \\ \sqrt{3}, & \text{otherwise.} \end{cases}$$

Let the graph G be defined with the vertex set S and edge set

$$E(G) = \big\{(j,p) : j = p^k, \text{ where } k \text{ is real}\big\}.$$

Let the map be given by $Tj = \sqrt{j}$. The map T is a generalised G-contraction and hence it has a fixed point.

4 Conclusion

We have conferred the generalised contractions endowed with graph and have examined fixed point results for these contractions under the frame of altering JS-metric spaces. We have defined several new notions of fixed point theory by introducing altering JS-metric space. We proved some fixed point theorem using generalised G-contraction mapping, generalised G-Kannan mapping, generalised G-Chatterjea mapping. Further, in the future we can extend the applications for the proposed results.

References

[1] Acar O., Altun I. *Multivalued F-contractive mappings with a graph and some fixed point results*. Publ. Math. Debrecen 2016, **88** (3-4), 305–317. doi:10.5486/PMD.2016.7308

[2] Acar O. *Rational Type Multivalued F_G-Contractive Mappings with a Graph*. Results Math. 2018, **73** (52), 1–9. doi:10.1007/s00025-018-0813-x

[3] Alfuraidan M.R., Bachar M., Khamsi M.A. *Almost monotone contractions on weighted graphs*. J. Nonlinear Sci. Appl. 2016, **9** (8), 5189–5195. doi:10.22436/jnsa.009.08.04

[4] Bojor F. *Fixed points of Kannan mappings in metric spaces endowed with a graph*. An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 2012, **20** (1), 31–40. doi:10.2478/v10309-012-0003-x

[5] Delbosco D. *Un'estensione di un teorema sul punto fisso di S. Reich*. Rend. Semin. Mat. Univ. Politec. Torino 1976, **35**, 233–238. (in Italian)

[6] Fallahi K., Aghanians A. *Fixed points for Chatterjea contractions on a metric space with a graph*. Int. J. Nonlinear Anal. Appl. 2016, **7** (2), 49–58. doi:10.22075/ijnaa.2016.449

[7] Jachymski J. *The contraction principle for mappings on a metric space with a graph*. Proc. Amer. Math. Soc. 2008, **136** (4), 1359–1373. doi:10.1090/S0002-9939-07-09110-1

[8] Jleli M., Samet B. *A generalized metric space and related fixed point theorems*. Fixed Point Theory Appl. 2015, **61** (2015), article number 61. doi:10.1186/s13663-015-0312-7

[9] Karapinar E. *A Short Survey on the Recent Fixed Point Results on b-Metric Spaces*. Constructive Math. Anal. 2018, **1** (1), 15–44.

[10] Khan M.S., Swaleh M., Sessa S. *Fixed point theorems by altering distances between the points*. Bull. Aust. Math. Soc. 1984, **30** (1), 1–9. doi:10.1017/S0004972700001659

[11] Nieto J.J., Rodríguez-López R. *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*. Order 2005, **22** (3), 223–239. doi:10.1007/s11083-005-9018-5

[12] Nieto J.J., Rodríguez-López R. *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*. Acta Math. Sinica (Chin. Ser.) 2007, **23** (12), 2205–2212. doi:10.1007/s10114-005-0769-0

[13] Petruşel A., Rus I.A. *Fixed point theorems in ordered L-spaces*. Proc. Amer. Math. Soc. 2006, **134** (2), 411–418. doi:10.1090/S0002-9939-05-07982-7

[14] Skof F. *Teorema di punti fisso per applicazioni negli spazi metrici*. Atti. Accad. Aci. Torino 1977, **111** (3–4), 323–329. (in Italian)

[15] Vetro C. *A Fixed-Point Problem with Mixed-Type Contractive Condition*. Constructive Math. Anal. 2020, **3** (1), 45–52. doi:10.33205/cma.684638

Received 24.03.2023

Аль-Шаммарі І., Езель С., Гнаначандра П., Варуння Х.М.Дж. *Теореми про нерухомі точки для узагальнених стисків в зміненому JS-метричному просторі з графом* // Карпатські матем. публ. — 2024. — Т.16, №2. — C. 606–616.

У цій статті ми пропонуємо розширення метричного простору, введеного М. Джлелі та Б. Саметом (JS-метричний простір). Це розширення здійснюється за допомогою зміненої функції відстані, тому його названо "змінена JS-метрика". Ідея нерухомих точок для стискаючих відображень, наділених графом, була вперше представлена Я. Яхимським у 2008 році. Після цього багато дослідників розвивали цю ідею та запропонували різноманітні стискаючі відображення, наділені графом. Натхненні цими ідеями, ми узагальнили деякі відомі стискаючі відображення, описані в літературі, які наділені графом. Крім того, ми навели результати щодо нерухомих точок для цих стискань у зміненому JS-метричному просторі.

Ключові слова і фрази: JS-метрика, змінена JS-метрика, узагальнений G-стиск, узагальнене відображення G-Каннана, узагальнене відображення G-Чаттерджі.