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Wick multiplication and its relationship with integration and stochastic differentiation on spaces of nonregular test functions in the Lévy white noise analysis

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We deal with spaces of nonregular test functions in the Lévy white noise analysis, which are constructed using Lytvynov's generalization of a chaotic representation property. Our goal is to study properties of a natural multiplication — a Wick multiplication on these spaces, and to describe the relationship of this multiplication with integration and stochastic differentiation. More exactly, we establish that the Wick product of nonregular test functions is a nonregular test function; show that when employing the Wick multiplication, it is possible to take a time-independent multiplier out of the sign of a generalized stochastic integral; establish an analog of this result for a Pettis integral (a weak integral); obtain a representation of the generalized stochastic integral via formal Pettis integral from the Wick product of the original integrand by a Lévy white noise; and prove that the operator of stochastic differentiation of first order on the spaces of nonregular test functions satisfies the Leibnitz rule with respect to the Wick multiplication.

Key words and phrases: Lévy process, Wick product, stochastic integral, Pettis integral, operator of stochastic differentiation.

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Introduction

Many problems in modern mathematics and physics require a theory of test and generalized functions depending on infinitely many variables, i.e. with arguments belonging to infinite-dimensional spaces. Such a theory can be constructed by different ways. One of the most successful of them consists in introducing of spaces of test and generalized functions such that the pairing between elements of these spaces is generated by integration with respect to a probability measure on some dual nuclear space. At first, it was the Gaussian measure (see, e.g., [2, 17, 30, 31]), the corresponding theory is called the *Gaussian white noise analysis*, afterwards it were realized numerous generalizations. In particular, important results were obtained using the generalized Meixner measure [35], and the Lévy white noise measure (see, e.g., [7, 8, 32]), the corresponding theories are called the *Miexner* and *Lévy white noise analysis*, respectively.

An important role in the Gaussian analysis belongs to a so-called *chaotic representation property* (CRP): roughly speaking, any square integrable with respect to the Gaussian measure random variable can be presented as a series of repeated Itô's stochastic integrals with nonrandom integrands (see, e.g., [33]). Using CRP, one can construct spaces of test and generalized func-

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tions, introduce and study different operators and operations on these spaces, in particular, stochastic integrals and derivatives, a Wick multiplication etc. Unfortunately, in the Meixner and Lévy analysis there is no CRP, generally speaking [39]; nevertheless, there are various generalizations of this property. Specifically, in the Meixner analysis a square integrable random variable can be decomposed in series of generalized Meixner polynomials [35]; in the Lévy analysis there are decompositions connected with a Lévy-Khintchine representation of a Lévy process (Itô's approach [19], see also [6]), decompositions by repeated stochastic integrals from nonrandom integrands with respect to so-called orthogonalized centered power jump processes (Nualart-Schoutens' approach [34], see also [36]), decompositions by special orthogonal functions (Lytvynov's approach [32], see also [5]), special orthogonal decompositions with numeric coefficients (Øksendal's approach [8], see also [7]) etc. The relationships between these generalizations of CRP are described in, e.g., [1,7,8,23,32,38,40].

In this paper, we deal with one of the most useful generalizations of CRP in the Lévy analysis, which is proposed by E.W. Lytvynov [32]. The idea of this generalization is to decompose square integrable random variables in series of special orthogonal functions with nonrandom kernels, by analogy with decompositions of square integrable random variables by Hermite polynomials in the Gaussian analysis (remind that the last decompositions are equivalent to the decompositions by repeated Itô's stochastic integrals). As in the Gaussian analysis, it is possible to use Lytvynov's generalization of CRP, in particular, in order to construct and study spaces of regular and nonregular test and generalized functions [20], introduce and investigate various operators and operations on these spaces etc. It should be noted that the extended stochastic integral and the Hida stochastic derivative on the spaces of *regular* test and generalized functions are introduced and studied in [11, 20], operators of stochastic differentiation — in [9, 10, 14], elements of a Wick calculus and its relationship with operators of stochastic differentiation and integration on the spaces of *regular generalized* functions — in [12, 13], on the spaces of *regular test* functions — in [25]. As for the spaces of *nonregular* test and generalized functions, the corresponding results are presented in [20, 26–29]. The paper [24] is a survey of some author's results related to the development of the Lévy white noise analysis in terms of Lytvynov's generalization of CRP.

In order to build a meaningful and applicable theory of test and generalized functions it is necessary to introduce a natural multiplication on spaces of the mentioned functions. As is known, in various versions of the white noise analysis such a multiplication is a so-called *Wick multiplication*. In particular, using the Wick multiplication, one can take a time-independent multiplier out of the sign of an extended stochastic integral and of a Pettis integral (a weak integral). In addition, an extended stochastic integral can be presented as a Pettis integral (or a *formal* Pettis integral, depending on the concrete situation) from the Wick product of the original integrand by the corresponding white noise. Also an operator of stochastic differentiation is a differentiation (i.e. satisfies the Leibnitz rule) with respect to the Wick multiplication. On the above-mentioned spaces of *nonregular generalized* functions in the Lévy analysis such results were obtained in [26,29], on the spaces of *regular generalized* functions — in [12, 13], on the spaces of *regular test* functions — in [25].

The aim of the present paper is to introduce and to study by analogy with [21, 25] the Wick product on the spaces of *nonregular test* functions in the Lévy analysis; to transfer some results of [13, 25] to these spaces; and to consider certain related topics, in particular, the relationship between Wick multiplication and stochastic differentiation.

The paper is organized in the following manner. In the first section, we consider a Lévy process L and recall the construction of a required probability triplet connected with L; afterwards we describe Lytvynov's generalization of CRP and recall the construction of a nonregular rigging of the space of square integrable random variables (the positive and negative spaces of this rigging are the spaces of nonregular test and generalized functions respectively). In the second section, we introduce the Wick product on the spaces of nonregular test functions and study its main properties. In the third section, we recall the definition of a generalized stochastic integral (a natural analog of the extended stochastic integral on the spaces of nonregular test functions); show that when employing the Wick multiplication, it is possible to take a time-independent multiplier out of the sign of this integral; obtain an analog of this result for the Pettis integral; and prove a theorem about a representation of the generalized stochastic integral via a formal Pettis integral. In the fourth section, we recall the definition of operators of stochastic differentiation on the spaces of nonregular test functions and establish that the operator of stochastic differentiation of first order is a differentiation with respect to the Wick multiplication.

1 Preliminaries

We denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the real, i.e. bilinear, scalar product in H; by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_H$ the dual pairing generated by the scalar product in H; by $\mathcal{B}$ the Borel σ-algebra and by 1_Δ the indicator of a set Δ. Further, we use a designation pr lim (respectively, ind lim) for a projective (respectively, inductive) limit of a family of spaces, this designation implies that the limit space is endowed with the projective (respectively, inductive) limit topology (see, e.g., [3] for the detailed description).

1.1 A Lévy process and its probability space

Denote $\mathbb{R}_+ := [0,+\infty)$. Let $L = (L_u)_{u\in\mathbb{R}_+}$ be a real-valued locally square integrable Lévy process (i.e. a continuous in probability random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$, see, e.g., [4] for the detailed description) without Gaussian part and drift. As is known (e.g., [8]), the characteristic function of L is

$$\mathbb{E}\left[e^{i\theta L_u}\right] = \exp\left[u\int_{\mathbb{R}}\left(e^{i\theta x} - 1 - i\theta x\right)\nu(dx)\right], \tag{1}$$

where ν is the Lévy measure of L, which is a measure on $\big(\mathbb{R},\mathcal{B}(\mathbb{R})\big)$, $\mathbb{E}$ denotes the expectation. We assume that ν *is a Radon measure, whose support contains an infinite number of points,* $\nu\big(\{0\}\big) = 0$, *there exists* $\varepsilon > 0$ *such that* $\int_{\mathbb{R}} x^2 e^{\varepsilon|x|}\nu(dx) < \infty$, *and* $\int_{\mathbb{R}} x^2\nu(dx) = 1$.

Let us define a measure of the white noise of L. Denote by $\mathcal{D}$ the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [3], see also Subsection 1.3). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. Note that $\mathcal{D}$ and $\mathcal{D}'$ are the positive and negative spaces of a chain

$$\mathcal{D}' \supset L^2(\mathbb{R}_+) \supset \mathcal{D}, \tag{2}$$

where $L^2(\mathbb{R}_+)$ is the space of (classes of) real-valued functions on $\mathbb{R}_+$, square integrable with respect to the Lebesgue measure (e.g., [3]). Denote by $\langle\cdot,\cdot\rangle$ the dual pairing between elements of $\mathcal{D}'$ and $\mathcal{D}$, generated by the scalar product in $L^2(\mathbb{R}_+)$.

Definition 1. *A probability measure μ on $\left(\mathcal{D}',\mathcal{C}\left(\mathcal{D}'\right)\right)$, where $\mathcal{C}$ denotes the cylindrical σ-algebra, with the Fourier transform*

$$\int_{\mathcal{D}'} e^{i\langle\omega,\varphi\rangle}\mu(d\omega)=\exp\left[\int_{\mathbb{R}_+\times\mathbb{R}}\left(e^{i\varphi(u)x}-1-i\varphi(u)x\right)du\nu(dx)\right],\qquad \varphi\in\mathcal{D}, \tag{3}$$

is called the measure of the Lévy white noise.

The existence of μ follows from the Bochner-Minlos theorem (e.g., [18]), see [32]. Below we assume that the *σ-algebra $\mathcal{C}\left(\mathcal{D}'\right)$ is completed with respect to μ.*

Consider a probability space (probability triplet) $\left(\mathcal{D}',\mathcal{C}\left(\mathcal{D}'\right),\mu\right)$. Let us denote by $(L^2):=L^2\big(\mathcal{D}',\mathcal{C}(\mathcal{D}'),\mu\big)$ the space of square integrable random variables, i.e. the space of (classes of) complex-valued functions on $\mathcal{D}'$, square integrable with respect to μ. Let $f\in L^2\left(\mathbb{R}_+\right)$ and a sequence $(\varphi_k\in\mathcal{D})_{k\in\mathbb{N}}$ converges to f in $L^2\left(\mathbb{R}_+\right)$ as $k\to\infty$ (remind that $\mathcal{D}$ is a dense set in $L^2\left(\mathbb{R}_+\right)$). One can show [7,8,23,32] that $\langle\circ,f\rangle:=\left(L^2\right)\text{-}\lim_{k\to\infty}\langle\circ,\varphi_k\rangle$ is a well-defined element of $\left(L^2\right)$.

Put $1_{[0,0)}\equiv 0$. It follows from (1) and (3) that $\left(\langle\circ,1_{[0,u)}\rangle\right)_{u\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space $\left(\mathcal{D}',\mathcal{C}\left(\mathcal{D}'\right),\mu\right)$ (see, e.g., [7,8]). So, for each $u\in\mathbb{R}_+$ we have $L_u=\langle\circ,1_{[0,u)}\rangle\in\left(L^2\right)$.

Note that the derivative in the sense of generalized functions of a Lévy process (the Lévy white noise) is $\dot{L}_\cdot(\omega)=\langle\omega,\delta_\cdot\rangle\equiv\omega(\cdot)$, where δ is the Dirac delta-function. Therefore $\dot{L}$ is a generalized random process in the sense of [15] with trajectories from $\mathcal{D}'$, and μ is the measure of $\dot{L}$ in the classical sense of this notion [16].

1.2 Lytvynov's generalization of the chaotic representation property

In what follows, we denote by a subscript $\mathbb{C}$ complexifications of spaces; by a symbol $\widehat{\otimes}$ the symmetric tensor multiplication; and preserve the above-introduced notation $\langle\cdot,\cdot\rangle$ for the dual pairings in symmetric tensor powers of the complexification of chain (2) (actually, of more general chain (7), i.e. for the dual pairings between elements of negative and positive spaces from chains (40)). Designate $\mathbb{Z}_+:=\mathbb{N}\cup\{0\}$. Let $\mathcal{P}$ be the set of complex-valued polynomials on $\mathcal{D}'$ that consists of zero and elements of the form

$$f(\omega)=\sum_{n=0}^{N_f}\left\langle\omega^{\otimes n},f^{(n)}\right\rangle,\quad \omega\in\mathcal{D}',\ f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n},\ N_f\in\mathbb{Z}_+,\ f^{(N_f)}\neq 0,$$

here N_f is called the *power of a polynomial f*; $\left\langle\omega^{\otimes 0},f^{(0)}\right\rangle:=f^{(0)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}0}:=\mathbb{C}$. The measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (3) and properties of the measure ν, see also [32]), therefore $\mathcal{P}$ is a dense set in $\left(L^2\right)$ [37]. Let $\mathcal{P}_n$, $n\in\mathbb{Z}_+$, be the set of polynomials of power smaller than or equal to n, by $\overline{\mathcal{P}}_n$ we denote the closure of $\mathcal{P}_n$ in $\left(L^2\right)$. Let $\mathbf{P}_n:=\overline{\mathcal{P}}_n\ominus\overline{\mathcal{P}}_{n-1}$ for all $n\in\mathbb{N}$ be the orthogonal difference in $\left(L^2\right)$; put $\mathbf{P}_0:=\overline{\mathcal{P}}_0$. It is clear that

$$\left(L^2\right)=\mathop{\oplus}_{n=0}^{\infty}\mathbf{P}_n. \tag{4}$$

Let $f^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n\in\mathbb{Z}_+$. Denote by $:\left\langle\circ^{\otimes n},f^{(n)}\right\rangle:\ \in\left(L^2\right)$ the orthogonal projection of a monomial $\left\langle\circ^{\otimes n},f^{(n)}\right\rangle$ onto $\mathbf{P}_n$. We define real (bilinear) scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$, $n\in\mathbb{Z}_+$, by setting

$$\left(f^{(n)},g^{(n)}\right)_{ext}:=\frac{1}{n!}\int_{\mathcal{D}'}:\left\langle\omega^{\otimes n},f^{(n)}\right\rangle::\left\langle\omega^{\otimes n},g^{(n)}\right\rangle:\mu(d\omega),\qquad f^{(n)},g^{(n)}\in\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}. \tag{5}$$

The well-posedness of this definition is proved (up to obvious modifications) in [32].

Denote by $|\cdot|_{ext}$ the norms corresponding to scalar products (5), i.e. $|\cdot|_{ext} := \sqrt{(\cdot,\bar{\cdot})_{ext}}$. Let $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, be the completions of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n}$ with respect to these norms. For each $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ we define a *Wick monomial* $:\left\langle \circ^{\otimes n}, F^{(n)} \right\rangle: \stackrel{\text{def}}{=} (L^2)\text{-}\lim_{k\to\infty} :\left\langle \circ^{\otimes n}, f_k^{(n)} \right\rangle:$, where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n} \ni f_k^{(n)} \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}_{ext}^{(n)}$. It is easy to prove by the method of "mixed sequences" that this definition is well-posed, and to show that $:\left\langle \circ^{\otimes 0}, F^{(0)} \right\rangle: = \left\langle \circ^{\otimes 0}, F^{(0)} \right\rangle = F^{(0)}$ and $:\left\langle \circ, F^{(1)} \right\rangle: = \left\langle \circ, F^{(1)} \right\rangle$ (cf. [32]).

In the next statement, which follows from (4) and the fact that for each $n \in \mathbb{Z}_+$ the set $\left\{ :\left\langle \circ^{\otimes n}, f^{(n)} \right\rangle: \mid f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes}n} \right\}$ is dense in $\mathbf{P}_n$, and therefore $\mathbf{P}_n = \left\{ :\left\langle \circ^{\otimes n}, F^{(n)} \right\rangle: \mid F^{(n)} \in \mathcal{H}_{ext}^{(n)} \right\}$, Lytvynov's generalization of the chaotic representation property (CRP) is described.

Theorem 1 ([32])**.** *A random variable F belongs to (L^2) if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} :\left\langle \circ^{\otimes n}, F^{(n)} \right\rangle: \tag{6}$$

(the series converges in (L^2)) and

$$\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} \left|F(\omega)\right|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! \left|F^{(n)}\right|_{ext}^2 < \infty.$$

Remark 1. *In this paper, we do not use directly an explicit (calculation-friendly) formula for scalar products* (5) *and therefore we prefer not to write it down. The interested reader can find this formula in [32]; in another record form it is given, e.g., in [12,14,20,23,24].*

Denote $\mathcal{H} := L^2(\mathbb{R}_+)$, then $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}_+)_{\mathbb{C}}$. It follows from the explicit formula for $(\cdot,\cdot)_{ext}$ that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, and for $n \in \mathbb{N}\setminus\{1\}$ one can identify $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (roughly speaking, such that $F^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k, j \in \{1,\dots,n\}$, $k \neq j$, but $u_k = u_j$). In this sense the *space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}n}$*, this explains why we use the subscript "*ext*" in our designations. Also we note that for each $n \in \mathbb{N}\setminus\{1\}$ the space $\mathcal{H}_{ext}^{(n)}$ is the symmetric subspace of the space of (classes of) complex-valued functions on $\mathbb{R}_+^n$, square integrable with respect to a certain Radon measure.

1.3 A nonregular rigging of the space of square integrable random variables

Let T be the set of indexes $\tau = (\tau_1, \tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Denote by $\mathcal{H}_\tau$ the real Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e. $\mathcal{H}_\tau$ is the completion of $\mathcal{D}$ with respect to the norm generated by the scalar product

$$(\varphi, \psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+} \left(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1} \varphi^{[k]}(u)\psi^{[k]}(u) \right) \tau_2(u) du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ, respectively. As is known (see, e.g., [3]), $\mathcal{D} = \operatorname{pr}\lim_{\tau\in T} \mathcal{H}_\tau$ (moreover, for any $n \in \mathbb{N}$ we have $\mathcal{D}^{\widehat{\otimes}n} = \operatorname{pr}\lim_{\tau\in T} \mathcal{H}_\tau^{\widehat{\otimes}n}$), and for each $\tau \in T$ the space $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$. Therefore one can consider a chain (cf. (2))

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D}, \tag{7}$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Note that by the Schwartz theorem [3], we have $\mathcal{D}' = \operatorname{ind\,lim}_{\tau \in T} \mathcal{H}_{-\tau}$ (in what follows we will consider $\mathcal{D}'$ as a topological space with the inductive limit topology).

Denote the norms in $\mathcal{H}_{\tau,\mathbb{C}}$ and its symmetric tensor powers by $|\cdot|_\tau$, i.e. for $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, $|f^{(n)}|_\tau = \sqrt{\left(f^{(n)}, \overline{f^{(n)}}\right)_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}}$ (note that $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$ and $|f^{(0)}|_\tau = |f^{(0)}|$).

It follows from results of [20] that one can modify T (it is necessary to remove from T some "bad" indexes; and we further assume that T is modified) in order to obtain the following statement.

Proposition 1. *1) For each $\tau \in T$ the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\tau}$, i.e. $\mu(\mathcal{H}_{-\tau}) = 1$.*

2) For each $\tau \in T$ and each $n \in \mathbb{Z}_+$ the space $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ we have $|f^{(n)}|_{ext}^2 \le n! c(\tau)^n |f^{(n)}|_\tau^2$.

Let $\tau \in T$ and $q \in \mathbb{Z}_+$. Denote

$$\mathcal{P}_W := \left\{ f = \sum_{n=0}^{N_f} :\left\langle \circ^{\otimes n}, f^{(n)} \right\rangle: \mid f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, N_f \in \mathbb{Z}_+ \right\} \subset (L^2).$$

For $f = \sum_{n=0}^{N_f} :\left\langle \circ^{\otimes n}, f^{(n)} \right\rangle:$, $g = \sum_{n=0}^{N_g} :\left\langle \circ^{\otimes n}, g^{(n)} \right\rangle: \in \mathcal{P}_W$, we define real (bilinear) scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ by setting

$$(f, g)_{\tau,q} := \sum_{n=0}^{\min(N_f, N_g)} (n!)^2 2^{qn} \left(f^{(n)}, g^{(n)}\right)_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}}. \tag{8}$$

The well-posedness of this definition is proved in [26].

Remark 2. *One can introduce more general scalar products on $\mathcal{P}_W$, writting in* (8) *K^{qn} with arbitrary $K > 1$ instead of 2^{qn}. But such a generalization is not essential for our considerations, so, for simplification of presentation we will restrict ourselves to the case $K = 2$.*

Denote by $\|\cdot\|_{\tau,q}$ the norms corresponding to scalar products (8), i.e. $\|\cdot\|_{\tau,q} := \sqrt{(\cdot, \bar{\cdot})_{\tau,q}}$. Let $(\mathcal{H}_\tau)_q$ be the completions of $\mathcal{P}_W$ with respect to these norms. Set $(\mathcal{H}_\tau) := \operatorname{pr\,lim}_{q\to\infty} (\mathcal{H}_\tau)_q$, $(\mathcal{D}) := \operatorname{pr\,lim}_{\tau\in T, q\to\infty} (\mathcal{H}_\tau)_q$. As is easy to see, f belongs to $(\mathcal{H}_\tau)_q$ if and only if it can be uniquely presented in the form (cf. (6))

$$f = \sum_{n=0}^{\infty} :\left\langle \circ^{\otimes n}, f^{(n)} \right\rangle:, \quad f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \tag{9}$$

$\big($the series converges in $(\mathcal{H}_\tau)_q\big)$, with

$$\|f\|^2_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} \left|f^{(n)}\right|^2_\tau < \infty \tag{10}$$

(since by Proposition 1 for each $n \in \mathbb{Z}_+$ we have $\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}} \subseteq \mathcal{H}^{(n)}_{ext}$, for $f^{(n)} \in \mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}$: $\left\langle \circ^{\otimes n}, f^{(n)} \right\rangle$: is a well defined Wick monomial, see Subsection 1.2). Further, $f \in (\mathcal{H}_\tau)$ $\big(f \in (\mathcal{D})\big)$ if and only if f can be uniquely presented in form (9) and norm (10) is finite for each $q \in \mathbb{Z}_+$ (for each $\tau \in T$ and each $q \in \mathbb{Z}_+$).

Proposition 2 ([20, 26])**.** *For each $\tau \in T$ there exists $q_0(\tau) \in \mathbb{Z}_+$ such that for each $q \in \mathbb{N}_{q_0(\tau)} := \{q_0(\tau), q_0(\tau)+1, \dots\}$ the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into $\left(L^2\right)$.*

In view of this proposition one can consider a chain

$$\left(\mathcal{D}'\right) \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset \left(L^2\right) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}), \ \tau \in T, \ q \in \mathbb{N}_{q_0(\tau)}, \tag{11}$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \underset{q' \to \infty}{\operatorname{ind\,lim}} (\mathcal{H}_{-\tau})_{-q'}$ and $(\mathcal{D}') = \underset{\widetilde{\tau} \in T, q' \to \infty}{\operatorname{ind\,lim}} (\mathcal{H}_{-\widetilde{\tau}})_{-q'}$ are the spaces dual of $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ with respect to $\left(L^2\right)$.

In what follows, we assume that $\tau \in T$, $q \in \mathbb{N}_{q_0(\tau)}$.

Definition 2. *Chain* (11) *is called a nonregular rigging of the space $\left(L^2\right)$. The positive spaces of this rigging $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ are called (Kondratiev-type) spaces of nonregular test functions. The negative spaces of this rigging $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$ and $(\mathcal{D}')$ are called (Kondratiev-type) spaces of nonregular generalized functions.*

Remark 3. *Let $\tau \in T$, $q \in \mathbb{Z}_+$ and $\beta \in [0,1]$. By analogy with the regular case one can introduce on $\mathcal{P}_W$ scalar products $(\cdot,\cdot)_{\tau,q,\beta}$ by setting*

$$(f,g)_{\tau,q,\beta} := \sum_{n=0}^{\min\left(N_f,N_g\right)} (n!)^{1+\beta} 2^{qn} \left(f^{(n)}, g^{(n)}\right)_{\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}}, \quad f,g \in \mathcal{P}_W,$$

$\big($cf. (8)$\big)$, and define "parametrized spaces of nonregular test functions" $(\mathcal{H}_\tau)^\beta_q$ as completions of $\mathcal{P}_W$ with respect to the norms generated by these scalar products. It is possible to study properties of the spaces $(\mathcal{H}_\tau)^\beta_q$ and their projective limits, to introduce and to study different operators and operations on them; such considerations are interesting by itself and can be useful for applications. But $(\mathcal{H}_\tau)^\beta_q \not\subset \left(L^2\right)$ if $\beta < 1$ (except for the Gaussian and Poissonian special cases, which we do not consider in this paper), so we cannot consider $(\mathcal{H}_\tau)^\beta_q$ with $\beta < 1$ as spaces of test functions in the framework of the Lévy white noise analysis.

Finally, for completeness and to introduce some necessary concepts we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider chains

$$\mathcal{D}'^{(n)}_{\mathbb{C}} \supset \mathcal{H}^{(n)}_{-\tau,\mathbb{C}} \supset \mathcal{H}^{(n)}_{ext} \supset \mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}} \supset \mathcal{D}^{\widehat{\otimes} n}_{\mathbb{C}}, \quad n \in \mathbb{N}, \tag{12}$$

where $\mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ and $\mathcal{D}'^{(n)}_{\mathbb{C}} = \underset{\widetilde{\tau} \in T}{\operatorname{ind\,lim}} \mathcal{H}^{(n)}_{-\widetilde{\tau},\mathbb{C}}$ are the spaces dual of $\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}$ and $\mathcal{D}^{\widehat{\otimes} n}_{\mathbb{C}} = \underset{\widetilde{\tau} \in T}{\operatorname{pr\,lim}} \mathcal{H}^{\widehat{\otimes} n}_{\widetilde{\tau},\mathbb{C}}$ with respect to $\mathcal{H}^{(n)}_{ext}$. Set $\mathcal{D}^{\widehat{\otimes} 0}_{\mathbb{C}} = \mathcal{H}^{\widehat{\otimes} 0}_{\tau,\mathbb{C}} = \mathcal{H}^{(0)}_{ext} = \mathcal{H}^{(0)}_{-\tau,\mathbb{C}} = \mathcal{D}'^{(0)}_{\mathbb{C}} := \mathbb{C}$. In what follows we denote by $\langle\cdot,\cdot\rangle_{ext}$ the real (bilinear) dual pairings between elements of negative and positive spaces from chains (12), these pairings are generated by the scalar products in $\mathcal{H}_{ext}^{(n)}$.

The next statement follows from the definition of the spaces $(\mathcal{H}_{-\tau})_{-q}$ and the general duality theory (cf. [20, 22]).

Proposition 3. *There exists a system of generalized functions*

$$\left\{:\left\langle\circ^{\otimes n}, F_{ext}^{(n)}\right\rangle:\in(\mathcal{H}_{-\tau})_{-q} \mid F_{ext}^{(n)}\in\mathcal{H}_{-\tau,\mathbb{C}}^{(n)},\ n\in\mathbb{Z}_+\right\}$$

such that

1) *for* $F_{ext}^{(n)}\in\mathcal{H}_{ext}^{(n)}\subset\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $:\left\langle\circ^{\otimes n}, F_{ext}^{(n)}\right\rangle:$ *is a Wick monomial that is defined in Subsection 1.2;*

2) *any generalized function* $F\in(\mathcal{H}_{-\tau})_{-q}$ *can be uniquely presented as a series*

$$F=\sum_{n=0}^{\infty}:\left\langle\circ^{\otimes n}, F_{ext}^{(n)}\right\rangle:,\quad F_{ext}^{(n)}\in\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}, \tag{13}$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$, *i.e.*

$$\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}}=\sum_{n=0}^{\infty}2^{-qn}\left|F_{ext}^{(n)}\right|^2_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}}<\infty; \tag{14}$$

and, vice versa, any series (13) *with finite norm* (14) *is a generalized function from* $(\mathcal{H}_{-\tau})_{-q}$, *i.e. such a series converges in* $(\mathcal{H}_{-\tau})_{-q}$;

3) *the dual pairing between* $F\in(\mathcal{H}_{-\tau})_{-q}$ *and* $f\in(\mathcal{H}_{\tau})_{q}$, *that is generated by the scalar product in* (L^2), *has the form*

$$\langle\!\langle F,f\rangle\!\rangle_{(L^2)}=\sum_{n=0}^{\infty}n!\left\langle F_{ext}^{(n)}, f^{(n)}\right\rangle_{ext},$$

where $F_{ext}^{(n)}\in\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ *and* $f^{(n)}\in\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}$ *are the kernels from decompositions* (13) *and* (9) *for* F *and* f, *respectively.*

It is clear that F belongs to $(\mathcal{H}_{-\tau})$ (respectevely, to $(\mathcal{D}')$) if and only if it can be uniquely presented in form (13) and norm (14) is finite for some $q\in\mathbb{N}_{q_0(\tau)}$ (respectevely, for some $\tau\in T$ and some $q\in\mathbb{N}_{q_0(\tau)}$).

2 Wick product on the spaces of nonregular test functions

It is known that the development of a meaningful and applicable theory of test and generalized functions depending on infinitely many variables requires a natural multiplication on spaces of the mentioned functions. Unfortunately, the classical pointwise multiplication is not suitable for this role: the pointwise product of test functions may not belong to the corresponding space, and the pointwise product of generalized functions is undefined, in general. However, this is not a serious problem, since on the spaces of test and generalized functions it is possible to introduce a natural multiplication, called a *Wick multiplication*.

Recall that the Wick multiplication on the spaces of *regular generalized* functions in the Lévy white noise analysis is introduced and studied in [12] (see also [13]), on the spaces of *regular test* functions — in [25], on the spaces of *nonregular generalized* functions — in [26] (see also [29]). Now our goal is to introduce and study the Wick multiplication on the spaces of *nonregular test* functions. It is worth noting that an important feature of the *regular* case is the fact that the Wick multiplication on the spaces of test functions is a restriction to these spaces of the Wick multiplication, defined on the spaces of generalized functions. In the *nonregular* case there is no such a property, which complicates the theory somewhat.

Remark 4. *Unfortunately, the spaces from nonregular rigging* (11) *of* (L^2)*, in particular, the spaces of nonregular test and generalized functions in the Lévy analysis (except of the Gaussian and Poissonian special cases, which, as mentioned above, we do not consider in this paper), have the following unpleasant feature: not all operators and operations can be naturally continued (respectively, restricted) from a narrower to a wider (respectively, from a wider to a narrower) space. Specifically, a stochastic derivative and operators of stochastic differentiation cannot be naturally continued from* (L^2) *to* $(\mathcal{H}_{-\tau})_{-q}$*; an extended stochastic integral cannot be naturally restricted from* (L^2) *to* $(\mathcal{H}_\tau)_q$*; for elements of* $(\mathcal{H}_\tau) \subset (\mathcal{H}_{-\tau})$ *the Wick product, introduced on* $(\mathcal{H}_{-\tau})$*, does not belong to* $(\mathcal{H}_\tau)$*, in general; etc. However, this problem has a solution: one can introduce natural analogs of the stochastic derivative and of the operators of stochastic differentiation on* $(\mathcal{H}_{-\tau})_{-q}$*,* $(\mathcal{H}_{-\tau})$ *and* $(\mathcal{D}')$ *[27]; a natural analog of the extended stochastic integral on* $(\mathcal{H}_\tau)_q$*,* $(\mathcal{H}_\tau)$ *and* $(\mathcal{D})$ *[27]; a natural Wick multiplication on* $(\mathcal{H}_\tau)$ *and* $(\mathcal{D})$ *(see below); etc.*

The classical definition of a Wick product is based on a so-called S-transform. In particular, for $F, G \in (\mathcal{H}_{-\tau})$ we can define the Wick product $F\lozenge G \in (\mathcal{H}_{-\tau})$ as $F\lozenge G := S^{-1}(SF \cdot SG)$, where $(SF)(\lambda) := \sum_{n=0}^{\infty} \left\langle F_{ext}^{(n)}, \lambda^{\otimes n} \right\rangle_{ext}$, $\lambda \in \mathcal{D}_{\mathbb{C}}$, $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$ are the kernels from decomposition (13) for F (see [26]). But, as noted above, such a definition is not appropriate for the spaces $(\mathcal{H}_\tau)$. Indeed, as established in [26], the Wick product on $(\mathcal{H}_{-\tau})$ can be presented as

$$F\lozenge G = \sum_{n=0}^{\infty} :\left\langle \circ^{\otimes n}, \sum_{k=0}^{n} F_{ext}^{(k)} \diamond G_{ext}^{(n-k)} \right\rangle:, \tag{15}$$

where a multiplication $\diamond$ is a natural analog of the symmetric tensor multiplication on the negative spaces from chains (12) (this multiplication will be introduced in Subection 4.1, see (42)), $F_{ext}^{(k)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(k)}$ and $G_{ext}^{(n-k)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n-k)}$ are the kernels from decompositions (13) for F and G, respectively. But for $f^{(k)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} k} \subset \mathcal{H}_{-\tau,\mathbb{C}}^{(k)}$ and $g^{(n-k)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n-k} \subset \mathcal{H}_{-\tau,\mathbb{C}}^{(n-k)}$ we have $f^{(k)} \diamond g^{(n-k)} \notin \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, generally speaking, therefore $f\lozenge g \notin (\mathcal{H}_\tau)$ for $f, g \in (\mathcal{H}_\tau) \subset (\mathcal{H}_{-\tau})$ in the general case. Nevertheless, a natural Wick product on $(\mathcal{H}_\tau)$ can be defined using a suitable analog of the S-transform. Namely, for $f \in (\mathcal{H}_\tau)$ we set formally

$$(\widehat{S}f)(\lambda) := \sum_{n=0}^{\infty} \left\langle f^{(n)}, \lambda^{\otimes n} \right\rangle, \tag{16}$$

where $\lambda \in \mathcal{D}_{\mathbb{C}}$ and $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ are the kernels from decomposition (9) for f. Note that, as in the classical case, $(\widehat{S}f)(0) = f^{(0)}$, $\widehat{S}1 \equiv 1$.

Definition 3. *For* $f_1, \ldots, f_m \in (\mathcal{H}_\tau)$ *we define the Wick product* $f_1 \blacklozenge \cdots \blacklozenge f_m$ *by setting formally*

$$f_1 \blacklozenge \cdots \blacklozenge f_m := \widehat{S}^{-1}\left(\widehat{S}f_1 \cdot \cdots \cdot \widehat{S}f_m\right).$$

It is easy to see that the Wick multiplication $\blacklozenge$ is commutative, associative, distributive, and for any $\alpha \in \mathbb{C}$ we have

$$(\alpha f_1)\blacklozenge f_2\blacklozenge\cdots\blacklozenge f_m = f_1\blacklozenge(\alpha f_2)\blacklozenge\cdots\blacklozenge f_m = \cdots$$
$$= f_1\blacklozenge\cdots\blacklozenge f_{m-1}\blacklozenge(\alpha f_m) = \alpha\left(f_1\blacklozenge\cdots\blacklozenge f_m\right) \equiv \alpha f_1\blacklozenge\cdots\blacklozenge f_m.$$

Remark 5. *By the generalized and classical Cauchy-Bunyakovsky inequalities and* (10) *for each* $\lambda \in \mathcal{D}_{\mathbb{C}}$ *and each* $q \in \mathbb{Z}_+$ *we have*

$$\left|(\widehat{S}f)(\lambda)\right| \le \sum_{n=0}^{\infty} |f^{(n)}|_{\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}} |\lambda^{\otimes n}|_{\mathcal{H}^{\widehat{\otimes} n}_{-\tau,\mathbb{C}}} = \sum_{n=0}^{\infty}\left(n!2^{qn/2}|f^{(n)}|_{\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}}\right)\left(\left(n!2^{qn/2}\right)^{-1}|\lambda|^n_{\mathcal{H}_{-\tau,\mathbb{C}}}\right)$$
$$\le \sqrt{\sum_{n=0}^{\infty}(n!)^2 2^{qn}\,|f^{(n)}|^2_{\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}}}\sqrt{\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}_{\mathcal{H}_{-\tau,\mathbb{C}}}}{(n!)^2\,2^{qn}}} = \|f\|_{(\mathcal{H}_\tau)_q}\sqrt{\sum_{n=0}^{\infty}\frac{|\lambda|^{2n}_{\mathcal{H}_{-\tau,\mathbb{C}}}}{(n!)^2\,2^{qn}}} < \infty,$$

where $\mathcal{H}^{\widehat{\otimes} n}_{-\tau,\mathbb{C}}$, $n \in \mathbb{N}$, *are the symmetric tensor powers of the complexifications of negative spaces from chain* (7) *(these spaces are dual of* $\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}$ *with respect to* $\mathcal{H}^{\widehat{\otimes} n}_{\mathbb{C}}$*),* $\mathcal{H}^{\widehat{\otimes} 0}_{-\tau,\mathbb{C}} := \mathbb{C}$. *Actually,* $\widehat{S}f$ *is a well-defined complex-valued function on* $\mathcal{D}_{\mathbb{C}}$ *for each* $f \in (\mathcal{H}_\tau)$ *(cf. [26]). Moreover, it can be proved that for* $f_1,\ldots,f_m \in (\mathcal{H}_\tau)$ *the pointwise product of functions* $\widehat{S}f_1,\ldots,\widehat{S}f_m$ *can be decomposed in a pointwise convergent series of form* (16).

Following the classical scheme of studying the Wick multiplication, let us write out a "coordinate formula" for the Wick product on $(\mathcal{H}_\tau)$, i.e. a representation of $f_1\blacklozenge\cdots\blacklozenge f_m$ via kernels from decompositions (9) for $f_1,\ldots,f_m$. Direct calculation by analogy with the Meixner analysis [22] gives the following result.

Proposition 4. *For* $f_1,\ldots,f_m \in (\mathcal{H}_\tau)$ *we have*

$$f_1\blacklozenge\cdots\blacklozenge f_m = \sum_{n=0}^{\infty} :\left\langle \circ^{\otimes n}, \sum_{\substack{k_1,\ldots,k_m\in\mathbb{Z}_+\\ k_1+\cdots+k_m=n}} f_1^{(k_1)}\widehat{\otimes}\cdots\widehat{\otimes} f_m^{(k_m)}\right\rangle:, \tag{17}$$

in particular, for $f,g \in (\mathcal{H}_\tau)$ *we get*

$$f\blacklozenge g = \sum_{n=0}^{\infty} :\left\langle \circ^{\otimes n}, \sum_{k=0}^{n} f^{(k)}\widehat{\otimes} g^{(n-k)}\right\rangle: \tag{18}$$

(cf. (15)*). Here* $f_j^{(k_j)} \in \mathcal{H}^{\widehat{\otimes} k_j}_{\tau,\mathbb{C}}$, $j \in \{1,\ldots,m\}$, *are the kernels from decompositions* (9) *for* f_j*;* $f^{(k)} \in \mathcal{H}^{\widehat{\otimes} k}_{\tau,\mathbb{C}}$ *and* $g^{(n-k)} \in \mathcal{H}^{\widehat{\otimes} n-k}_{\tau,\mathbb{C}}$ *are the kernels from the same decompositions for* f *and* g*, respectively.*

Note that formula (17) can be used as an alternative definition of the Wick product.

Further, it is clear that in order to give an informal sense to a notion "the Wick product" (this is necessary for the construction of a meaningful theory), we have to study a question about convergence of series (17) in the spaces $(\mathcal{H}_\tau)$.

Theorem 2. *Let* $f_1,\ldots,f_m \in (\mathcal{H}_\tau)$. *Then* $f_1\blacklozenge\cdots\blacklozenge f_m \in (\mathcal{H}_\tau)$. *Moreover, the Wick multiplication is continuous in the sense that for any* $q \in \mathbb{Z}_+$ *we have*

$$\|f_1\blacklozenge\cdots\blacklozenge f_m\|_{(\mathcal{H}_\tau)_q} \le \sqrt{\max_{n\in\mathbb{Z}_+}\left[2^{-n}(n+1)^{m-1}\right]}\,\|f_1\|_{(\mathcal{H}_\tau)_{q_1}}\cdots\|f_m\|_{(\mathcal{H}_\tau)_{q_1}}, \tag{19}$$

where $q_1 \ge q + 2\log_2 m + 1$.

Proof. It is clear that it is sufficient to establish estimate (19). One can make this by direct calculation by using (9), (17), (10) and the estimate $\left|f_1^{(k_1)}\widehat{\otimes}\cdots\widehat{\otimes}f_m^{(k_m)}\right|_\tau \le \left|f_1^{(k_1)}\right|_\tau\cdot\cdots\cdot\left|f_m^{(k_m)}\right|_\tau$, which follows from the definition of the symmetric tensor multiplication, by analogy with the proof of Theorem 2.1 in [21]. □

Remark 6. *In the case* $m = 2$ *estimate* (19) *reduces to*

$$\|f_1\blacklozenge f_2\|_{(\mathcal{H}_\tau)_q} \le \|f_1\|_{(\mathcal{H}_\tau)_{q_1}}\|f_2\|_{(\mathcal{H}_\tau)_{q_1}}, \tag{20}$$

$q_1 \ge q+3$. *Using this result and the associativity of the Wick multiplication, one can prove by the mathematical induction method that for* $f_1,\dots,f_m \in (\mathcal{H}_\tau)$ *and any* $q \in \mathbb{Z}_+$ *we get*

$$\|f_1\blacklozenge\cdots\blacklozenge f_m\|_{(\mathcal{H}_\tau)_q} \le \|f_1\|_{(\mathcal{H}_\tau)_{q_1}}\|f_2\|_{(\mathcal{H}_\tau)_{q_2}}\cdots\|f_{m-1}\|_{(\mathcal{H}_\tau)_{q_{m-1}}}\|f_m\|_{(\mathcal{H}_\tau)_{q_{m-1}}},$$

where $q_l \ge q_{l-1}+3$, $l \in \{1,\dots,m-1\}$, $q_0 := q$ *(cf. [25, Remark 7]).*

Finally, we note that by analogy with the regular case [25] (see also [21]) one can introduce so-called *Wick versions of holomorphic functions* on $(\mathcal{H}_\tau)$. Namely, let $f \in (\mathcal{H}_\tau)$ and $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic at $(\widehat{S}f)(0)$ function. We define a Wick version of h as

$$h^\blacklozenge(f) := \widehat{S}^{-1}h(\widehat{S}f).$$

It is not difficult to verify that $h^\blacklozenge$ can be presented in the form

$$h^\blacklozenge(f) = \sum_{n=0}^{\infty} h_n\left(f - (\widehat{S}f)(0)\right)^{\blacklozenge n}, \tag{21}$$

where $f^{\blacklozenge n} := \underbrace{f\blacklozenge\cdots\blacklozenge f}_{n \text{ times}}$, $f^{\blacklozenge 0} := 1$, $h_n \in \mathbb{C}$ are the coefficients from the Taylor decomposition

$$h(u) = \sum_{n=0}^{\infty} h_n\left(u - (\widehat{S}f)(0)\right)^n \tag{22}$$

(cf. [25]). It follows from Theorem 2 and (21), that for a *polynomial* h and a test function $f \in (\mathcal{H}_\tau)$ we have $h^\blacklozenge(f) \in (\mathcal{H}_\tau)$. But, unfortunately, in a general case for $f \in (\mathcal{H}_\tau)$ and a holomorphic at $(\widehat{S}f)(0)$ function $h : \mathbb{C} \to \mathbb{C}$ $h^\blacklozenge(f)$ may not belong to $(\mathcal{H}_\tau)$ (however, one can prove that $h^\blacklozenge(f) \in (\mathcal{H}_\tau)$ if $f \in (\mathcal{H}_\tau)$ is a *polynomial*, i.e. decomposition (9) for f contains only a finite number of nonzero terms, and coefficients h_n from decomposition (22) for h tend to zero quickly enough as $n \to \infty$). Actually, properties of Wick versions of holomorphic functions on the spaces of test functions are quite similar in the nonregular and regular cases. So, the interested reader can find more information in [25] (see also [21]), here we only note that the spaces $(\mathcal{H}_\tau)$ correspond to the space of regular test functions with the parameter $\beta = 1$ (cf. Remark 3), in this case Wick versions of holomorphic functions, unfortunately, have, in a sense, the worst properties.

3 The relationship between Wick multiplication and integration

3.1 Generalized stochastic integral

Denote by $\int \circ(u)\widehat{d}L_u$ the extended stochastic integral on $(L^2)\otimes\mathcal{H}_{\mathbb{C}}$ with respect to a Lévy process L [23]. As it has been stated above (see Remark 4), this integral cannot be naturally restricted to the spaces of nonregular test functions: for $f \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}} \subset \big(L^2\big) \otimes \mathcal{H}_{\mathbb{C}}$ the integral $\int f(u)\widehat{d}L_u$ is not necessary a nonregular test function. One can show that for any $\tau \in T$ and $q \in \mathbb{Z}_+$ such that $q > \log_2 c(\tau)$, where $c(\tau) > 0$ is described in Proposition 1, if $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ then $\int f(u)\widehat{d}L_u \in \big(L^2\big)$; and for q sufficiently large, in particular, if $f \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$, this integral is a *regular* test function. Nevertheless, it is possible to introduce on the spaces of nonregular test functions a linear operator that has properties quite analogous to the properties of the extended stochastic integral. Now we recall the construction of such an operator, which will be called a *generalized stochastic integral*, following [27].

Let $\mathbf{I} : (\mathcal{H}_\tau)_q \to \bigoplus_{n=0}^{\infty} (n!)^2 2^{qn} \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$ be the generalized Wiener-Itô-Sigal isometrical isomorphism, generated by decomposition (9) $\big($see also (10)$\big)$, $\mathbf{1} : \mathcal{H}_{\tau,\mathbb{C}} \to \mathcal{H}_{\tau,\mathbb{C}}$ be the identity operator. For each $f_\cdot^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}$, $n \in \mathbb{Z}_+$, we define a Wick monomial by

$$:\left\langle \circ^{\otimes n}, f_\cdot^{(n)} \right\rangle: \stackrel{def}{=} (\mathbf{I} \otimes \mathbf{1})^{-1} \left(\underbrace{0, \dots, 0}_{n}, f_\cdot^{(n)}, 0, \dots \right) \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}. \tag{23}$$

By analogy with the regular case [14], it is easy to prove that such Wick monomials form orthogonal bases in the spaces $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ in the sense that any $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$ can be uniquely presented as

$$f(\cdot) = \sum_{n=0}^{\infty} :\left\langle \circ^{\otimes n}, f_\cdot^{(n)} \right\rangle:, \quad f_\cdot^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}, \tag{24}$$

where the series converges in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}$, with

$$\|f\|^2_{(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} \left| f_\cdot^{(n)} \right|^2_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}} < \infty. \tag{25}$$

Definition 4. *We define a generalized stochastic integral*

$$\int \circ(u) \widetilde{d}L_u : (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_{\tau,\mathbb{C}} \to (\mathcal{H}_\tau)_q$$

as a linear continuous operator given for $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ by the formula

$$\int f(u) \widetilde{d}L_u := \sum_{n=0}^{\infty} :\left\langle \circ^{\otimes n+1}, \widehat{f}^{(n)} \right\rangle:, \tag{26}$$

where $\widehat{f}^{(n)} := Pr f_\cdot^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+1}$ are the orthoprojections onto $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+1}$, i.e. the symmetrizations by all variables, of the kernels $f_\cdot^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ from decomposition (24) *for f.*

By (26), (10), the obvious estimate $\big|\widehat{f}^{(n)}\big|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+1}} \le \big|f_\cdot^{(n)}\big|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}}$ and (25), we get

$$\begin{aligned} \left\| \int f(u) \widetilde{d}L_u \right\|^2_{(\mathcal{H}_\tau)_q} &= \sum_{n=0}^{\infty} \big((n+1)!\big)^2 2^{q(n+1)} \left| \widehat{f}^{(n)} \right|^2_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+1}} \\ &\le 2^q \sum_{n=0}^{\infty} (n!)^2 2^{(q+1)n} \left[(n+1)^2 2^{-n} \right] \left| f_\cdot^{(n)} \right|^2_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}} \le 9 \cdot 2^{q-2} \|f\|^2_{(\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_{\tau,\mathbb{C}}}, \end{aligned}$$

so this definition is well-posed. It is clear that the restriction of the operator $\int \circ(u) \widetilde{d}L_u$ to the space $(\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$ is a linear continuous operator acting from $(\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$ to $(\mathcal{H}_\tau)$.

3.2 Wick multiplication under the sign of the generalized stochastic integral

As is known, some properties of stochastic integrals are quite unusual. In particular, for $f \in (L^2)$ and $h^{(1)} \in \mathcal{H}_{\mathbb{C}}$ such that $f \otimes h^{(1)}$ is integrable with respect to a Lévy process L in the extended sense we have

$$\int \left(f \otimes h^{(1)}\right)(u)\widehat{d}L_u \equiv \int f \cdot h^{(1)}(u)\widehat{d}L_u \neq f \cdot \int h^{(1)}(u)\widehat{d}L_u,$$

generally speaking, although f does not depend on u. Moreover, in general, the product $f \cdot \int h^{(1)}(u)\widehat{d}L_u$ is undefined. This property of the extended stochastic integral holds true if f is a regular test or generalized function, or a nonregular generalized function. But if one uses the Wick multiplication instead of the pointwise multiplication, it becomes possible to take a time-independent, i.e. independent on u, multiplier out of the sign of the extended stochastic integral, as in the Lebesgue integration theory. The same can be said about the generalized stochastic integral, now we will explain this in detail.

Let us begin with a preparation. We need to introduce a Wick product for elements of $(\mathcal{H}_\tau)$ and $(\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$. Let $n, m \in \mathbb{Z}_+$, $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, $g_\cdot^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}$. We put

$$f^{(n)}\overline{\otimes} g_\cdot^{(m)} := (Pr \otimes \mathbf{1})\left(f^{(n)} \otimes g_\cdot^{(m)}\right) \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{\tau,\mathbb{C}}, \tag{27}$$

where $Pr \otimes \mathbf{1}$ is the orthoprojector acting from $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ to $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ or, which is the same, the operator of symmetrization by $n + m$ variables, except for the variable "$\cdot$" (of course, this operator depends on n and m, but we simplify the nonation). Clearly that

$$|f^{(n)}\overline{\otimes} g_\cdot^{(m)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{\tau,\mathbb{C}}} \leq |f^{(n)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}} |g_\cdot^{(m)}|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau,\mathbb{C}}}, \tag{28}$$

and for $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n}$, $g^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}$ and $h^{(1)} \in \mathcal{H}_{\tau,\mathbb{C}}$ we have

$$f^{(n)}\overline{\otimes}\left(g^{(m)} \otimes h^{(1)}\right) = \left(f^{(n)}\widehat{\otimes} g^{(m)}\right) \otimes h^{(1)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{\tau,\mathbb{C}}. \tag{29}$$

Now we can accept the following definition based on "coordinate formula" (18).

Definition 5. *Let $f \in (\mathcal{H}_\tau)$, $g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$. We define a Wick product $f\overline{\blacklozenge} g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$, setting*

$$(f\overline{\blacklozenge} g)(\cdot) := \sum_{n=0}^{\infty} :\left\langle \circ^{\otimes n}, \sum_{k=0}^{n} f^{(k)}\overline{\otimes} g_\cdot^{(n-k)}\right\rangle:, \tag{30}$$

where $f^{(k)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} k}$ and $g_\cdot^{(n-k)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n-k} \otimes \mathcal{H}_{\tau,\mathbb{C}}$ are the kernels from decompositions (9) *and* (24) *for f and g, respectively.*

Using estimate (28), one can prove by analogy with [21], that this definition is well-posed and the Wick multiplication $\overline{\blacklozenge}$ is continuous in the sense that for all $f \in (\mathcal{H}_\tau)$, $g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$ and $q, q_1 \in \mathbb{Z}_+$ such that $q_1 \geq q + 3$ we have

$$\|f\overline{\blacklozenge} g\|_{(\mathcal{H}_\tau)_q \otimes \mathcal{H}_{\tau,\mathbb{C}}} \leq \|f\|_{(\mathcal{H}_\tau)_{q_1}} \|g\|_{(\mathcal{H}_\tau)_{q_1} \otimes \mathcal{H}_{\tau,\mathbb{C}}}$$

(cf. (20)).

Remark 7. *Let $f, g \in (\mathcal{H}_\tau)$, $h^{(1)} \in \mathcal{H}_{\tau,\mathbb{C}}$. Using* (30), (29) *and* (18), *one can show that*

$$f\overline{\blacklozenge}(g \otimes h^{(1)}) = (f\blacklozenge g) \otimes h^{(1)} \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}. \tag{31}$$

Theorem 3. *Let $f \in (\mathcal{H}_\tau)$ and $g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$. Then*

$$\int (f\overline{\blacklozenge}g)(u)\widetilde{d}L_u = f\blacklozenge \int g(u)\widetilde{d}L_u \in (\mathcal{H}_\tau). \tag{32}$$

Proof. First, we note that the expressions in the left hand side and in the right hand side of (32) belong to $(\mathcal{H}_\tau)$, this follows from the properties of the generalized stochastic integral and of the Wick multiplications $\overline{\blacklozenge}$ and $\blacklozenge$. Now, let us prove *equality* (32). We begin from the special case $f = :\left\langle \circ^{\otimes n}, f^{(n)}\right\rangle:$, $g(\cdot) = :\left\langle \circ^{\otimes m}, g_\cdot^{(m)}\right\rangle:$, $f^{(n)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n}$, $g_\cdot^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}m} \otimes \mathcal{H}_{\tau,\mathbb{C}}$, $n, m \in \mathbb{Z}_+$. By (30), we have $(f\overline{\blacklozenge}g)(\cdot) = :\left\langle \circ^{\otimes n+m}, f^{(n)}\overline{\otimes}g_\cdot^{(m)}\right\rangle:$, hence $\int (f\overline{\blacklozenge}g)(u)\widetilde{d}L_u = :\left\langle \circ^{\otimes n+m+1}, \widehat{f^{(n)}\overline{\otimes}g_\cdot^{(m)}}\right\rangle:$ (see (26)). On the other hand, by (26) we have $\int g(u)\widetilde{d}L_u = :\left\langle \circ^{\otimes m+1}, \widehat{g}^{(m)}\right\rangle:$, therefore by (18) we obtain $f\blacklozenge \int g(u)\widetilde{d}L_u = :\left\langle \circ^{\otimes n+m+1}, f^{(n)}\widehat{\otimes}\widehat{g}^{(m)}\right\rangle:$. So, we have to prove that $\widehat{f^{(n)}\overline{\otimes}g_\cdot^{(m)}} = f^{(n)}\widehat{\otimes}\widehat{g}^{(m)}$ in $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes}n+m+1}$. But this follows from the properties of orthoprojectors (it does not matter in which order to symmetrize functions). Thus, in our special case the statement of the theorem is proved. In the general case, equality (32) follows from the just proved result, continuity of the Wick multiplications $\overline{\blacklozenge}$ and $\blacklozenge$, and continuity of the generalized stochastic integral. □

Remark 8. *One can interpret g as a function on $\mathbb{R}_+$ with values in $(\mathcal{H}_\tau)$ and, taking into account the construction of the Wick multiplications $\blacklozenge$ and $\overline{\blacklozenge}$, rewrite equality* (32) *in a classical form $\int f\blacklozenge g(u)\widetilde{d}L_u = f\blacklozenge \int g(u)\widetilde{d}L_u$.*

3.3 Wick multiplication under the sign of a Pettis integral

As in the regular case [25], let us obtain an analog of property (32) for a Pettis integral (a weak integral) on the spaces of nonregular test functions.

First, we consider the Pettis integral over a set of finite Lebesgue measure ρ.

Definition 6. *For all $\Delta \in \mathcal{B}(\mathbb{R}_+)$ with $\rho(\Delta) < \infty$ and $g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_\mathbb{C}$ we define a Pettis integral $\int_\Delta g(u)du \in (\mathcal{H}_\tau)$ as a unique element of $(\mathcal{H}_\tau)$ such that for each $F \in (\mathcal{H}_{-\tau})$ we have*

$$\left\langle\!\!\left\langle F, \int_\Delta g(u)du \right\rangle\!\!\right\rangle_{(L^2)} = \langle\!\langle F \otimes 1_\Delta, g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_\mathbb{C}}. \tag{33}$$

By the generalized Cauchy-Bunyakovsky inequality, for any $q \in \mathbb{N}_{q_0(\tau)}$ (see Proposition 2) such that $F \in (\mathcal{H}_{-\tau})_{-q}$ we have

$$\left|\langle\!\langle F \otimes 1_\Delta, g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_\mathbb{C}}\right| \le \|F\|_{(\mathcal{H}_{-\tau})_{-q}} \sqrt{\rho(\Delta)} \|g\|_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\mathbb{C}},$$

therefore this definition is well-posed and a Pettis integral

$$\int_\Delta \circ(u)du : (\mathcal{H}_\tau) \otimes \mathcal{H}_\mathbb{C} \to (\mathcal{H}_\tau) \tag{34}$$

is a linear *continuous* operator.

Let us show that

$$\int_\Delta \left(f \otimes h^{(1)}\right)(u)du \equiv \int_\Delta f \cdot h^{(1)}(u)du = f \cdot \int_\Delta h^{(1)}(u)du \tag{35}$$

for arbitrary $f \in (\mathcal{H}_\tau)$ and $h^{(1)} \in \mathcal{H}_\mathbb{C}$.

Indeed, for each $F \in (\mathcal{H}_{-\tau})$ by (33) we have

$$\left\langle\!\!\left\langle F, \int_\Delta f \cdot h^{(1)}(u)du \right\rangle\!\!\right\rangle_{(L^2)} = \left\langle\!\!\left\langle F \otimes 1_\Delta, f \otimes h^{(1)} \right\rangle\!\!\right\rangle_{(L^2)\otimes\mathcal{H}_\mathbb{C}}$$
$$= \langle\!\langle F, f\rangle\!\rangle_{(L^2)} \int_\Delta h^{(1)}(u)du = \left\langle\!\!\left\langle F, f \cdot \int_\Delta h^{(1)}(u)du \right\rangle\!\!\right\rangle_{(L^2)}.$$

Further, let $f \in (\mathcal{H}_\tau)$, $g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_\mathbb{C}$. One can show as in Subsection 3.1 that g can be decomposed in series (24) with kernels $g^{(n)}_{\cdot} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_\mathbb{C}$ and Wick monomials defined by formula (23) with the identity operator $\mathbf{1} : \mathcal{H}_\mathbb{C} \to \mathcal{H}_\mathbb{C}$. Define a Wick product $f\overline{\blacklozenge}g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_\mathbb{C}$ by formula (30) (it is clear that now in the definition of the tensor multiplication $\overline{\otimes}$ (27) an operator $Pr \otimes \mathbf{1}$ is the orthoprojector acting from $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m} \otimes \mathcal{H}_\mathbb{C}$ to $\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_\mathbb{C}$). Since, obviously, this multiplication is a natural extension of multiplication (30) (so we keep for it the designation $\overline{\blacklozenge}$), everything said above about multiplication (30) holds true up to obvious modifications.

Let now $f, g \in (\mathcal{H}_\tau)$ and $h^{(1)} \in \mathcal{H}_\mathbb{C}$. Using equalities (31) and (35) we obtain

$$\int_\Delta \left(f\overline{\blacklozenge}(g \otimes h^{(1)})\right)(u)du = \int_\Delta \left((f\blacklozenge g) \otimes h^{(1)}\right)(u)du \equiv \int_\Delta (f\blacklozenge g) \cdot h^{(1)}(u)du$$
$$= (f\blacklozenge g) \cdot \int_\Delta h^{(1)}(u)du = f\blacklozenge \left(g \cdot \int_\Delta h^{(1)}(u)du\right)$$
$$= f\blacklozenge \int_\Delta g \cdot h^{(1)}(u)du \equiv f\blacklozenge \int_\Delta (g \otimes h^{(1)})(u)du.$$

Hence, by virtue of continuity of the Wick multiplications $\overline{\blacklozenge}$ and $\blacklozenge$, and continuity of Pettis integral (34), we obtain the following statement (cf. Theorem 3).

Theorem 4. *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$ be such that $\rho(\Delta) < \infty$, $f \in (\mathcal{H}_\tau)$ and $g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_\mathbb{C}$. Then*

$$\int_\Delta (f\overline{\blacklozenge}g)(u)du = f\blacklozenge \int_\Delta g(u)du \in (\mathcal{H}_\tau). \tag{36}$$

Note that, as in the case of the generalized stochastic integral, one can interpret g as a function acting from $\mathbb{R}_+$ to $(\mathcal{H}_\tau)$, and rewrite (36) in a classical form $\int_\Delta f\blacklozenge g(u)du = f\blacklozenge \int_\Delta g(u)du$.

Now, we consider a Pettis integral over a set of infinite Lebesgue measure.

Definition 7 (cf. Definition 6)**.** *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$ be such that $\rho(\Delta) = \infty$, and $g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_\mathbb{C}$ satisfy the condition*

$$\forall q \in \mathbb{N}_{q_0(\tau)} \qquad \int_\Delta \|g(u)\|_{(\mathcal{H}_\tau)_q} du < \infty \tag{37}$$

(here we interpret g as a function acting from $\mathbb{R}_+$ to $(\mathcal{H}_\tau)$).

Define a Pettis integral $\int_\Delta g(u)du \in (\mathcal{H}_\tau)$ as a unique element of $(\mathcal{H}_\tau)$ such that for each $F \in (\mathcal{H}_{-\tau})$ equality (33) *is fulfilled.*

The well-posedness of this definition follows from the estimate (see (33))

$$\left|\langle\!\langle F \otimes 1_\Delta, g\rangle\!\rangle_{(L^2)\otimes\mathcal{H}_\mathbb{C}}\right| = \left|\int_\Delta \langle\!\langle F, g(u)\rangle\!\rangle_{(L^2)} du\right| \le \int_\Delta \left|\langle\!\langle F, g(u)\rangle\!\rangle_{(L^2)}\right| du$$
$$\le \|F\|_{(\mathcal{H}_{-\tau})_{-q}} \int_\Delta \|g(u)\|_{(\mathcal{H}_\tau)_q} du$$

(here we used the generalized Cauchy-Bunyakovsky inequality), where $q \in \mathbb{N}_{q_0(\tau)}$ is such that $F \in (\mathcal{H}_{-\tau})_{-q}$.

By approximation the Pettis integral over a set of infinite Lebesgue measure by Pettis integrals over sets of finite Lebesgue measure, quite analogously to the regular case [25] one can prove the following statement (cf. Theorem 4).

Theorem 5. *Let $\Delta \in \mathcal{B}(\mathbb{R}_+)$ be such that $\rho(\Delta) = \infty$, $f \in (\mathcal{H}_\tau)$ and $g \in (\mathcal{H}_\tau) \otimes \mathcal{H}_\mathbb{C}$ satisfy condition* (37). *Then representation* (36) *is fulfilled.*

Remark 9. *Since $\mathcal{H}_{\tau,\mathbb{C}} \subset \mathcal{H}_\mathbb{C}$, then, of course, it is possible to consider the Pettis integral on the spaces $(\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}} \subset (\mathcal{H}_\tau) \otimes \mathcal{H}_\mathbb{C}$. It is clear that all the above statements hold true for this integral.*

3.4 Representation of the generalized stochastic integral via the Pettis integral

As is known, in different versions of the infinite-dimensional white noise analysis an extended stochastic integral can be presented as a Pettis integral from a Wick product of the original integrand by the corresponding white noise. In particular, in the Lévy analysis this representation has the form

$$\int f(u)\widehat{d}L_u = \int f(u)\Diamond \dot{L}_u du, \tag{38}$$

where $\dot{L}$ is a Lévy white noise. Depending on spaces in which integration is considered, equality (38) can be formal (e.g., on the spaces of regular test and generalized functions, see [25] and [13], respectively) or can have a rigorous sense (e.g., on the spaces of nonregular generalized functions, see [29]). In any case this equality is very useful for applications. Note that, in a sense, representation (38) is an analog of a formula for replacement of a measure in the Lebesgue integration theory (in particular, $\dot{L}$ is an analog of a Radon-Nikodym derivative).

Let us obtain a natural analog of representation (38) for the generalized stochastic integral. As noted in Subsection 1.1, a Lévy white noise $\dot{L}$ can be presented as $\dot{L}_u(\circ) = \langle \circ, \delta_u \rangle$, where δ_u is the Dirac delta-function, concentrated at u. Since for each $u \in \mathbb{R}_+$ (and for each $\tau \in T$) $\delta_u \in \mathcal{H}_{-\tau,\mathbb{C}}$, for each $q \in \mathbb{N}_{q_0(\tau)}$ (see Proposition 2) $\langle \circ, \delta_u \rangle = :\left\langle \circ, \delta_u \right\rangle: \in (\mathcal{H}_{-\tau})_{-q}$, i.e. the Lévy white noise can be considered as a function on $\mathbb{R}_+$ with values in a space of nonregular generalized functions (so, as noted above, in the analysis on the spaces of nonregular generalized functions representation (38) has a rigorous sense). At the same time $\delta_u \notin \mathcal{H}_{\tau,\mathbb{C}}$, $\tau \in T$, therefore $\dot{L}_u$ is not a nonregular test function and, as in the regular case, the analog of representation (38) in the analysis on the spaces of nonregular test functions turns out to be a *formal*.

Theorem 6. *For arbitrary $f \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$ the generalized stochastic integral $\int f(u)\widetilde{d}L_u$ can be formally presented as*

$$\int f(u)\widetilde{d}L_u = \int f(u)\blacklozenge \dot{L}_u du \equiv \int f(u)\blacklozenge \langle \circ, \delta_u \rangle \, du \in (\mathcal{H}_\tau), \tag{39}$$

where the integral in the right hand side is a formal Pettis integral over the set $\mathbb{R}_+$.

Remark 10. *It is clear that in the right hand side of* (39) *f is interpreted as a function on $\mathbb{R}_+$ with valuses in $(\mathcal{H}_\tau)$ $\big($a similar remark is true for representation (38)$\big)$.*

Proof. Let $f \in (\mathcal{H}_\tau) \otimes \mathcal{H}_{\tau,\mathbb{C}}$. Using (formally) (18), we obtain

$$\int f(u)\blacklozenge \dot{L}_u du \equiv \int f(u)\blacklozenge \langle \circ, \delta_u \rangle du = \int f(u)\blacklozenge{:}\left\langle \circ, \delta_u \right\rangle{:}du$$
$$= \int \sum_{n=0}^{\infty} {:}\left\langle \circ^{\otimes n+1}, f_u^{(n)} \widehat{\otimes} \delta_u \right\rangle{:}du = \sum_{n=0}^{\infty} {:}\left\langle \circ^{\otimes n+1}, \int f_u^{(n)} \widehat{\otimes} \delta_u du \right\rangle{:}.$$

But

$$f_u^{(n)} \widehat{\otimes} \delta_u = \frac{1}{n+1} \Big(f_u^{(n)}(\cdot_1, \dots, \cdot_n)\delta_u(\cdot) + f_u^{(n)}(\cdot_2, \dots, \cdot_n, \cdot)\delta_u(\cdot_1) + \dots + f_u^{(n)}(\cdot, \cdot_1, \dots, \cdot_{n-1})\delta_u(\cdot_n) \Big),$$

hence

$$\int f_u^{(n)} \widehat{\otimes} \delta_u du = \frac{1}{n+1} \Big(f_{\cdot}^{(n)}(\cdot_1, \dots, \cdot_n) + f_{\cdot_1}^{(n)}(\cdot_2, \dots, \cdot_n, \cdot) + \dots + f_{\cdot_n}^{(n)}(\cdot, \cdot_1, \dots, \cdot_{n-1}) \Big)$$
$$= Prf_{\cdot}^{(n)} = \widehat{f}^{(n)},$$

and therefore $\int f(u)\blacklozenge \dot{L}_u du = \sum_{n=0}^{\infty} {:}\left\langle \circ^{\otimes n+1}, \widehat{f}^{(n)} \right\rangle{:} = \int f(u)\widetilde{d}L_u$ (see Definition 4). $\square$

4 The relationship between Wick multiplication and stochastic differentiation

4.1 Operators of stochastic differentiation

In different versions of the white noise analysis one can introduce and study so-called operators of stochastic differentiation on spaces of test and generalized functions. These operators are closely related with stochastic integrals and derivatives, and can be used, in particular, in order to study properties of stochastic integrals and properties of solutions of certain stochastic equations. In this subsection, we recall the definition of the mentioned operators on the spaces of nonregular test functions, following [28].

Let us start with the necessary preparation. Consider a family of chains

$$\mathcal{D}'^{\widehat{\otimes} n}_{\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes} n}_{-\tau,\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes} n}_{\mathbb{C}} \equiv L^2(\mathbb{R}_+)^{\widehat{\otimes} n}_{\mathbb{C}} \supset \mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}} \supset \mathcal{D}^{\widehat{\otimes} n}_{\mathbb{C}}, \quad n \in \mathbb{Z}_+ \tag{40}$$

(as is known, $\mathcal{H}^{\widehat{\otimes} n}_{-\tau,\mathbb{C}}$ and $\mathcal{D}'^{\widehat{\otimes} n}_{\mathbb{C}} = \operatorname{ind\,lim}_{\tilde{\tau} \in T} \mathcal{H}^{\widehat{\otimes} n}_{-\tilde{\tau},\mathbb{C}}$ are the spaces dual of $\mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}$ and $\mathcal{D}^{\widehat{\otimes} n}_{\mathbb{C}}$ with respect to $\mathcal{H}^{\widehat{\otimes} n}_{\mathbb{C}}$; in the case $n = 0$ all spaces from chain (40) are equal to $\mathbb{C}$). Since the spaces of test functions in chains (40) and (12) coincide, there exists a family of natural isomorphisms

$$U_n : \mathcal{D}'^{(n)}_{\mathbb{C}} \to \mathcal{D}'^{\widehat{\otimes} n}_{\mathbb{C}}$$

such that for all $F^{(n)}_{ext} \in \mathcal{D}'^{(n)}_{\mathbb{C}}$ and $f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}_{\mathbb{C}}$ we have

$$\left\langle F^{(n)}_{ext}, f^{(n)} \right\rangle_{ext} = \left\langle U_n F^{(n)}_{ext}, f^{(n)} \right\rangle. \tag{41}$$

It is easy to verify that the restrictions of U_n to $\mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ are isometric isomorphisms between the spaces $\mathcal{H}^{(n)}_{-\tau,\mathbb{C}}$ and $\mathcal{H}^{\widehat{\otimes} n}_{-\tau,\mathbb{C}}$.

Remark 11. *Remind that $\mathcal{H}^{(1)}_{ext} = \mathcal{H}_{\mathbb{C}}$, therefore in the case $n = 1$ chains* (40) *and* (12) *coincide. Hence U_1 is the identity operator on $\mathcal{D}'^{(1)}_{\mathbb{C}} = \mathcal{D}'_{\mathbb{C}}$. In the case $n = 0$, U_0 is, obviously, the identity operator on $\mathbb{C}$.*

Now we can define a natural analog of the symmetric tensor multiplication on the negative spaces of chains (12). Let $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $G_{ext}^{(m)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m)}$, $n, m \in \mathbb{Z}_+$. We put

$$F_{ext}^{(n)} \diamond G_{ext}^{(m)} := U_{n+m}^{-1}\Big[\left(U_n F_{ext}^{(n)}\right)\widehat{\otimes}\left(U_m G_{ext}^{(m)}\right)\Big] \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)} \tag{42}$$

$\big($recall (15)$\big)$. It is easy to verify that a multiplication $\diamond$ is commutative, associative and distributive; for any $\alpha \in \mathbb{C}$ we have

$$\left(\alpha F_{ext}^{(n)}\right)\diamond G_{ext}^{(m)} = F_{ext}^{(n)} \diamond \left(\alpha G_{ext}^{(m)}\right) = \alpha\left(F_{ext}^{(n)} \diamond G_{ext}^{(m)}\right) \equiv \alpha F_{ext}^{(n)} \diamond G_{ext}^{(m)},$$

and

$$\left|F_{ext}^{(n)}) \diamond G_{ext}^{(m)}\right|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n+m)}} \le \left|F_{ext}^{(n)})\right|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}}\left|G_{ext}^{(m)}\right|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}}, \tag{43}$$

see [28] for more details.

Let now $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$, $f^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}$, $n \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $m > n$. We define a generalized partial pairing $\left\langle F_{ext}^{(n)}, f^{(m)}\right\rangle_{ext} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m-n}$ by setting

$$\left\langle G_{ext}^{(m-n)}, \left\langle F_{ext}^{(n)}, f^{(m)}\right\rangle_{ext}\right\rangle_{ext} = \left\langle F_{ext}^{(n)} \diamond G_{ext}^{(m-n)}, f^{(m)}\right\rangle_{ext} \tag{44}$$

for any $G_{ext}^{(m-n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(m-n)}$. By the generalized Cauchy-Bunyakovsky inequality and (43), we get

$$\begin{aligned}\left|\left\langle F_{ext}^{(n)} \diamond G_{ext}^{(m-n)}, f^{(m)}\right\rangle_{ext}\right| &\le \left|F_{ext}^{(n)} \diamond G_{ext}^{(m-n)}\right|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}}\left|f^{(m)}\right|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}} \\ &\le \left|F_{ext}^{(n)}\right|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}}\left|G_{ext}^{(m-n)}\right|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m-n)}}\left|f^{(m)}\right|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}},\end{aligned}$$

which implies that this definition is well-posed and

$$\left|\left\langle F_{ext}^{(n)}, f^{(m)}\right\rangle_{ext}\right|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m-n}} \le \left|F_{ext}^{(n)}\right|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(n)}}\left|f^{(m)}\right|_{\mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}}. \tag{45}$$

Remark 12. *We recall that in the case $m = n \in \mathbb{Z}_+$ the pairings $\left\langle F_{ext}^{(n)}, f^{(n)}\right\rangle_{ext}$ are defined in Subsection 1.3 as the real dual pairings between elements of negative and positive spaces from chains* (12)*, now estimate* (45) *holds true because it is nothing but the generalized Cauchy-Bunyakovsky inequality.*

Definition 8. *Let $n \in \mathbb{N}$ and $F_{ext}^{(n)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(n)}$. Let us define a linear continuous operator $(D^n\circ)\left(F_{ext}^{(n)}\right) : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q$ by setting*

$$(D^n f)\left(F_{ext}^{(n)}\right) := \sum_{m=n}^{\infty}\frac{m!}{(m-n)!}{:}\left\langle \circ^{\otimes m-n}, \left\langle F_{ext}^{(n)}, f^{(m)}\right\rangle_{ext}\right\rangle{:}, \quad f \in (\mathcal{H}_\tau)_q, \tag{46}$$

where $f^{(m)} \in \mathcal{H}_{\tau,\mathbb{C}}^{\widehat{\otimes} m}$ are the kernels from decomposition (9) *for f.*

Using estimate (45) one can show that this definition is well-posed. Clearly that the restriction of the operator $(D^n\circ)\left(F_{ext}^{(n)}\right)$ to the space $(\mathcal{H}_\tau)$ is a linear continuous operator on $(\mathcal{H}_\tau)$. The interested reader can find more information about operators of stochastic differentiation on the spaces of nonregular test functions in [28].

4.2 The Leibnitz rule

One of the important properties of the operators of stochastic differentiation in the Lévy analysis (as well as in other versions of the white noise analysis) is that the operator of stochastic differentiation of first order is a differentiation, i.e. satisfies the Leibnitz rule, with respect to the Wick multiplication. For the above operator on the spaces of *regular test and generalized* functions this fact is established in [25] and [12], respectively, on the spaces of *nonregular generalized* functions — in [26]. Now our goal is to obtain the corresponding result on the spaces of *nonregular test* functions $(\mathcal{H}_\tau)$.

Let $F^{(1)}_{ext} \equiv F^{(1)} \in \mathcal{H}^{(1)}_{-\tau,\mathbb{C}} = \mathcal{H}_{-\tau,\mathbb{C}}$. Denote $(D\circ)\left(F^{(1)}\right) := (D^1\circ)\left(F^{(1)}\right)$. For arbitrary $f \in (\mathcal{H}_\tau)$ according to (46) we obtain

$$(Df)\left(F^{(1)}\right) = \sum_{m=1}^{\infty} m{:}\left\langle \circ^{\otimes m-1}, \left\langle F^{(1)}, f^{(m)}\right\rangle_{ext}\right\rangle{:}, \tag{47}$$

where $f^{(m)} \in \mathcal{H}^{\widehat{\otimes} m}_{\tau,\mathbb{C}}$ are the kernels from decomposition (9) for f.

Theorem 7. *The operator of stochastic differentiation of first order on $(\mathcal{H}_\tau)$ is a differentiation, i.e. satisfies the Leibnitz rule, with respect to the Wick multiplication, that is for all $f, g \in (\mathcal{H}_\tau)$ and $F^{(1)} \in \mathcal{H}_{-\tau,\mathbb{C}}$ we have*

$$\left(D\left(f\blacklozenge g\right)\right)\left(F^{(1)}\right) = (Df)\left(F^{(1)}\right)\blacklozenge g + f\blacklozenge(Dg)\left(F^{(1)}\right) \in (\mathcal{H}_\tau). \tag{48}$$

Remark 13. *Since, in contrast to the regular case, the constructions of the Wick product (as well as of the operators of stochastic differentiation) on the spaces $(\mathcal{H}_{-\tau})$ and $(\mathcal{H}_\tau)$ are different, we cannot use the corresponding result from [26] and have to prove* (48) *directly.*

Proof. First we note that the expressions in the left hand side and in the right hand side of (48) belong to $(\mathcal{H}_\tau)$, this follows from the properties of the operator D and Theorem 2. Now, let us prove *equality* (48). Note, that if $f = {:}\left\langle \circ^{\otimes 0}, f^{(0)}\right\rangle{:} = f^{(0)} \in \mathbb{C}$ or $g = {:}\left\langle \circ^{\otimes 0}, g^{(0)}\right\rangle{:} = g^{(0)} \in \mathbb{C}$, then equality (48) trivially holds. We begin from the special case $f = {:}\left\langle \circ^{\otimes n}, f^{(n)}\right\rangle{:}$, $g = {:}\left\langle \circ^{\otimes m}, g^{(m)}\right\rangle{:}$, $f^{(n)} \in \mathcal{H}^{\widehat{\otimes} n}_{\tau,\mathbb{C}}$, $g^{(m)} \in \mathcal{H}^{\widehat{\otimes} m}_{\tau,\mathbb{C}}$, $n, m \in \mathbb{N}$. Let $F^{(1)}$ be an arbitrary element of $\mathcal{H}_{-\tau,\mathbb{C}}$. By (18) and (47) we obtain

$$f\blacklozenge g = {:}\left\langle \circ^{\otimes n+m}, f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle{:}, \quad \left(D\left(f\blacklozenge g\right)\right)\left(F^{(1)}\right) = (n+m){:}\left\langle \circ^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle_{ext}\right\rangle{:},$$

$$(Df)\left(F^{(1)}\right) = n{:}\left\langle \circ^{\otimes n-1}, \left\langle F^{(1)}, f^{(n)}\right\rangle_{ext}\right\rangle{:},\ (Df)\left(F^{(1)}\right)\blacklozenge g = n{:}\left\langle \circ^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)}\right\rangle_{ext}\widehat{\otimes} g^{(m)}\right\rangle{:},$$

$$(Dg)\left(F^{(1)}\right) = m{:}\left\langle \circ^{\otimes m-1}, \left\langle F^{(1)}, g^{(m)}\right\rangle_{ext}\right\rangle{:}, f\blacklozenge(Dg)\left(F^{(1)}\right) = m{:}\left\langle \circ^{\otimes n+m-1}, f^{(n)}\widehat{\otimes}\left\langle F^{(1)}, g^{(m)}\right\rangle_{ext}\right\rangle{:}.$$

So, we have to prove that

$$(n+m)\left\langle F^{(1)}, f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle_{ext} = n\left\langle F^{(1)}, f^{(n)}\right\rangle_{ext}\widehat{\otimes} g^{(m)} + m f^{(n)}\widehat{\otimes}\left\langle F^{(1)}, g^{(m)}\right\rangle_{ext}. \tag{49}$$

Since the set $\left\{G^{\otimes n+m-1} : G \in \mathcal{H}_{-\tau,\mathbb{C}}\right\}$ is total in the space $\mathcal{H}^{\widehat{\otimes} n+m-1}_{-\tau,\mathbb{C}}$, in order to prove (49) it is sufficient to prove that for each $G \in \mathcal{H}_{-\tau,\mathbb{C}}$ we have

$$\begin{aligned}(n+m)\left\langle G^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle_{ext}\right\rangle &= n\left\langle G^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)}\right\rangle_{ext}\widehat{\otimes} g^{(m)}\right\rangle \\ &+ m\left\langle G^{\otimes n+m-1}, f^{(n)}\widehat{\otimes}\left\langle F^{(1)}, g^{(m)}\right\rangle_{ext}\right\rangle.\end{aligned} \tag{50}$$

Using (41), (42), (44) and considering that $U_1 = \mathbf{1}$ (see Remark 11), we obtain

$$\begin{aligned}
(n+m)\left\langle G^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle_{ext}\right\rangle &= (n+m)\left\langle U_{n+m-1}^{-1}G^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle_{ext}\right\rangle_{ext} \\
&= (n+m)\left\langle F^{(1)} \diamond \left(U_{n+m-1}^{-1}G^{\otimes n+m-1}\right), f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle_{ext} \\
&= (n+m)\left\langle U_{n+m}^{-1}\left[F^{(1)}\widehat{\otimes} G^{\otimes n+m-1}\right], f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle_{ext} \\
&= (n+m)\left\langle F^{(1)}\widehat{\otimes} G^{\otimes n+m-1}, f^{(n)}\widehat{\otimes} g^{(m)}\right\rangle,
\end{aligned} \tag{51}$$

$$\begin{aligned}
& n\left\langle G^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)}\right\rangle_{ext}\widehat{\otimes} g^{(m)}\right\rangle \\
&= n\left\langle G^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)}\right\rangle_{ext}\otimes g^{(m)}\right\rangle = n\left\langle G^{\otimes n-1}, \left\langle F^{(1)}, f^{(n)}\right\rangle_{ext}\right\rangle\left\langle G^{\otimes m}, g^{(m)}\right\rangle \\
&= n\left\langle U_{n-1}^{-1}G^{\otimes n-1}, \left\langle F^{(1)}, f^{(n)}\right\rangle_{ext}\right\rangle_{ext}\left\langle G^{\otimes m}, g^{(m)}\right\rangle \\
&= n\left\langle F^{(1)} \diamond \left(U_{n-1}^{-1}G^{\otimes n-1}\right), f^{(n)}\right\rangle_{ext}\left\langle G^{\otimes m}, g^{(m)}\right\rangle \\
&= n\left\langle U_n^{-1}\left[F^{(1)}\widehat{\otimes} G^{\otimes n-1}\right], f^{(n)}\right\rangle_{ext}\left\langle G^{\otimes m}, g^{(m)}\right\rangle = n\left\langle F^{(1)}\widehat{\otimes} G^{\otimes n-1}, f^{(n)}\right\rangle\left\langle G^{\otimes m}, g^{(m)}\right\rangle \\
&= \Big\langle F^{(1)}(\cdot_1)G(\cdot_2)\cdots G(\cdot_n) + F^{(1)}(\cdot_2)G(\cdot_3)\cdots G(\cdot_n)G(\cdot_1) \\
&\quad + \cdots + F^{(1)}(\cdot_n)G(\cdot_1)\cdots G(\cdot_{n-1}), f^{(n)}(\cdot_1,\ldots,\cdot_n)\Big\rangle\left\langle G^{\otimes m}, g^{(m)}\right\rangle \\
&= \Big\langle F^{(1)}(\cdot_1)G(\cdot_2)\cdots G(\cdot_n)G(\cdot_{n+1})\cdots G(\cdot_{n+m}) + F^{(1)}(\cdot_2)G(\cdot_3)\cdots G(\cdot_n)G(\cdot_1)G(\cdot_{n+1})\cdots G(\cdot_{n+m}) \\
&\quad + \cdots + F^{(1)}(\cdot_n)G(\cdot_1)\cdots G(\cdot_{n-1})G(\cdot_{n+1})\cdots G(\cdot_{n+m}), f^{(n)}(\cdot_1,\ldots,\cdot_n)g^{(m)}(\cdot_{n+1},\ldots,\cdot_{n+m})\Big\rangle,
\end{aligned}$$

$$\begin{aligned}
& m\left\langle G^{\otimes n+m-1}, f^{(n)}\widehat{\otimes}\left\langle F^{(1)}, g^{(m)}\right\rangle_{ext}\right\rangle = m\left\langle G^{\otimes n+m-1}, f^{(n)}\otimes\left\langle F^{(1)}, g^{(m)}\right\rangle_{ext}\right\rangle \\
&= m\left\langle G^{\otimes n}, f^{(n)}\right\rangle\left\langle G^{\otimes m-1}, \left\langle F^{(1)}, g^{(m)}\right\rangle_{ext}\right\rangle \\
&= m\left\langle G^{\otimes n}, f^{(n)}\right\rangle\left\langle U_{m-1}^{-1}G^{\otimes m-1}, \left\langle F^{(1)}, g^{(m)}\right\rangle_{ext}\right\rangle_{ext} \\
&= m\left\langle G^{\otimes n}, f^{(n)}\right\rangle\left\langle F^{(1)} \diamond \left(U_{m-1}^{-1}G^{\otimes m-1}\right), g^{(m)}\right\rangle_{ext} \\
&= m\left\langle G^{\otimes n}, f^{(n)}\right\rangle\left\langle U_m^{-1}\left[F^{(1)}\widehat{\otimes} G^{\otimes m-1}\right], g^{(m)}\right\rangle_{ext} = m\left\langle G^{\otimes n}, f^{(n)}\right\rangle\left\langle F^{(1)}\widehat{\otimes} G^{\otimes m-1}, g^{(m)}\right\rangle \\
&= \left\langle G^{\otimes n}, f^{(n)}\right\rangle\Big\langle F^{(1)}(\cdot_{n+1})G(\cdot_{n+2})\cdots G(\cdot_{n+m}) + F^{(1)}(\cdot_{n+2})G(\cdot_{n+3})\cdots G(\cdot_{n+m})G(\cdot_{n+1}) \\
&\quad + \cdots + F^{(1)}(\cdot_{n+m})G(\cdot_{n+1})\cdots G(\cdot_{n+m-1}), g^{(m)}(\cdot_{n+1},\ldots,\cdot_{n+m})\Big\rangle \\
&= \Big\langle G(\cdot_1)\cdots G(\cdot_n)F^{(1)}(\cdot_{n+1})G(\cdot_{n+2})\cdots G(\cdot_{n+m}) \\
&\quad + G(\cdot_1)\cdots G(\cdot_n)F^{(1)}(\cdot_{n+2})G(\cdot_{n+3})\cdots G(\cdot_{n+m})G(\cdot_{n+1}) \\
&\quad + \cdots + G(\cdot_1)\cdots G(\cdot_n)F^{(1)}(\cdot_{n+m})G(\cdot_{n+1})\cdots G(\cdot_{n+m-1}), f^{(n)}(\cdot_1,\ldots,\cdot_n)g^{(m)}(\cdot_{n+1},\ldots,\cdot_{n+m})\Big\rangle,
\end{aligned}$$

therefore

$$n\left\langle G^{\otimes n+m-1}, \left\langle F^{(1)}, f^{(n)} \right\rangle_{ext} \widehat{\otimes} g^{(m)} \right\rangle + m\left\langle G^{\otimes n+m-1}, f^{(n)} \widehat{\otimes} \left\langle F^{(1)}, g^{(m)} \right\rangle_{ext} \right\rangle$$
$$= \Big\langle F^{(1)}(\cdot_1)\, G(\cdot_2) \cdot \dots \cdot G(\cdot_{n+m}) + F^{(1)}(\cdot_2)\, G(\cdot_3) \cdot \dots \cdot G(\cdot_{n+m})\, G(\cdot_1)$$
$$+ \dots + F^{(1)}(\cdot_{n+m})\, G(\cdot_1) \cdot \dots \cdot G(\cdot_{n+m-1}),\, f^{(n)}(\cdot_1, \dots, \cdot_n)\, g^{(m)}(\cdot_{n+1}, \dots, \cdot_{n+m}) \Big\rangle$$
$$= (n+m) \left\langle F^{(1)} \widehat{\otimes} G^{\otimes n+m-1}, f^{(n)} \otimes g^{(m)} \right\rangle = (n+m) \left\langle F^{(1)} \widehat{\otimes} G^{\otimes n+m-1}, f^{(n)} \widehat{\otimes} g^{(m)} \right\rangle,$$

whence, by virtue of (51), equality (50) follows. Thus, in our special case equality (48) is proved. In the general case, (48) follows from this result and from linearity and continuity of the operator of stochastic differentiation and of the Wick multiplication on $(\mathcal{H}_\tau)$. □

Remark 14. *Let $f \in (\mathcal{H}_\tau)$ and $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic at $(\widehat{S}f)(0)$ function such that its Wick version $h^{\blacklozenge}(f)$ belongs to $(\mathcal{H}_\tau)$ (recall Section 2). It can be proved that in this case $h'^{\blacklozenge}(f) \in (\mathcal{H}_\tau)$ and the following corollary of Theorem 7 is true: for any $F^{(1)} \in \mathcal{H}_{-\tau,\mathbb{C}}$ we have*

$$\left(Dh^{\blacklozenge}(f)\right)\left(F^{(1)}\right) = h'^{\blacklozenge}(f) \blacklozenge (Df)\left(F^{(1)}\right) \in (\mathcal{H}_\tau),$$

here and above $h'^{\blacklozenge}(f)$ is the Wick version of the usual derivative of a function h.

Finally, we note that all results of this paper hold true (up to obvious modifications) if we consider the space $(\mathcal{D})$ instead of $(\mathcal{H}_\tau)$ (recall chain (11)).

References

[1] Benth F.E., Di Nunno G., Lokka A., Øksendal B., Proske F. *Explicit representation of the minimal variance portfolio in markets driven by Lévy processes*. Math. Finance 2003, **13** (1), 55–72. doi:10.1111/1467-9965.t01-1-00005

[2] Berezansky Yu.M., Samoilenko Yu.S. *Nuclear spaces of functions of infinitely many variables*. Ukrainian Math. J. 1973, **25** (6), 599–609. doi:10.1007/BF01090792

[3] Berezansky Yu.M., Sheftel Z.G., Us G.F. Functional Analysis, Vol. 2. Birkhäuser Verlag, Basel-Boston-Berlin, 1996.

[4] Bertoin J. Lévy Processes. Cambridge University Press, Cambridge, 1996.

[5] Bożejko M., Lytvynov E.W., Rodionova I.V. *An extended anyon Fock space and noncommutative Meixner-type orthogonal polynomials in infinite dimensions*. Russian Math. Surveys 2015, **70** (5), 857–899.

[6] Dermoune A. *Distributions sur l'espace de P. Lévy et calcul stochastic*. Ann. Inst. Henri Poincaré Anal. Probab. Stat. 1990, **26** (1), 101–119.

[7] Di Nunno G., Øksendal B., Proske F. Malliavin Calculus for Lévy Processes with Applications to Finance. In: Berestycki N., Casacuberta C., Greenlees J., MacIntyre A., Sabbah C., Süli E. (Eds.) Universitext. Springer-Verlag, Berlin, 2009.

[8] Di Nunno G., Øksendal B., Proske F. *White noise analysis for Lévy processes*. J. Funct. Anal. 2004, **206** (1), 109–148.

[9] Dyriv M.M., Kachanovsky N.A. *On operators of stochastic differentiation on spaces of regular test and generalized functions of Lévy white noise analysis*. Carpathian Math. Publ. 2014, **6** (2), 212–229. doi:10.15330/cmp.6.2.212-229

[10] Dyriv M.M., Kachanovsky N.A. *Operators of stochastic differentiation on spaces of regular test and generalized functions in the Lévy white noise analysis*. Research Bulletin of National Technical University of Ukraine "Kyiv Polytechnic Institute" 2014, (4), 36–40.

[11] Dyriv M.M., Kachanovsky N.A. *Stochastic integrals with respect to a Levy process and stochastic derivatives on spaces of regular test and generalized functions*. Research Bulletin of National Technical University of Ukraine "Kyiv Polytechnic Institute" 2013, (4), 27–30.

[12] Frei M.M. *Wick calculus on spaces of regular generalized functions of Lévy white noise analysis*. Carpathian Math. Publ. 2018, **10** (1), 82–104. doi:10.15330/cmp.10.1.82-104

[13] Frei M.M., Kachanovsky N.A. *On the relationship between Wick calculus and stochastic integration in the Lévy white noise analysis*. Eur. J. Math. 2020, **6** (1), 179–196. doi:10.1007/s40879-019-00317-8

[14] Frei M.M., Kachanovsky N.A. *Some remarks on operators of stochastic differentiation in the Lévy white noise analysis*. Methods Funct. Anal. Topology 2017, **23** (4), 320–345.

[15] Gelfand I.M., Vilenkin N.Ya. Generalized Functions, Vol. IV: Applications of Harmonic Analysis. Academic Press, New York, London, 1964.

[16] Gihman I.I., Skorohod A.V. Theory of Random Processes, Vol. 2. Nauka, Moscow, 1973.

[17] Hida T. Analysis of Brownian Functionals. In: Mathematical Programming Studies, 5. North-Holland Publishing Company, 1976.

[18] Holden H., Øksendal B., Uboe J., Zhang T.-S. Stochastic Partial Differential Equations: A Modeling, White Noise Functional Approach. In: Berestycki N., Casacuberta C., Greenlees J., MacIntyre A., Sabbah C., Süli E. (Eds.) Universitext. Birkhäuser, Boston, 1996.

[19] Itô K. *Spectral type of the shift transformation of differential processes with stationary increments*. Trans. Amer. Math. Soc. 1956, **81** (2), 253–263. doi:10.2307/1992916

[20] Kachanovsky N.A. *Extended stochastic integrals with respect to a Lévy process on spaces of generalized functions*. Mathematical Bulletin of Taras Shevchenko Scientific Society 2013, **10**, 169–188.

[21] Kachanovsky N.A. *Notes on Wick calculus on parametrized spaces of test functions of Meixner white noise*. Methods Funct. Anal. Topology 2011, **17** (2), 150–167.

[22] Kachanovsky N.A. *On an extended stochastic integral and the Wick calculus on the connected with the generalized Meixner measure Kondratiev-type spaces*. Methods Funct. Anal. Topology 2007, **13** (4), 338–379.

[23] Kachanovsky N.A. *On extended stochastic integrals with respect to Lévy processes*. Carpathian Math. Publ. 2013, **5** (2), 256–278. doi:10.15330/cmp.5.2.256-278

[24] Kachanovsky N.A. *On stochastic integration, differentiation and Wick calculus in the Lévy white noise analysis*. The collection of works of Institute of Mathematics, NASU 2021, **18** (1), 456–507 (in Ukrainian).

[25] Kachanovsky N.A. *On Wick calculus and its relationship with stochastic integration on spaces of regular test functions in the Lévy white noise analysis*. Carpathian Math. Publ. 2022, **14** (1), 194–212. doi:10.15330/cmp.14.1.194-212

[26] Kachanovsky N.A. *On Wick calculus on spaces of nonregular generalized functions of Lévy white noise analysis*. Carpathian Math. Publ. 2018, **10** (1), 114–132. doi:10.15330/cmp.10.1.114-132

[27] Kachanovsky N.A. *Operators of stochastic differentiation on spaces of nonregular generalized functions of Lévy white noise analysis*. Carpathian Math. Publ. 2016, **8** (1), 83–106. doi:10.15330/cmp.8.1.83-106

[28] Kachanovsky N.A. *Operators of stochastic differentiation on spaces of nonregular test functions of Lévy white noise analysis*. Methods Funct. Anal. Topology 2015, **21** (4), 336–360.

[29] Kachanovsky N.A., Kachanovska T.O. *Interconnection between Wick multiplication and integration on spaces of nonregular generalized functions in the Lévy white noise analysis*. Carpathian Math. Publ. 2019, **11** (1), 70–88. doi:10.15330/cmp.11.1.70-88

[30] Kondratiev Yu.G. Generalized Functions in Problems of Infinite-dimensional Analysis. PhD Thesis, Kyiv, 1978. (Russian)

[31] Koshmanenko V.D., Samoilenko Yu.S. *Isomorphism of Fock space with a space of functions of infinitely many variables*. Ukrainian Math. J. 1975, **27** (5), 552–555.

[32] Lytvynov E.W. *Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 2003, **6** (1), 73–102.

[33] Meyer P.A. Quantum Probability for Probabilists. In: Morel J.-M., Teissier B. (Eds.) Lecture Notes in Mathematics, 1538. Springer-Verlag, Berlin, 1993.

[34] Nualart D., Schoutens W. *Chaotic and predictable representations for Lévy processes*. Stochastic Process. Appl. 2000, **90** (1), 109–122. doi:10.1016/S0304-4149(00)00035-1

[35] Rodionova I.V. *Analysis connected with generating functions of exponential type in one and infinite dimensions*. Methods Funct. Anal. Topology 2005, **11** (3), 275–297.

[36] Schoutens W. Stochastic Processes and Orthogonal Polynomials. In: Diggle P., Zeger S., Gather U., Bühlmann P. (Eds.) Lecture Notes in Statistics, 146. Springer-Verlag, New York, 2000.

[37] Skorohod A.V. Integration in Hilbert Space. In: Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge, 79. Springer-Verlag, Berlin-New York-Heidelberg, 1974.

[38] Solé J.L., Utzet F., Vives J. Chaos expansions and Malliavin calculus for Lévy processes. In: Benth F.E., Di Nunno G., Lindstrøm T., Øksendal B., Zhang T. (Eds.) Stochastic Analysis and Applications, Abel Symposia, 2. Springer, Berlin, 2007, 595–612. doi:10.1007/978-3-540-70847-6_27

[39] Surgailis D. *On L^2 and non-L^2 multiple stochastic integration*. In: Allgöwer F., Deshpande R.D. (Eds.) Lecture Notes in Control and Information Sciences, 36. Springer-Verlag., Berlin-Heidelberg, 1981.

[40] Vershik A.M., Tsilevich N.V. *Fock factorizations and decompositions of the L^2 spaces over general Lévy processes*. Russian Math. Surveys 2003, **58** (3), 427–472.

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Качановський М.О. *Віківське множення та його зв'язок з інтегруванням та стохастичним диференціюванням на просторах нерегулярних основних функцій в аналізі білого шуму Леві* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 61–83.

Ми працюємо з просторами нерегулярних основних функцій в аналізі білого шуму Леві, побудованими з використанням узагальнення властивості хаотичного розкладу, запропонованого Є.В. Литвиновим. Нашою метою є вивчення властивостей природного множення — віківського множення на цих просторах, а також опис взаємозв'язку цього множення з інтегруванням та стохастичним диференціюванням. Більш точно, ми встановлюємо, що віківський добуток нерегулярних основних функцій є нерегулярною основною функцією; показуємо, що, використовуючи віківське множення, можна виносити незалежний від часу множник з-під знаку узагальненого стохастичного інтеграла; встановлюємо аналог цього результату для інтеграла Петтіса (слабкого інтеграла); отримуємо представлення узагальненого стохастичного інтеграла через формальний інтеграл Петтіса від віківського добутку вихідної підінтегральної функції на білий шум Леві; та доводимо, що оператор стохастичного диференціювання першого порядку на просторах нерегулярних основних функцій задовольняє правило Лейбніца відносно віківського множення.

Ключові слова і фрази: процес Леві, віківський добуток, стохастичний інтеграл, інтеграл Петтіса, оператор стохастичного диференціювання.