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# A uniqueness theorem for Sturm-Liouville equations with a spectral parameter nonlinearly contained in the boundary condition

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In this work, we consider the inverse scattering problem for Sturm-Liouville equation on the semi-infinite interval with a nonlinear spectral parameter in the boundary condition. The scattering data of the problem is defined and the properties of the scattering data are investigated. The fundamental equation is obtained and uniqueness of the algorithm to the potential with given scattering data is studied.

*Key words and phrases:* inverse scattering problem, normalization polynomial, scattering data, fundamental equation, uniqueness theorem.

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## 1 Introduction

On the half line $[0,\infty)$, we consider the differential equation

$$-z'' + p(x)z = \lambda^2 z \tag{1}$$

and the boundary condition

$$\left(\alpha_0 + \alpha_2\lambda^2\right) z'(0) - \left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right) z(0) = 0. \tag{2}$$

Here $p(x)$ is a real valued function satisfying the condition

$$\int_0^\infty (1+x)\left|p(x)\right| dx < \infty, \tag{3}$$

$\lambda$ is a complex parameter, $\alpha_j$, $\beta_j$, $j = 0,1,2$, are real numbers and $\alpha_0$, $\alpha_2$, $\beta_0 > 0$.

Boundary value problem with spectral parameter dependence in the boundary condition may come across in problems as well as in applications. The physical applications of these types of boundary value problems on the half line $[0,\infty)$ are given in [1,12,13,15].

In this paper, we study the *inverse scattering problem* for the equation (1) with spectral parameter contained in the boundary condition. We note that direct and inverse scattering problems for the boundary value problem in the case $\alpha_2 = \beta_1 = \beta_2 = 0$ are completely solved in [3,4,11]. Also, when spectral parameter contained in the boundary condition was investigated in [2,6–10,16].

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Note that, as different from the previous work (different from the self-adjoint case) the zeros of the Jost function do not lie on the imaginary axis, lie on the complex plane and these zeros are not simple or the boundary value problem (1)–(3) may have complex eigenvalues.

Depending on the coefficients in the boundary condition, selfadjoint and nonselfadjoint cases are encountered. Therefore, when solving the problem, it is necessary to use different methods.

It is well known from [11] if the condition (3) is satisfied, then (for all $\operatorname{Im}\lambda \geqslant 0$ ) exists a unique solution $f(x,\lambda)$ regular (with respect to $\lambda$) in half plane $\operatorname{Im}\lambda > 0$, continuous on the real line and can be expressed as

$$f(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt. \tag{4}$$

The kernel $K(x,t)$ satisfies the inequality

$$|K(x,t)| \leqslant \frac{1}{2}\sigma\left(\frac{x+t}{2}\right)\exp\left\{\sigma_1(x) - \sigma_1\left(\frac{x+t}{2}\right)\right\}$$

and satisfies the equality

$$K(x,x) = \frac{1}{2}\int_x^\infty p(t)dt, \tag{5}$$

where

$$\sigma(x) \equiv \int_x^\infty |p(t)|dt, \quad \sigma_1(x) \equiv \int_x^\infty \sigma(t)dt.$$

Moreover, the function $f(x,\lambda)$ has the following properties in $\operatorname{Im}\lambda \geqslant 0$ (see [11])

$$|f(x,\lambda)| \leqslant \exp\{-\operatorname{Im}\lambda x + \sigma_1(x)\}, \tag{6}$$

$$\left|f(x,\lambda) - e^{i\lambda x}\right| \leqslant \left\{\sigma_1(x) - \sigma_1\left(x + \frac{1}{|\lambda|}\right)\right\}\exp\{-\operatorname{Im}\lambda x + \sigma_1(x)\}, \tag{7}$$

$$\left|f'(x,\lambda) - i\lambda e^{i\lambda x}\right| \leqslant \sigma(x)\exp\{-\operatorname{Im}\lambda x + \sigma_1(x)\}. \tag{8}$$

Using the above properties it was shown for real $\lambda \neq 0$ the functions $f(x,\lambda)$ and $\overline{f(x,\lambda)} = f(x,-\lambda)$ form a fundamental system of solutions of the equation (1) and their Wronskian is independent of variable $x$ and is equal to $2i\lambda$ (see [11, p. 168]), i.e.

$$W\{f(x,\lambda),\overline{f(x,\lambda)}\} = f'(x,\lambda)\overline{f(x,\lambda)} - f(x,\lambda)\overline{f'(x,\lambda)} = 2i\lambda. \tag{9}$$

Denote by $\phi(x,\lambda)$ the solution of the equation (1) satisfying the initial-value conditions

$$\phi(0,\lambda) = \alpha_0 + \alpha_2\lambda^2, \quad \phi'(0,\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2. \tag{10}$$

Let

$$E(\lambda) \equiv \left(\alpha_0 + \alpha_2\lambda^2\right)f'(0,\lambda) - \left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right)f(0,\lambda),$$

$$E_1(\lambda) \equiv \left(\alpha_0 + \alpha_2\lambda^2\right)f'(0,-\lambda) - \left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right)f(0,-\lambda).$$

## 2 Scattering data

Now we introduce the scattering data for the boundary value problem.

**Lemma 1.** *For any real $\lambda \neq 0$ the inequality $E(\lambda) \neq 0$ holds.*

*Proof.* Assume the contrary. Let $E(\lambda_0) = 0$ for $\lambda_0 \in \mathbb{R}$, $\lambda_0 \neq 0$. Then

$$f'(0,\lambda_0) = \frac{\beta_0 + \beta_1\lambda_0 + \beta_2\lambda_0^2}{\alpha_0 + \alpha_2\lambda_0^2} f(0,\lambda_0). \tag{11}$$

From (9) and (11) we have

$$\frac{\beta_0 + \beta_1\lambda_0 + \beta_2\lambda_0^2}{\alpha_0 + \alpha_2\lambda_0^2} f(0,\lambda_0)\overline{f(0,\lambda_0)} - \frac{\beta_0 + \beta_1\lambda_0 + \beta_2\lambda_0^2}{\alpha_0 + \alpha_2\lambda_0^2}\overline{f(0,\lambda_0)} f(0,\lambda_0) = 2i\lambda_0$$

or $0 = 2i\lambda_0$. This is a contradiction for $\lambda_0 \neq 0$. Lemma 1 is proved. □

**Lemma 2.** *The identity*

$$\frac{2i\lambda\phi(x,\lambda)}{E(\lambda)} = f(x,-\lambda) - S(\lambda) f(x,\lambda) \tag{12}$$

*holds for real $\lambda \neq 0$, where*

$$S(\lambda) = \frac{E_1(\lambda)}{E(\lambda)}. \tag{13}$$

*Moreover,*

$$\left|S(\lambda)\right| = 1.$$

*Proof.* Since two functions $f(x,\lambda)$ and $f(x,-\lambda)$ form a fundamental system of solutions to the equation (1) for all real $\lambda \neq 0$, we can write

$$\phi(x,\lambda) = c_1(\lambda) f(x,\lambda) + c_2(\lambda) f(x,-\lambda). \tag{14}$$

Using (9) and the initial data (10), we obtain

$$c_1(\lambda) = -\frac{E_1(\lambda)}{2i\lambda}, \quad c_2(\lambda) = \frac{E(\lambda)}{2i\lambda}.$$

After substituting $c_1(\lambda)$ and $c_2(\lambda)$ in (14), we get

$$\phi(x,\lambda) = -\frac{E_1(\lambda)}{2i\lambda} f(x,\lambda) + \frac{E(\lambda)}{2i\lambda} f(x,-\lambda). \tag{15}$$

By Lemma 1 for real $\lambda \neq 0$, $E(\lambda) \neq 0$. Then dividing (15) by $\dfrac{1}{2i\lambda}E(\lambda)$, we obtain (12), which can be expressed as (13). Since

$$\begin{aligned}\overline{E(\lambda)} &= \left(\alpha_0 + \alpha_2\lambda^2\right)\overline{f'(0,\lambda)} - \left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right)\overline{f(0,\lambda)} \\ &= \left(\alpha_0 + \alpha_2\lambda^2\right) f'(0,-\lambda) - \left(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\right) f(0,-\lambda) = E_1(\lambda),\end{aligned}$$

then clearly, implies that $\left|S(\lambda)\right| = 1$. Lemma 2 is proved. □

The function $S(\lambda)$ is called the *scattering function* of the boundary value problem (1)–(3).

**Lemma 3.** *The function $E(\lambda)$ may have only a finite number of zeros in the half plane Im $\lambda > 0$. The function $\lambda\left[E(\lambda)\right]^{-1}$ is bounded in a neighborhood point $\lambda = 0$.*

*Proof.* Since $E(\lambda) \neq 0$ for $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the point $\lambda = 0$ is possible real zero of the function $E(\lambda)$. It follows from the analyticity of the function $E(\lambda)$ on upper half plane. Zeros of $E(\lambda)$ form at the most countable set. Let us show that this set is bounded. We assume that $E\left(\lambda_k\right) = 0$ and $|\lambda_k| \to \infty$ as $k \to \infty$. Then we have

$$f'\left(0, \lambda_k\right) = \frac{\beta_0 + \beta_1\lambda_k + \beta_2\lambda_k^2}{\alpha_0 + \alpha_2\lambda_k^2} f\left(0, \lambda_k\right).$$

For $x = 0$ and $\lambda = \lambda_k$ it follows from (8) that

$$\left|\frac{\beta_0 + \beta_1\lambda_k + \beta_2\lambda_k^2}{\alpha_0 + \alpha_2\lambda_k^2} f\left(0, \lambda_k\right) - i\lambda_k\right| \leqslant \sigma(0)\exp\left\{\sigma_1(0)\right\}.$$

From here,

$$|\lambda_k| \leqslant \left|\frac{\beta_0 + \beta_1\lambda_k + \beta_2\lambda_k^2}{\alpha_0 + \alpha_2\lambda_k^2} f\left(0, \lambda_k\right)\right| + \sigma(0)\exp\left\{\sigma_1(0)\right\}.$$

Since $|\lambda_k| \to \infty$ as $k \to \infty$, then it follows from (6) and (7) that $f\left(0, \lambda_k\right) = 1$ as $k \to \infty$. The resulting contradiction shows that the set $\{\lambda_k\}$ is bounded. Therefore, the zeros of $E(\lambda)$ form at most countable and bounded set having $\lambda = 0$ as the possible limit point.

Now let us show that the function $E(\lambda)$ may have a finite number of zeros $\lambda = \lambda_n$, $n = 1, 2, \ldots$. Then the function $z_n = z\left(x, \lambda_n\right)$ satisfies the equation

$$-z_n'' + p(x)z_n = \lambda_n^2 z_n \tag{16}$$

and the boundary condition

$$\left(\alpha_0 + \alpha_2\lambda_n^2\right) z_n'(0) - \left(\beta_0 + \beta_1\lambda_n + \beta_2\lambda_n^2\right) z_n(0) = 0. \tag{17}$$

Let us multiply both sides of the equation (16) by $\overline{z_n}$ and integrate this equation over $x$ from 0 to $\infty$. In this connection, using (17) and integrating by parts, we have

$$\lambda_n^2 - \frac{\beta_0 + \beta_1\lambda_n + \beta_2\lambda_n^2}{\alpha_0 + \alpha_2\lambda_n^2} \left|z_n(0)\right|^2 - L\left(z_n, z_n\right) = 0, \tag{18}$$

where

$$L\left(z_n, z_n\right) = \int_0^\infty \left(\left|z_n'\right|^2 + p(x)\left|z_n\right|^2\right) dx \equiv \Phi\left(z_n\right), \quad \left\langle z_n, z_n\right\rangle = 1.$$

Investigating the roots of the equation (18) of the fourth degree (applying Vieta's theorem) and taking into account the conditions on the coefficients ($\alpha_0$, $\alpha_2$, $\beta_0 > 0$), we obtain $\Phi\left(z_n\right) < 0$, $n = 1, 2, \ldots$.

From the asymptotic formulas (as $x \to \infty$) for the solutions of equation (1) it follows that the system $\{z_n(x)\}_{n=1}^{\infty}$ of functions is linearly independent (see [14, p. 445]).

Now, we construct the following series of functions

$$u_j(x) = a_j z_j(x) + b_j z_{j+1}(x), \quad j = 1, 2, \ldots,$$

where $a_j$, $b_j$, $j = 1, 2, \ldots$, are complex numbers and can be chosen such that the condition $u_j(0) = 0$ holds. In this case, we shall show that the functions $u_j(x)$ are linearly independent.

Thus, the relations

$$-u_n'' + p(x)u_n = \lambda_n u_n,$$
$$u_n(0) = 0$$

hold.

Denote by $L_\lambda$ the operator in the space $L_2(0, \infty)$ acting as $L_\lambda z = -z'' + p(x)z$ on the domain

$$D(L_\lambda) = \left\{ z(x) : z'(x) \in AC[0, \infty),\ -z'' + p(x)z \in L_2(0, \infty),\ z(0) = 0 \right\}.$$

From this it follows that $u_n(x) \in D(L_\lambda)$ and $\langle L_\lambda u_n, u_n \rangle < 0$. We obtain that the operator $L_\lambda$ has only an infinite number of negative eigenvalues. But this is impossible due to the condition (3) on $p(x)$ (see [11]). Consequently, we obtain a contradiction. It follows that the function $E(\lambda)$ may have a finite number of zeros in upper half plane $\operatorname{Im} \lambda > 0$.

Similarly to [11, Lemma 3.1.3], it is obtained that the function $\lambda \left[ E(\lambda) \right]^{-1}$ is bounded in half sphere $\left\{ \lambda : |\lambda| \leqslant \rho, \operatorname{Im} \lambda \geqslant 0 \right\}$. Lemma 3 is proved. □

**Corollary 1.** *The zeros of the function $E(\lambda)$ and $E_1(\lambda)$ are complex conjugate each other and the number of these zeros is equal.*

*Proof.* According to Lemma 3 the function $E(\lambda)$ in upper half plane $\operatorname{Im} \lambda > 0$ has finitely many zeros $\lambda_j$, $j = 1, 2, \ldots, n$. From the properties $\overline{f(0, \lambda_j)} = f(0, -\overline{\lambda_j})$, $\overline{f'(0, \lambda_j)} = f'(0, -\overline{\lambda_j})$ of the function $f(x, \lambda)$ we have

$$\overline{E(\lambda_j)} = \left( \alpha_0 + \alpha_2 \overline{\lambda_j^2} \right) f'\left(0, -\overline{\lambda_j}\right) - \left( \beta_0 + \beta_1 \overline{\lambda_j} + \beta_2 \overline{\lambda_j^2} \right) f\left(0, -\overline{\lambda_j}\right) = E_1\left(\overline{\lambda_j}\right), \quad j = 1, 2, \ldots, n.$$

Therefore, the zeros of the functions $E(\lambda)$ and $E_1(\lambda)$ are complex conjugate, and the number of these zeros is equal. Corollary 1 is proved. □

**Lemma 4.** *From the properties of the function $E(\lambda)$ for $|\lambda| \to \infty$ we get*

$$S(\lambda) = -1 + O\left(\frac{1}{\lambda}\right).$$

From the relation for $\operatorname{Im} \lambda \geqslant 0$ we get $-1 - S(\lambda) \in L_2(-\infty, \infty)$ and hence the function

$$F_s(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ -1 - S(\lambda) \right] e^{i\lambda x} d\lambda$$

also belongs to the space $L_2(-\infty, \infty)$.

It is known (see [14, p. 299]) that the equation (1) has a solution $\hat{f}(x, \lambda)$, which satisfies for every $\theta > 0$ and $\delta > 0$ the relation

$$\hat{f}(x, \lambda) = e^{-i\lambda x}\left[1 + o(1)\right] \quad \text{as} \quad x \to \infty$$

uniformly in the domain $\operatorname{Im} \lambda \geqslant \theta$, $|\lambda| \geqslant \delta$.

We denote

$$r_j(x) = i \operatorname*{Res}_{\lambda = \lambda_j} \frac{\hat{E}(\lambda)}{E(\lambda)} e^{i\lambda x}, \tag{19}$$

where $\hat{E}(\lambda) = (\alpha_0 + \alpha_2\lambda^2)\hat{f}'(0,\lambda) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)\hat{f}(0,\lambda)$, $j = 1,2,\dots,n$. According to [5] (see also [14, p. 327]) we call the polynomial

$$P_j(x) = e^{-i\lambda_j x} r_j(x), \quad j = 1,2,\dots,n,$$

with the degree of $k_j - 1$ the *normalization polynomial* for the boundary value problem (1)–(3), where $k_j$ is the multiplicity of the numbers $\lambda_j$, $j = 1,2,\dots,n$.

The set of values $\{S(\lambda), \lambda_j, P_j(x) : j = 1,2,\dots,n\}$ is called the *scattering data* of the boundary value problem (1)–(3).

The inverse scattering problem for the boundary value problem (1)–(3) consists in recovering the coefficient $p(x)$ from the scattering data.

Similarly, for the solution $f(x,\lambda)$ and $\hat{f}(x,\lambda)$,

$$\frac{2i\lambda\phi(x,\lambda)}{E(\lambda)} = \hat{f}(x,\lambda) - \frac{\hat{E}(\lambda)}{E(\lambda)} f(x,\lambda) \tag{20}$$

is obtained.

## 3 The fundamental equation

The *fundamental equation* has played an important role in solving of the inverse scattering problem. We use the identity (12), which was obtained in Lemma 2. Rewriting the identity (12), we get the following form

$$\begin{aligned}\frac{2i\lambda\phi(x,\lambda)}{E(\lambda)} - e^{i\lambda x} - e^{-i\lambda x} =& \left[-1 - S(\lambda)\right]\left\{e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt\right\} \\ &+ \int_x^\infty K(x,t)e^{-i\lambda t}dt + \int_x^\infty K(x,t)e^{i\lambda t}dt.\end{aligned}$$

Let us multiply both sides of the last relation by $\frac{1}{2\pi}e^{i\lambda\tau}$ with $\tau > x$ and integrate over $\lambda$ from $-\infty$ to $\infty$, we obtain

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x,\lambda)\left[\frac{i\lambda}{E(\lambda)} - \frac{1}{\alpha_2\lambda}\right]e^{i\lambda\tau}d\lambda + \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{\phi(x,\lambda)}{\alpha_2\lambda} - \cos\lambda x\right]e^{i\lambda\tau}d\lambda \\ = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[-1 - S(\lambda)\right]e^{i\lambda(x+\tau)}d\lambda \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\left[-1 - S(\lambda)\right]\int_x^\infty K(x,t)e^{i\lambda(t+\tau)}dt\right\}d\lambda \\ + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_x^\infty K(x,t)e^{-i\lambda(t+\tau)}dt\right\}d\lambda.\end{aligned} \tag{21}$$

On the right-hand side of (21), taking $K(x,t) = 0$ for $x > t$ into account, we obtain

$$F_s(x+\tau) + K(x,\tau) + \int_x^\infty K(x,t)F_s(t+\tau)\,dt.$$

Taking into account (20) in the first integral of (21), we find

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x,\lambda)\left[\frac{i\lambda}{E(\lambda)} - \frac{1}{\alpha_2\lambda}\right]e^{i\lambda\tau}d\lambda = i\sum_{\operatorname{Im}\lambda>0}\operatorname*{Res}_{\lambda=\lambda_j}\left[\hat{f}(x,\lambda) - \frac{\hat{E}(\lambda)}{E(\lambda)}f(x,\lambda) - \frac{2i}{\alpha_2\lambda}\phi(x,\lambda)\right]e^{i\lambda\tau}.$$

It is clear that for $\mathrm{Im}\,\lambda > 0$, $\hat{f}(x,\lambda)$ is holomorphic function and $\phi(x,\lambda)$ is entire function with respect to $\lambda$. Then according to Lemma 3, we have

$$\begin{aligned}
&\frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x,\lambda)\left[\frac{i\lambda}{E(\lambda)}-\frac{1}{\alpha_2\lambda}\right]e^{i\lambda\tau}d\lambda \\
&\quad = -i\left\{\sum_{\mathrm{Im}\,\lambda>0}\operatorname*{Res}_{\lambda=\lambda_j}\frac{\hat{E}(\lambda)}{E(\lambda)}e^{i\lambda(x+\tau)}\right\} - i\int_x^{\infty}K(x,t)\left\{\operatorname*{Res}_{\lambda=\lambda_j}\frac{\hat{E}(\lambda)}{E(\lambda)}e^{i\lambda(t+\tau)}\right\}dt \\
&\quad = -\sum_{j=1}^{n}r_j(x+\tau) - \int_x^{\infty}K(x,t)\sum_{j=1}^{n}r_j(t+\tau)dt,
\end{aligned} \tag{22}$$

where $r_j(x)$ is defined by (19).

Therefore, for $\tau > x$, taking the equality (22) into account, from (21) we derive the relation

$$-\sum_{j=1}^{n}r_j(x+\tau) - \int_x^{\infty}K(x,t)\sum_{j=1}^{n}r_j(t+\tau)dt = F_s(x+\tau) + K(x,\tau) + \int_x^{\infty}K(x,t)F_s(t+\tau)dt.$$

Finally, we obtain the integral equation

$$F(x+\tau) + K(x,\tau) + \int_x^{\infty}K(x,t)F(t+\tau)dt = 0, \quad x < \tau < \infty, \tag{23}$$

where

$$F(x) = \sum_{j=1}^{n}r_j(x) + F_s(x). \tag{24}$$

The equation (23) is called the fundamental equation or Gelfand-Levitan-Marchenko equation of the inverse problem of scattering theory for the boundary value problem (1) – (3). Therefore we have the assertion.

**Theorem 1.** *For each fixed $x \geqslant 0$, the kernel $K(x,t)$ of the representation* (4) *satisfies the fundamental equation* (23).

As in [11], using the fundamental equation (23), we show that for all $x \geqslant 0$ the function $F_s(x)$ has a derivative and the following inequalities hold:

$$\begin{aligned}
&\left|F(2x)\right| \leqslant C\sigma(x), \\
&\left|F'(2x) - \frac{1}{4}p(x)\right| \leqslant C\left[\sigma(x)\right]^2,
\end{aligned} \tag{25}$$

where $C > 0$. Since $(1+x)\sigma(x) < \infty$, from (25) we obtain that the function $(1+x)\left|F'(x)\right|$ is also summable on $[0,\infty)$.

## 4 Uniqueness theorem

It is shown above that the function $F(x)$ is differentiable and

$$\int_0^{\infty}(1+x)\left|F'(x)\right|dx < \infty.$$

Let

$$\Omega(x) = \int_x^{\infty}\left|F'(t)\right|dt, \quad \Omega_1(x) = \int_x^{\infty}\Omega(t)dt. \tag{26}$$

Clearly,

$$\left|F(x)\right| \leqslant \Omega(x) \tag{27}$$

and $\Omega_1(0) < \infty$. In fact, according to (26), we have

$$\left|F(x)\right| \leqslant \int_0^\infty \left|F'(t)\right| dt = \Omega(x).$$

**Theorem 2.** *Assume that the function $\varphi_x(t)$ is summable on the half line $t > x$ and*

$$\varphi_x(t) + \int_x^\infty \varphi_x(\xi)F(\xi+t)d\xi = 0, \quad t > x. \tag{28}$$

*Then $\varphi_x(t) \equiv 0$ for $t > x$.*

*Proof.* Assume that $y_x(t)$ is the solution of the equation

$$\varphi_x(t) = y_x(t) + \int_x^t K(\tau,t)y_x(\tau)d\tau, \quad t \geqslant x, \tag{29}$$

where the function $K(\tau,t)$ satisfies the equation (23) and for this function the inequality $\left|K(x,t)\right| \leqslant C\sigma\left(\frac{x+t}{2}\right)$ is valid. Then according to (27)

$$\begin{aligned}\int_\tau^\infty \left|F(t+\xi)K(\tau,\xi)\right|d\xi &\leqslant \int_\tau^\infty \frac{1}{2}e^{\sigma_1(0)}\sigma\left(\frac{\tau+\xi}{2}\right)\Omega(t+\xi)d\xi \\ &\leqslant \frac{1}{2}e^{\sigma_1(0)}\sigma(0)\int_\tau^\infty \Omega(\xi)d\xi = \frac{1}{2}e^{\sigma_1(0)}\sigma(0)\Omega_1(\tau),\end{aligned}$$

$$\int_\tau^\infty \left|F(t+\xi)K(\tau,\xi)\right|d\xi \leqslant C\sigma_1(\tau), \tag{30}$$

where $C = \dfrac{1}{2}e^{\sigma_1(0)}\sigma(0) > 0$. It follows from (29) and (30) that

$$\begin{aligned}y_x(t) + \int_x^t K(\tau,t)y_x(\tau)d\tau = &-\int_x^\infty y_x(\tau)F(\tau+t)d\tau \\ &-\left(\int_x^t + \int_t^\infty\right)y_x(\tau)d\tau\int_\tau^\infty K(\tau,\xi)F(\xi+t)d\xi.\end{aligned}$$

According to the fundamental equation (23), for $t \geqslant \tau$ we get

$$\int_\tau^\infty K(\tau,\xi)F(t+\xi)d\xi = -\left[F(t+\tau) + K(\tau,t)\right].$$

From this equality, it follows that the relation

$$y_x(t) = -\int_t^\infty y_x(\tau)\left[F(\tau+t) + \int_\tau^\infty K(\tau,\xi)F(\xi+t)d\xi\right]d\tau, \quad t \geqslant \tau, \tag{31}$$

holds. From the inequalities (28) and (29), we have

$$\left|F(t+\tau) + \int_\tau^\infty K(\tau,\xi)F(t+\xi)d\xi\right| \leqslant \Omega(t+\tau) + C\Omega_1(\tau) \leqslant \Omega(\tau) + C\Omega_1(\tau) \leqslant C_1\Omega_1(\tau),$$

where $C, C_1 > 0$. Using this relation and (31), we find

$$|y_x(t)| \leqslant C_1\int_t^\infty |y_x(\tau)|\,\Omega(t+\tau)d\tau \leqslant C_1\int_t^\infty |y_x(\tau)|\,\Omega(\tau)d\tau,$$

$$|y_x(t)| \leqslant \varepsilon + C_1 \int_t^\infty |y_x(\tau)|\,\Omega(\tau)d\tau \quad \text{for all } \varepsilon > 0.$$

Applying Gronwall's lemma,

$$|y_x(t)| \leqslant \varepsilon \exp\left\{C_1 \int_t^\infty \Omega(\tau)d\tau\right\} \leqslant \varepsilon \exp\left\{C_1 \int_0^\infty \Omega(\tau)d\tau\right\}$$

is obtained. Since $\varepsilon > 0$ is arbitrary real number, it follows that for $t > x$, $y_x(t) \equiv 0$. Then from (29), for $t \geqslant x$ we have $\varphi_x(t) \equiv 0$. Theorem 2 is proved. □

**Theorem 3.** *The equation* (23) *has the unique solution* $K(x,\cdot) \in L_1(0,\infty)$ *for each* $x \geqslant 0$.

*Proof.* To prove the theorem, it suffices to show that the homogeneous equation

$$\varphi_x(t) + \int_x^\infty \varphi_x(\xi)F(\xi+t)dt = 0, \quad t > x,$$

has only the trivial solution $g_x(t) \in L_1(0,\infty)$. Since the operator

$$(\mathbb{F}\varphi_x)(t) = \int_x^\infty \varphi_x(\xi)F(\xi+t)d\xi$$

is the compact operator (for the compactness of $\mathbb{F}$, see [11, Lemma 3.3.1]). From the properties of the function $F(y)$, it is obtained that the function $F(t)$ and the corresponding solution $g_x(t) = K(x,t)$ are bounded in the half axis $x \leqslant t < \infty$. Therefore, $g_x(t) \in L_2(x,\infty)$. For each fixed $x$, we can consider the fundamental equation as Fredholm type equation. Then from Theorem 2 we obtain that the equation has a unique solution. Theorem 3 is proved. □

For $x \geqslant 0$, from the continuity of the function $F(x)$ it follows that the fundamental equation is also valid for $t = x$.

According to Theorem 2 for arbitrary $x \geqslant 0$, fundamental equation (23) has not any solution other than the function $K(x,t)$ satisfying the condition (4).

Assume that the collection $\{\tilde{S}(\lambda), \tilde{\lambda}_j, \tilde{P}_j(x) : -\infty < \lambda < \infty, j = 1,2,\ldots,n\}$ is the scattering data of the boundary value problem (1), (2) whose potential $\tilde{p}(x)$ satisfies condition (3). Then the following corollary is obtained.

**Corollary 2.** *The potential from class* (3) *in the problem* (1), (2) *is uniquely defined by the scattering data, i.e. if* $S(\lambda) = \tilde{S}(\lambda)$, $-\infty < \lambda < \infty$, $\lambda_j = \tilde{\lambda}_j$, $P_j(x) = \tilde{P}_j(x)$, $j = 1,2,\ldots,n$, *then* $p(x) = \tilde{p}(x)$ *almost everywhere on the half line* $[0,\infty)$.

*Proof.* Given the scattering data, we can use formula (24) to construct the function $F(x)$ and write out the fundamental equation (23) for the unknown function $K(x,t)$. It follows from Theorem 3 that the fundamental equation has unique solution. Solving this equation, we find the function $K(x,t)$ and by (5) the potential is $p(x) = -2\dfrac{d}{dx}K(x,x)$. □

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Фарзуллазаде Р.Г., Мамедов Х.Р. *Теорема єдиності для рівнянь Штурма-Ліувіля зі спектральним параметром, який нелінійно міститься в граничній умові* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 548–557.

У даній роботі розглядається обернена задача розсіяння для рівняння Штурма-Ліувілля на напівнескінченному інтервалі з нелінійним спектральним параметром у граничній умові. Визначено дані розсіювання задачі та досліджено властивості даних розсіювання. Отримано фундаментальне рівняння та досліджено унікальність алгоритму для потенціалу із заданими даними розсіювання.

*Ключові слова і фрази:* обернена задача розсіювання, нормалізаційний поліном, дані розсіювання, фундаментальне рівняння, теорема єдиності.