ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2023, **15** (1), 104–119
doi:10.15330/cmp.15.1.104-119

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2023, Т.15, №1, С.104–119

Spectra of algebras of analytic functions, generated by sequences of polynomials on Banach spaces, and operations on spectra

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We consider the subalgebra of the Fréchet algebra of entire functions of bounded type, generated by a countable set of algebraically independent homogeneous polynomials on the complex Banach space X. We investigate the spectrum of this subalgebra in the case $X = \ell_1$. We also consider some shift type operations that can be performed on the spectrum of this subalgebra in the case $X = \ell_p$ with $p \geq 1$.

Key words and phrases: n-homogeneous polynomial, analytic function, spectrum of an algebra.

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Introduction

Algebras of analytic functions on a Banach space generated by a countable number of polynomials naturally appear as algebras of invariants with respect to actions of groups of symmetry on the Banach space. Typical examples of such algebras are algebras of symmetric analytic functions on ℓ_p, $1 \leq p < \infty$, with respect to the group of permutations of the basis vectors [1–3, 10] or algebras of symmetric analytic functions of bounded type on $L_p(\Omega)$, $1 \leq p \leq \infty$, with respect to the group of automorphisms of a measure space Ω [5,6,14–17]. The problem of the description of spectra of such kind of algebras is non-trivial ever in the finite dimensional case. For example, the famous Hilbert's fourteenth problem on invariant theory has a negative solution due to the counterexample of Nagata.

Let X be a complex Banach space. Let us consider the sequence $\{P_n\}_{n=1}^{\infty}$ of n-homogeneous (see the definition below) continuous complex-valued polynomials on X, $n \in \mathbb{N}$. Let us define the multi-valued mapping $\beta : X \to \mathbb{C}^\infty$ by the formula

$$\beta(x) = \left\{ P_n^{\frac{1}{n}}(x) \right\}_{n=1}^{\infty} \quad \text{for every} \quad x \in X.$$

Note that $\left\{ P_n^{\frac{1}{n}}(x) \right\}_{n=1}^{\infty} \subset \ell_\infty$ for every $x \in X$. For every $x \in \ell_\infty$ and $n \in \mathbb{N}$ let us define the polynomials $\left\{ I_n^{(\infty)} \right\}_{n=1}^{\infty}$ by the formula $I_n^{(\infty)}(x) = x_n^n$.

It is easy to see that $P_n(x) = I_n^{(\infty)}\big(\beta(x)\big)$ for every $x \in X$. Taking into account such the representation of the polynomials $\{P_n\}_{n=1}^{\infty}$, it was interesting to investigate (see [18, 19]) the

УДК 517.98
2020 *Mathematics Subject Classification:* 46J20, 46E15, 46G20.

This research was funded by the National Research Foundation of Ukraine, 2020.02/0025, 0120U103996.

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existence of analytic functions of unbounded type, generated by the polynomials $\{P_n\}_{n=1}^{\infty}$, dependently on the image of the mapping β.

In this paper, we consider the particular case of countable generated algebras, namely the algebras generated by the polynomials $\{I_n^{(p)}\}_{n=1}^{\infty}$ on the spaces ℓ_p, $1 \le p < \infty$.

In Section 1, we recall some basic notions on the theory of analytic functions on a Banach space which are necessary for a full comprehension of the paper. In Section 2, we investigate the spectrum of the Fréchet algebra of entire functions of bounded type, generated by the sequence of polynomials on a complex Banach space ℓ_1. Spectra of algebras of analytic functions of bounded type were studied in [8,9]. In Section 3, we consider some shift type operations that can be performed on the spectra of the algebras of entire functions of bounded type, generated by the sequences of polynomials on complex Banach spaces ℓ_p, $p \ge 1$.

1 Preliminaries

In this section, we introduce the necessary background and establish some notations. Throughout the whole article, the letter X will always stand for a complex Banach space. The set of all positive integers will be denoted by $\mathbb{N}$, whereas the set $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. We denote by $\mathbb{Q}^+$ the set of all positive rationals. The set of complex numbers will be denoted by $\mathbb{C}$. We also denote by ℓ_p, $1 \le p < \infty$, the complex Banach space consisting of all sequences of complex numbers $x = \{x_n\}_{n=1}^{\infty}$ satisfying the condition $\sum_{n=1}^{\infty} |x_n|^p < \infty$ with the norm $\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$. If $p = \infty$, then ℓ_∞ is defined to be the space of all bounded sequences of complex numbers endowed with the norm $\|x\| = \sup\limits_{n\in\mathbb{N}} |x_n|$.

Definition 1. *For each $n \in \mathbb{N}$ a mapping $P : X \to \mathbb{C}$ is said to be an n-homogeneous polynomial if there exists some n-linear form $A_P : X^n \to \mathbb{C}$ such that*

$$P(x) = A_P(\underbrace{x, \ldots, x}_{n})$$

for every $x \in X$.

Let $\mathcal{P}^n(X)$ be the vector space of all n-homogeneous polynomials from X to $\mathbb{C}$ and $\mathcal{P}^0(X)$ be the vector space of all constant mappings from X to $\mathbb{C}$. For each $P \in \mathcal{P}^n(X)$ we set

$$\|P\|_1 = \sup\big\{|P(x)| : x \in X, \|x\| \le 1\big\}.$$

It is known that a polynomial $P \in \mathcal{P}^n(X)$ is continuous if and only if $\|P\|_1 < \infty$.

Definition 2. *A mapping $P : X \to \mathbb{C}$ is said to be a polynomial of degree at most n, $n \in \mathbb{N}_0$, if it can be represented as a sum $P = P_0 + P_1 + \cdots + P_n$, where $P_j \in \mathcal{P}^j(X)$ for $j = \overline{0, n}$.*

Definition 3. *Polynomials $P_1, P_2, \ldots, P_j \in \mathcal{P}^j(X)$, $j \in \mathbb{N}$, are called algebraically independent if for every $n \in \mathbb{N}$ and every polynomial $q : \mathbb{C}^n \to \mathbb{C}$ the equality $q(P_1(x), P_2(x), \ldots, P_n(x)) = 0$ for every $x \in X$ implies $q \equiv 0$.*

Definition 4. *A polynomial $P : X \to \mathbb{C}$ is called an algebraic combination of elements of $\mathbb{P} = \{P_1, P_2, \ldots\}$ if there exist $n \in \mathbb{N}$ and a polynomial $q : \mathbb{C}^n \to \mathbb{C}$ such that $P(x) = q\big(P_1(x), \ldots, P_n(x)\big)$ for every $x \in X$.*

Let us denote by $B(a,r)$ and $\bar{B}(a,r)$ an open and a closed balls of a radius r and a center $a \in X$, respectively.

Definition 5 ([11, p. 33, Definition 5.1]). *Let U be the open subset of X. A mapping $f : U \to \mathbb{C}$ is said to be holomorphic or analytic on U if for each $a \in U$ there exist an open ball $B(a;r) \subset U$ and a sequence of continuous polynomials $\{f_0, f_1, \dots\}$, where $f_0 \in \mathbb{C}$ and $f_j : X \to \mathbb{C}$ is a j-homogeneous polynomial for each $j \in \mathbb{N}$, such that*

$$f(x) = \sum_{n=0}^{\infty} f_n(x-a) \quad \textit{uniformly for} \;\; x \in B(a;r).$$

Note that the power series $\sum_{n=0}^{\infty} f_n(x-a)$ is called the Taylor series of the function f at the point a. If $U = X$, the function f is called an entire function.

Definition 6. *The radius of uniform convergence $R(f)$ of the function $f(x) = \sum_{n=0}^{\infty} f_n(x-a)$ is the supremum of all $r \geq 0$ such that the series $\sum_{n=0}^{\infty} f_n(x-a)$ converges uniformly on the ball $\overline{B}(a,r)$.*

According to [11, p. 27, Theorem 4.3], the radius of convergence of the power series is given by the Cauchy-Hadamard formula

$$R(f) = \frac{1}{\limsup\limits_{n\to\infty} \|f_n\|_1^{\frac{1}{n}}}.$$

Definition 7. *Let Y be a Fréchet linear space. A continuous linear operator $T : Y \to Y$ is hypercyclic if there is a vector $y_0 \in Y$ for which the orbit under T, $Orb(T, y_0) = \{y_0, Ty_0, T^2y_0, \dots\}$, is dense in Y. Every such vector y_0 is called a hypercyclic vector of T.*

Let $\mathbb{P} = \{P_1, P_2, \dots, P_n, \dots\}$ be a set of algebraically independent continuous n-homogeneous polynomials on X, such that $\|P_n\|_1 = 1$, $n \in \mathbb{N}$. We denote by $\mathcal{P}_{\mathbb{P}}(X)$ the minimal unital algebra containing polynomials $\mathbb{P}$. Clearly that $\mathbb{P}$ forms an algebraic basis in $\mathcal{P}_{\mathbb{P}}(X)$, that is, every polynomial in $\mathcal{P}_{\mathbb{P}}(X)$ can be uniquely represented as an algebraic combination of elements in $\mathcal{P}_{\mathbb{P}}(X)$. It is easy to see that the basis $\mathbb{P}$ is not unique.

Denote by $H_b(X)$ the Fréchet algebra of $\mathbb{C}$-valued entire functions of bounded type on X, that is, the algebra of all entire mappings from X to $\mathbb{C}$, which are bounded on bounded subsets. We endow the algebra $H_b(X)$ with the system of uniform norms

$$\|f\|_r = \sup\big\{|f(x)| : x \in X, \|x\| \leq r\big\}, \quad \text{where} \quad r \in \mathbb{Q}^+.$$

Let $H_{b\mathbb{P}}(X)$ be the closure of $\mathcal{P}_{\mathbb{P}}(X)$ in $H_b(X)$. Note that $H_{b\mathbb{P}}(X)$ is a subalgebra of $H_b(X)$. In [8], it is proved that each term of the Taylor series of a function $f \in H_{b\mathbb{P}}(X)$ can be uniquely represented as an algebraic combination of elements of the set $\mathbb{P}$. Consequently,

$$f(x) = f(0) + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\dots+nk_n=n} a_{k_1,k_2,\dots,k_n} P_1^{k_1}(x) P_2^{k_2}(x) \cdots P_n^{k_n}(x),$$

where $x \in X$, $a_{k_1,k_2,\dots,k_n} \in \mathbb{C}$ and $k_1, k_2 \dots, k_n \in \mathbb{N}_0$.

Algebras $H_{b\mathbb{P}}(X)$ for various sequences of polynomials were considered in [4,8,9,12,18]. We denote by $M_{b\mathbb{P}} = M_{b\mathbb{P}}(X)$ the spectrum (the set of all non-trivial continuous complex-valued linear multiplicative functionals = non-trivial continuous complex-valued homomorphisms = continuous characters) of $H_{b\mathbb{P}}(X)$. It is known [8] that the spectrum $M_{b\mathbb{P}}$ can be described as the set of sequences $\tau(M_{b\mathbb{P}})$, where the mapping $\tau : M_{b\mathbb{P}}(X) \mapsto \mathbb{C}^\infty$ is defined by

$$\tau(\varphi) = \big(\varphi(P_1), \varphi(P_2), \dots\big) \quad \text{for every} \quad \varphi \in M_{b\mathbb{P}}(X). \tag{1}$$

2 On the spectrum of the algebra $H_{b\mathbb{I}}(\ell_1)$

Let $\mathbb{I} = \left\{ I_1^{(p)}, I_2^{(p)}, \ldots \right\}$ be the set of n-homogeneous complex-valued polynomials on the space ℓ_p, $1 \le p \le +\infty$, defined as

$$I_n^{(p)}(x) = x_n^n \tag{2}$$

for all $n \in \mathbb{N}$ and $x \in \ell_p$. In this section, we provide some information about the spectrum of the algebra $H_{b\mathbb{I}}(\ell_1)$, where $\mathbb{I} = \left\{ I_n^{(1)} \right\}_{n=1}^{\infty}$. Note that for every $x \in \ell_1$ the point-evaluation functional δ_x defined by $\delta_x(f) = f(x)$ for every $f \in H_{b\mathbb{I}}(\ell_1)$ belongs to the spectrum $M_{b\mathbb{I}}(\ell_1)$ of the algebra $H_{b\mathbb{I}}(\ell_1)$. It is interesting whether there exist any other characters in the spectrum $M_{b\mathbb{I}}(\ell_1)$.

Let us suppose that there exists a non-trivial linear continuous multiplicative functional $\phi : H_{b\mathbb{I}}(\ell_1) \to \mathbb{C}$, which is not equal to any δ_x with $x \in \ell_1$. Let us consider the sequence $x_\phi = \left(\phi(I_1^{(1)}), \sqrt{\phi(I_2^{(1)})}, \ldots, \sqrt[n]{\phi(I_n^{(1)})}, \ldots \right)$. Note that $x_\phi \notin \ell_1$. Otherwise, if $x_\phi \in \ell_1$, the character ϕ would be equal to the point-evaluation functional δ_{x_ϕ}. Let $f \in H_{b\mathbb{I}}(\ell_1)$. Then

$$f(x) = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} \left(I_1^{(1)}(x) \right)^{k_1} \left(I_2^{(1)}(x) \right)^{k_2} \cdots \left(I_n^{(1)}(x) \right)^{k_n},$$

where $x \in \ell_1$, $a_{k_1,k_2,\ldots,k_n} \in \mathbb{C}$ and $k_1, k_2 \ldots, k_n \in \mathbb{N}_0$. Note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} \left(I_1^{(1)}(x_\phi) \right)^{k_1} \left(I_2^{(1)}(x_\phi) \right)^{k_2} \cdots \left(I_n^{(1)}(x_\phi) \right)^{k_n} \\
&\quad = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} \left(\phi(I_1^{(1)}) \right)^{k_1} \left(\phi(I_2^{(1)}) \right)^{k_2} \cdots \left(\phi(I_n^{(1)}) \right)^{k_n} = \phi(f).
\end{aligned}$$

Therefore the function f can be naturally extended to the set $\ell_1 \bigcup \{x_\phi\}$. In general, every $f \in H_{b\mathbb{I}}(\ell_1)$ can be extended to each point of the set $\nu^{-1}\big(\tau\big(M_{b\mathbb{I}}(\ell_1)\big)\big)$, where the mapping τ is defined by (1) and $\nu : \mathbb{C}^\infty \to \mathbb{C}^\infty$ is defined by

$$\nu\big((x_1, x_2, \ldots, x_n, \ldots)\big) = (x_1, x_2^2, \ldots, x_n^n, \ldots), \tag{3}$$

where $(x_1, x_2, \ldots, x_n, \ldots) \in \mathbb{C}^\infty$. Thus, the question of the description of the spectrum of the algebra $H_{b\mathbb{I}}(\ell_1)$ is closely related to the question of the extension of elements of this algebra to the set that is wider than ℓ_1. In this section we investigate which points do not belong to the spectrum $M_{b\mathbb{I}}(\ell_1)$. We also construct an example of a function that belongs to the algebra $H_{b\mathbb{I}}(\ell_1)$ and is well-defined on the element $x_0 = (1, 1/2, \ldots, 1/n, \ldots)$, but cannot be extended to the analytic function of bounded type on the spaces ℓ_p, $1 < p < \infty$.

Proposition 1. *The polynomials $I_1^{(p)}, I_2^{(p)}, \ldots$, defined by the formula (2), are algebraically independent.*

Proof. Let $n \in \mathbb{N}$. Let $q : \mathbb{C}^n \to \mathbb{C}$ be a polynomial such that

$$q\left(I_1^{(p)}(x), \ldots, I_n^{(p)}(x) \right) = 0 \tag{4}$$

for all $x \in \ell_p$, $1 \le p \le +\infty$. Since q is the polynomial, then there exist a finite set $A \subset \mathbb{N}_0^n$ and complex numbers α_k, $k = (k_1, \dots, k_n) \in A$, such that q can be represented in the form

$$q(z_1, z_2, \dots, z_n) = \sum_{k \in A} \alpha_k z_1^{k_1} z_2^{k_2} \cdot \ldots \cdot z_n^{k_n}, \quad z_1, \dots, z_n \in \mathbb{C}. \tag{5}$$

Taking into account the equalities (4) and (5) we have

$$q\big(I_1^{(p)}(x), \dots, I_n^{(p)}(x)\big) = \sum_{k \in A} \alpha_k x_1^{k_1} \cdot \ldots \cdot x_n^{nk_n} = 0 \quad \text{for all} \quad x \in \ell_p,\ 1 \le p \le +\infty.$$

It follows that all the coefficients α_k, $k \in A$, are equal to zero. Therefore, $q \equiv 0$ and the polynomials $I_1^{(p)}, I_2^{(p)}, \dots$ are algebraically independent. □

Proposition 2. *The polynomials $I_1^{(p)}, I_2^{(p)}, \dots$, defined by the formula* (2)*, are continuous.*

Proof. It is clear that

$$\big\|I_n^{(p)}\big\|_1 = \sup\Big\{\big|I_n^{(p)}(x)\big| : x \in \ell_p, \|x\| \le 1\Big\} = \sup\big\{|x_n|^n : x \in \ell_p, \|x\| \le 1\big\} = 1 < \infty$$

for every $n \in \mathbb{N}$. Since $\big\|I_n^{(p)}\big\|_1 < \infty$ for every $n \in \mathbb{N}$, the polynomials $I_1^{(p)}, I_2^{(p)}, \dots$ are continuous. □

Proposition 3. *For the polynomials $I_1^{(p)}, \dots, I_n^{(p)}, \dots$, which are defined by the formula* (2)*, the following equality*

$$\big\|I_1^{(p)} \cdot \ldots \cdot I_n^{(p)}\big\|_b = \sqrt[p]{\frac{H(n)}{\left(\frac{n(n+1)}{2}\right)^{\frac{n(n+1)}{2}}}}\, b^{\frac{n(n+1)}{2}}$$

is true for every $b > 0$ and $n \in \mathbb{N}$, where $H(n)$ is a hyperfactorial of n, i.e.

$$H(n) = 1^1 2^2 3^3 \dots n^n, \tag{6}$$

and the norm is attained at the point

$$x = \left(\sqrt[p]{\frac{2}{n(n+1)}}b, \sqrt[p]{\frac{4}{n(n+1)}}b, \dots, \sqrt[p]{\frac{2n}{n(n+1)}}b, 0, \dots\right).$$

Proof. Let us use the mathematical induction. Note that, according to the Maximum-Modulus Theorem (see, e.g., [13, Chapter 5]), we have

$$\begin{aligned}\big\|I_1^{(p)} \cdot \ldots \cdot I_n^{(p)}\big\|_b &= \sup_{\|x\| \le b} \big|I_1^{(p)}(x) \cdot \ldots \cdot I_n^{(p)}(x)\big| \\ &= \sup_{\|x\| \le b} \big|x_1 x_2^2 \cdot \ldots \cdot x_n^n\big| = \sup_{\|x\| = b} \big|x_1 x_2^2 \cdot \ldots \cdot x_n^n\big| = \sup_{\|x\| = b} \big|I_1^{(p)}(x) \cdot \ldots \cdot I_n^{(p)}(x)\big|,\end{aligned}$$

where $x \in \ell_p$, $1 \le p < \infty$. It is easy to see that the statement is true for $n = 1$. Indeed,

$$\big\|I_1^{(p)}\big\|_b = \sup_{\|x\| = b} |x_1| = b \quad \text{for} \quad x = (b, 0, \dots).$$

Let $n \in \mathbb{N}$ and $n \geq 2$. Suppose the statement holds for $n-1$, that is,

$$\left\|I_1^{(p)} \cdot \ldots \cdot I_{n-1}^{(p)}\right\|_b = \sup_{\|x\|=b} \left|x_1 \cdot \ldots \cdot x_{n-1}^{n-1}\right| = \sqrt[p]{\frac{H(n-1)}{\left(\frac{n(n-1)}{2}\right)^{\frac{n(n-1)}{2}}}} b^{\frac{n(n-1)}{2}} \tag{7}$$

and the norm is attained at the point

$$x = \left(\sqrt[p]{\frac{2}{n(n-1)}}b, \sqrt[p]{\frac{4}{n(n-1)}}b, \ldots, \sqrt[p]{\frac{2(n-1)}{n(n-1)}}b, 0, \ldots\right).$$

Let us prove that the statement is true for n. Let $a = |x_n|$. Since $\|x\| = b$ and $|x_n| \leq \|x\|$, it follows $a \leq b$. Thus, $a \in [0, b]$. Since $|x_n| = a$ and for every $x = (x_1, \ldots, x_n, 0, \ldots) \in \ell_p$ we have $\|x\| = \sqrt[p]{|x_1|^p + \ldots + |x_n|^p} = b$, then $\sqrt[p]{|x_1|^p + \ldots + |x_{n-1}|^p + a^p} = b$, therefore $|x_1|^p + \ldots + |x_{n-1}|^p = b^p - a^p$ and, consequently, $\sqrt[p]{|x_1|^p + \ldots + |x_{n-1}|^p} = \sqrt[p]{b^p - a^p}$.

Then, using (7), we have

$$\begin{aligned}\left\|I_1^{(p)} \cdot \ldots \cdot I_n^{(p)}\right\|_b = \sup_{\|x\|=b} |x_1 \cdot \ldots \cdot x_n^n| &= a^n \sup_{\left(|x_1|^p + \ldots + |x_{n-1}|^p\right)^{1/p} = \left(b^p - a^p\right)^{1/p}} \left|x_1 \cdot \ldots \cdot x_{n-1}^{n-1}\right| \\ &= a^n \sqrt[p]{\frac{H(n-1)}{\left(\frac{n(n-1)}{2}\right)^{\frac{n(n-1)}{2}}}} \left(b^p - a^p\right)^{\frac{n(n-1)}{2p}}.\end{aligned}$$

Let us consider the function

$$f(a) = a^n \sqrt[p]{\frac{H(n-1)}{\left(\frac{n(n-1)}{2}\right)^{\frac{n(n-1)}{2}}}} \left(b^p - a^p\right)^{\frac{n(n-1)}{2p}}.$$

It can be checked that the function f has its maximum value on the segment $[0, b]$ at the point $a_{max} = \sqrt[p]{\frac{2}{n+1}}b$. Then

$$\left\|I_1^{(p)} \cdot \ldots \cdot I_n^{(p)}\right\|_b = f(a_{max}) = \sqrt[p]{\left(\frac{2}{n+1}\right)^n} b^n \sqrt[p]{\frac{H(n-1)}{\left(\frac{n(n-1)}{2}\right)^{\frac{n(n-1)}{2}}}} \left(b^p - \frac{2}{n+1}b^p\right)^{\frac{n(n-1)}{2p}}.$$

After simplifying the latter expression we obtain the desired result

$$\left\|I_1^{(p)} \cdot \ldots \cdot I_n^{(p)}\right\|_b = \sqrt[p]{\frac{H(n)}{\left(\frac{n(n+1)}{2}\right)^{\frac{n(n+1)}{2}}}} b^{\frac{n(n+1)}{2}}.$$

This completes the proof. □

Corollary 1. *For the polynomials $I_1^{(1)}, \ldots, I_n^{(1)}, \ldots$, defined by the formula* (2)*, the following is true*

$$\left\|I_1^{(1)} \cdot \ldots \cdot I_n^{(1)}\right\|_1 = \frac{H(n)}{\left(\frac{n(n+1)}{2}\right)^{\frac{n(n+1)}{2}}}.$$

Corollary 2. *For the polynomials* $I_1^{(p)}, \ldots, I_n^{(p)}, \ldots$, *defined by the formula* (2)*, the following is true*

$$\|I_1^{(p)} \cdot \ldots \cdot I_n^{(p)}\|_1 = \sqrt[p]{\frac{H(n)}{\left(\frac{n(n+1)}{2}\right)^{\frac{n(n+1)}{2}}}}.$$

Proposition 4. *The sequence* $(c, c, \ldots)$ *with* $c \geq 1$ *does not belong to the set* $\nu^{-1}\big(\tau\big(M_{b\mathbb{I}}(\ell_1)\big)\big)$*, where the mapping* τ *is defined by the formula* (1) *and the mapping* ν *is defined by the formula* (3).

Proof. Let us consider the function

$$f(x) = \sum_{m=1}^{\infty} \left(\frac{m}{2}\right)^m I_1^{(1)}(x) I_2^{(1)}(x) \cdot \ldots \cdot I_m^{(1)}(x), \quad x \in \ell_1.$$

Note that the nth term of a Taylor series of the function f is defined by the formula

$$f_n(x) = \begin{cases} \left(\frac{m}{2}\right)^m I_1^{(1)}(x) \cdot \ldots \cdot I_m^{(1)}(x), & \text{if} \quad n = \frac{m(m+1)}{2} \quad \text{for some} \quad m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

It is known [7] that the asymptotic growth rate of $H(m)$ is

$$H(m) \sim A m^{\frac{6m^2+6m+1}{12}} e^{\frac{-m^2}{4}}, \tag{8}$$

where $A = 1.2824\ldots$ is the Glaisher-Kinkelin constant. Taking this into account, it can be checked that

$$\begin{aligned} \lim_{m\to\infty} \|f_{\frac{m(m+1)}{2}}\|_1^{\frac{2}{m(m+1)}} &= \lim_{m\to\infty} \left(\left(\frac{m}{2}\right)^m \left\| I_1^{(1)} I_2^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1 \right)^{\frac{2}{m(m+1)}} \\ &= \lim_{m\to\infty} \left(\left(\frac{m}{2}\right)^m \frac{H(m)}{\left(\frac{m(m+1)}{2}\right)^{\frac{m(m+1)}{2}}} \right)^{\frac{2}{m(m+1)}} \\ &= \lim_{m\to\infty} \left(\left(\frac{m}{2}\right)^m \frac{A m^{\frac{6m^2+6m+1}{12}} e^{\frac{-m^2}{4}}}{\left(\frac{m(m+1)}{2}\right)^{\frac{m(m+1)}{2}}} \right)^{\frac{2}{m(m+1)}} = 0. \end{aligned}$$

Therefore, the radius of uniform convergence of f is equal to infinity. And so, $f \in H_{b\mathbb{I}}(\ell_1)$.

Next let us prove that the given function f is not defined on the sequence $(c, c, \ldots)$, with $c \in [1, +\infty)$, that is the series $\sum_{m=1}^{\infty} \left(\frac{m}{2}\right)^m c^{\frac{m(m+1)}{2}}$ diverges. It is easy to see that

$$\lim_{m\to\infty} \sqrt[m]{\left(\frac{m}{2}\right)^m c^{\frac{m(m+1)}{2}}} = \lim_{m\to\infty} \frac{m}{2} c^{\frac{m+1}{2}} = \infty.$$

Therefore, by the Cauchy root test, the series $\sum_{m=1}^{\infty} \left(\frac{m}{2}\right)^m c^{\frac{m(m+1)}{2}}$ is not convergent. This completes the proof. $\square$

Proposition 5. *Let* $\{I_1^{(1)}, \ldots, I_m^{(1)}, \ldots\}$ *be the set of polynomials defined by the formula* (2). *Then every function* f *of the form* $f(x) = \sum_{m=1}^{\infty} \alpha(m) I_1^{(1)}(x) \cdot \ldots \cdot I_m^{(1)}(x)$, *where* $\alpha(m)$ *is some coefficient depending on* m, *such that* $f \in H_{b\mathbb{I}}(\ell_1)$, *is well-defined on the element* $x_0 = (1, 1/2, \ldots, 1/n, \ldots)$.

Proof. Let the function $f(x) = \sum_{m=1}^{\infty} \alpha(m) I_1^{(1)}(x) \cdot \ldots \cdot I_m^{(1)}(x)$, where $\alpha(m)$ is some coefficient depending on m, belongs to the algebra $H_{b\mathbb{I}}(\ell_1)$. Note that the nth term of the Taylor series of the function f is defined by the formula

$$f_n(x) = \begin{cases} \alpha(m) I_1^{(1)}(x) \cdot \ldots \cdot I_m^{(1)}(x), & \text{if} \quad n = \frac{m(m+1)}{2} \quad \text{for some} \quad m \in \mathbb{N}, \\ 0, \quad \text{otherwise.} \end{cases} \tag{9}$$

Since the given function f belongs to the algebra $H_{b\mathbb{I}}(\ell_1)$, the radius of uniform convergence of f is equal to infinity. Then $\limsup\limits_{n\to\infty} \|f_n\|_1^{1/n} = 0$ and, consequently, $\lim\limits_{n\to\infty} \|f_n\|_1^{1/n} = 0$. Therefore, taking into account (9), we have that

$$\lim_{m\to\infty} \left\| f_{\frac{m(m+1)}{2}} \right\|_1^{\frac{2}{m(m+1)}} = 0, \quad \text{i.e.} \quad \lim_{m\to\infty} \left\| \alpha(m) I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1^{\frac{2}{m(m+1)}} = 0.$$

Let us make the substitution $\alpha(m) = \frac{\left(\varepsilon(m)\right)^{\frac{m(m+1)}{2}}}{\left\| I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1}$. Then we obtain

$$\lim_{m\to\infty} \left(\frac{|\varepsilon(m)|^{\frac{m(m+1)}{2}}}{\left\| I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1} \left\| I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1 \right)^{\frac{2}{m(m+1)}} = \lim_{m\to\infty} |\varepsilon(m)| = 0.$$

Therefore $\lim\limits_{m\to\infty} \varepsilon(m) = 0$. The latter equality is a necessary and sufficient condition of belonging the given function f to the algebra $H_{b\mathbb{I}}(\ell_1)$.

Let us prove that the function $f(x) = \sum\limits_{m=1}^{\infty} \frac{\left(\varepsilon(m)\right)^{\frac{m(m+1)}{2}}}{\left\| I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1} I_1^{(1)}(x) \cdot \ldots \cdot I_m^{(1)}(x)$, such that $f \in H_{b\mathbb{I}}(\ell_1)$, is well-defined on the element $x_0 = (1, 1/2, \ldots, 1/n, \ldots)$ Note that, taking into account the Corollary 1, we have

$$\begin{aligned} f(x_0) &= \sum_{m=1}^{\infty} \frac{(\varepsilon(m))^{\frac{m(m+1)}{2}}}{\left\| I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1} 1 \cdot \frac{1}{2^2} \cdot \ldots \cdot \frac{1}{m^m} \\ &= \sum_{m=1}^{\infty} \frac{(\varepsilon(m))^{\frac{m(m+1)}{2}}}{\left\| I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1} \frac{1}{H(m)} = \sum_{m=1}^{\infty} \frac{(\varepsilon(m))^{\frac{m(m+1)}{2}} \left(\frac{m(m+1)}{2}\right)^{\frac{m(m+1)}{2}}}{H^2(m)}. \end{aligned}$$

So, we need to show that the series

$$\sum_{m=1}^{\infty} \frac{(\varepsilon(m))^{\frac{m(m+1)}{2}} \left(\frac{m(m+1)}{2}\right)^{\frac{m(m+1)}{2}}}{H^2(m)} \tag{10}$$

converges. Since $f \in H_{b\mathbb{I}}(\ell_1)$, it follows that $\lim\limits_{m\to\infty} \varepsilon(m) = 0$ and so $\left|\varepsilon(m)\right| \le \frac{1}{2}$, when m is large. Taking into account this fact and (8), let us use the Cauchy root test. Then we obtain

$$\lim_{m\to\infty} \sqrt[m]{\frac{\left|\varepsilon(m)\right|^{\frac{m(m+1)}{2}} \left(\frac{m(m+1)}{2}\right)^{\frac{m(m+1)}{2}}}{H^2(m)}} = \lim_{m\to\infty} \frac{\left|\varepsilon(m)\right|^{\frac{m+1}{2}} \left(m^2+m\right)^{\frac{m+1}{2}} e^{\frac{m}{2}}}{A^{\frac{2}{m}} 2^{\frac{m+1}{2}} m^{\frac{12m^2+12m+2}{12m}}} \le \lim_{m\to\infty} \frac{\left(m^2+m\right)^{\frac{m+1}{2}} e^{\frac{m}{2}}}{A^{\frac{2}{m}} 2^{m+1} m^{\frac{12m^2+12m+2}{12m}}} = 0.$$

On the other hand,

$$\lim_{m\to\infty} \sqrt[m]{\frac{\left|\varepsilon(m)\right|^{\frac{m(m+1)}{2}} \left(\frac{m(m+1)}{2}\right)^{\frac{m(m+1)}{2}}}{H^2(m)}} \ge 0.$$

Then, according to the squeeze theorem,

$$\lim_{m\to\infty} \sqrt[m]{\frac{\left|\varepsilon(m)\right|^{\frac{m(m+1)}{2}} \left(\frac{m(m+1)}{2}\right)^{\frac{m(m+1)}{2}}}{H^2(m)}} = 0.$$

Thus, according to the Cauchy root test, the series (10) is convergent.

Finally, let us replace $\frac{\left(\varepsilon(m)\right)^{\frac{m(m+1)}{2}}}{\left\|I_1^{(1)}\cdot\ldots\cdot I_m^{(1)}\right\|_1}$ with $\alpha(m)$. Then we obtain the desired. This completes the proof. □

Example 1. *The function* $f(x) = \sum_{m=1}^{\infty} \left(H(m)\right)^{1-\frac{1}{\log(\log m)}} I_1^{(1)}(x) I_2^{(1)}(x) \ldots I_m^{(1)}(x) \in H_{b\mathbb{I}}(\ell_1)$, *where the polynomials* $I_1^{(1)}, I_2^{(1)}, \ldots$ *are defined by* (2) *and* $H(m)$ *is a hyperfactorial of* $m \in \mathbb{N}$, *defined by* (6), *cannot be extended to the analytic function of bounded type to the space* ℓ_p, $1 < p < \infty$. *But* f *is well-defined on the element* $x_0 = (1, 1/2, \ldots, 1/n, \ldots)$.

Proof. Firstly, let us show that $f \in H_{b\mathbb{I}}(\ell_1)$. To do this, we need to prove that the radius of uniform convergence of f is equal to infinity. Note that the n-th term of the Taylor series of the function f is defined by the formula

$$f_n(x) = \begin{cases} \left(H(m)\right)^{1-\frac{1}{\log(\log m)}} I_1^{(1)}(x) \cdot \ldots \cdot I_m^{(1)}(x), & \text{if} \quad n = \frac{m(m+1)}{2} \quad \text{for some} \quad m \in \mathbb{N}, \\ 0, \quad \text{otherwise}. \end{cases} \tag{11}$$

Taking into account (11) and the Corollary 1, we have

$$\begin{aligned} \left\| f_{\frac{m(m+1)}{2}} \right\|_1^{\frac{2}{m(m+1)}} &= \left\| \left(H(m)\right)^{1-\frac{1}{\log(\log m)}} I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1 \\ &= \left(\left(H(m)\right)^{1-\frac{1}{\log(\log m)}} \left\| I_1^{(1)} \cdot \ldots \cdot I_m^{(1)} \right\|_1 \right)^{\frac{2}{m(m+1)}} \\ &= \left(\left(H(m)\right)^{1-\frac{1}{\log(\log m)}} \right)^{\frac{2}{m(m+1)}} \cdot \frac{\left(H(m)\right)^{\frac{2}{m(m+1)}}}{\frac{m(m+1)}{2}} \\ &= \frac{\left(H(m)\right)^{\left(2-\frac{1}{\log(\log m)}\right)\cdot\frac{2}{m(m+1)}}}{\frac{m(m+1)}{2}}. \end{aligned}$$

Taking into account (8), it can be checked that

$$\lim_{m\to\infty}\left\|f_{\frac{m(m+1)}{2}}\right\|_1^{\frac{2}{m(m+1)}}=\lim_{m\to\infty}\frac{\left(H(m)\right)^{\left(2-\frac{1}{\log(\log m)}\right)\cdot\frac{2}{m(m+1)}}}{\frac{m(m+1)}{2}}$$
$$=\lim_{m\to\infty}\frac{2\left(Am^{\frac{6m^2+6m+1}{12}}e^{\frac{-m^2}{4}}\right)^{\frac{2}{m(m+1)}\left(2-\frac{1}{\log(\log m)}\right)}}{m(m+1)}=0.$$

Therefore $R(f)=\infty$ and $f\in H_{b\mathbb{I}}(\ell_1)$.

Next let us prove that $f\notin H_{b\mathbb{I}}(\ell_p)$, where $p>1$, that is, $R(f)\neq\infty$. Using the Corollary 2, we have

$$\left\|f_{\frac{m(m+1)}{2}}\right\|_1^{\frac{2}{m(m+1)}}=\left(\left(H(m)\right)^{1-\frac{1}{\log(\log m)}}\sqrt[p]{\frac{H(m)}{\left(\frac{m(m+1)}{2}\right)^{\frac{m(m+1)}{2}}}}\right)^{\frac{2}{m(m+1)}}=\frac{2^{\frac{1}{p}}\left(H(m)\right)^{\frac{2}{m(m+1)}\left(1-\frac{1}{\log(\log m)}+\frac{1}{p}\right)}}{\left(m(m+1)\right)^{\frac{1}{p}}}.$$

Then it can be checked that

$$\lim_{m\to\infty}\left\|f_{\frac{m(m+1)}{2}}\right\|_1^{\frac{2}{m(m+1)}}=\lim_{m\to\infty}\frac{2^{\frac{1}{p}}\left(H(m)\right)^{\frac{2}{m(m+1)}\left(1-\frac{1}{\log(\log m)}+\frac{1}{p}\right)}}{\left(m(m+1)\right)^{\frac{1}{p}}}$$
$$=\lim_{m\to\infty}\frac{2^{\frac{1}{p}}\left(Am^{\frac{6m^2+6m+1}{12}}e^{\frac{-m^2}{4}}\right)^{\frac{2}{m(m+1)}\left(1-\frac{1}{\log(\log m)}+\frac{1}{p}\right)}}{\left(m^2+m\right)^{\frac{1}{p}}}=\infty.$$

Therefore $R(f)=0$ and $f\notin H_{b\mathbb{I}}(\ell_p)$, where $p>1$.

Besides, according to the Proposition 5, the given function f is well-defined on the element $x_0=(1,1/2,\ldots,1/n,\ldots)$. □

3 Operations on the set $\tau(M_{b\mathbb{I}})$

Let $\mathbb{P}=\{P_n\}_{n=1}^{\infty}$ be the set of continuous complex-valued n-homogeneous algebraically independent polynomials on X, such that $\|P_n\|_1=1$ for every $n\in\mathbb{N}$. Since every function $f\in H_{b\mathbb{P}}(X)$ can be uniquely represented in the form

$$f(x)=\alpha_0+\sum_{n=1}^{\infty}\sum_{k_1+2k_2+\ldots+nk_n=n}\alpha_{k_1\ldots k_n}P_1^{k_1}(x)P_2^{k_2}(x)\cdots P_n^{k_n}(x),$$

where $\alpha_{k_1\ldots k_n}\in\mathbb{C}$ and $k_1,k_2,\ldots,k_n\in\mathbb{N}\bigcup\{0\}$, then every character $\varphi\in M_{b\mathbb{P}}$ is uniquely determined by its values on the polynomials $\{P_n\}_{n=1}^{\infty}$. Therefore it is interesting to consider the image $\tau(M_{b\mathbb{P}})$ of the spectrum $M_{b\mathbb{P}}$ of the algebra $H_{b\mathbb{P}}(X)$, where the mapping τ is defined by the formula (1). It is also interesting to consider the subset $\tau\big(M_{b\mathbb{P}}^{(0)}\big)$ of the set $\tau\left(M_{b\mathbb{P}}\right)$, where

$$M_{b\mathbb{P}}^{(0)}=\{\delta_x:x\in X\}.\tag{12}$$

In this section, we show that $\tau\big(M^{(0)}_{b\mathbb{P}}\big)$ has a structure of the linear space in the case $X=\ell_p$, $1\le p\le+\infty$, and $\{P_n\}_{n=1}^{\infty}=\big\{I_n^{(p)}\big\}_{n=1}^{\infty}$, where the polynomials $I_1^{(p)}, I_2^{(p)},\ldots$ are defined by (2).

Let $\mathbb{I}=\{I_n\}_{n=1}^{\infty}$. As it was mentioned above, every element $f\in H_{b\mathbb{I}}(\ell_p)$, $1\le p\le+\infty$, has the following representation:

$$f(x)=\alpha_0+\sum_{n=1}^{\infty}\sum_{k_1+2k_2+\ldots+nk_n=n}\alpha_{k_1\ldots k_n}\big(I_1^{(p)}(x)\big)^{k_1}\big(I_2^{(p)}(x)\big)^{k_2}\cdots\big(I_n^{(p)}(x)\big)^{k_n},$$

where $\alpha_{k_1\ldots k_n}\in\mathbb{C}$ and $k_1,k_2,\ldots,k_n\in\mathbb{N}\bigcup\{0\}$.

Let us define the functions $J_\lambda(f):\ell_p\to\mathbb{C}$ and $J_{+c}(f):\ell_p\to\mathbb{C}$ by

$$J_\lambda(f)(x)=\alpha_0+\sum_{n=1}^{\infty}\sum_{k_1+2k_2+\ldots+nk_n=n}\alpha_{k_1\ldots k_n}\left(\lambda I_1^{(p)}(x)\right)^{k_1}\cdots\left(\lambda I_n^{(p)}(x)\right)^{k_n} \tag{13}$$

and

$$J_{+c}(f)(x)=\alpha_0+\sum_{n=1}^{\infty}\sum_{k_1+2k_2+\ldots+nk_n=n}\alpha_{k_1\ldots k_n}\left(I_1^{(p)}(x)+c_1\right)^{k_1}\cdots\left(I_n^{(p)}(x)+c_n\right)^{k_n} \tag{14}$$

for all $\lambda\in\mathbb{C}$ and $c=\{c_n\}_{n=1}^{\infty}=\big\{\varphi\big(I_n^{(p)}\big)\big\}_{n=1}^{\infty}\in\mathbb{C}^{\infty}$, where $\varphi=\delta_z\in M^{(0)}_{b\mathbb{I}}(\ell_p)$ and $z\in\ell_p$ is a fixed element.

Theorem 1. *The functions $J_\lambda(f)$ and $J_{+c}(f)$, defined by* (13) *and* (14)*, are entire functions of bounded type on ℓ_p for every $f\in H_{b\mathbb{I}}(\ell_p)$, $1\le p\le+\infty$. The mappings $J_\lambda: f\mapsto J_\lambda(f)$ and $J_{+c}: f\mapsto J_{+c}(f)$ are the homomorphisms from the algebra $H_{b\mathbb{I}}(\ell_p)$ to itself.*

Proof. Let $f\in H_{b\mathbb{I}}(\ell_p)$, $1\le p\le+\infty$. Firstly, let us show that for every $\lambda\in\mathbb{C}$ the mapping J_λ is well defined. Let us prove that for all $\lambda\in\mathbb{C}$ and $x=(x_1,x_2,\ldots,x_n,\ldots)\in\ell_p$ there exists $y_x\in\ell_p$ such that the equality $I_n^{(p)}(y_x)=\lambda I_n^{(p)}(x)$ holds. It is sufficient to consider the sequence $y_x=(\lambda x_1,\sqrt{\lambda}x_2,\ldots,\sqrt[n]{\lambda}x_n,\ldots)$.

Let us show that $y_x\in\ell_p$. In the case $1\le p<+\infty$ we have

$$\|y_x\|^p=\sum_{n=1}^{\infty}|\sqrt[n]{\lambda}x_n|^p=\sum_{n=1}^{\infty}|\sqrt[n]{\lambda}|^p|x_n|^p\le\big(\max\{|\lambda|,1\}\big)^p\sum_{n=1}^{\infty}|x_n|^p=\big(\max\{|\lambda|,1\}\big)^p\|x\|^p<+\infty,$$

since $x\in\ell_p$. Consequently, $y_x\in\ell_p$. Besides, $\|y_x\|\le\max\{|\lambda|,1\}\|x\|$.

In the case $p=\infty$, we have

$$\|y_x\|=\sup_{n\in\mathbb{N}}\big(|\sqrt[n]{\lambda}||x_n|\big)\le\sup_{n\in\mathbb{N}}\big(\max\{|\lambda|,1\}|x_n|\big)=\max\{|\lambda|,1\}\|x\|<\infty,$$

since $x\in\ell_\infty$, Consequently, $y_x\in\ell_\infty$ and $\|y_x\|\le\max\{|\lambda|,1\}\|x\|$. The fulfillment of the equality $I_n^{(p)}(y_x)=\lambda I_n^{(p)}(x)$ is obvious in both cases. Thus, the given element $y_x\in\ell_p$, where $1\le p\le+\infty$, is desired. Then, taking into account that the function $f\in H_{b\mathbb{I}}(\ell_p)$ is well defined, we can write the following:

$$\begin{aligned}J_\lambda(f)(x)&=\alpha_0+\sum_{n=1}^{\infty}\sum_{k_1+2k_2+\ldots+nk_n=n}\alpha_{k_1\ldots k_n}\left(\lambda I_1^{(p)}(x)\right)^{k_1}\cdots\left(\lambda I_n^{(p)}(x)\right)^{k_n}\\&=\alpha_0+\sum_{n=1}^{\infty}\sum_{k_1+2k_2+\ldots+nk_n=n}\alpha_{k_1\ldots k_n}\left(I_1^{(p)}(y_x)\right)^{k_1}\cdots\left(I_n^{(p)}(y_x)\right)^{k_n}\\&=f(y_x)<\infty.\end{aligned}$$

Therefore the function $J_\lambda(f)$ is well defined.

Next let us show that $J_\lambda(f) \in H_{b\mathbb{I}}(\ell_p)$, $1 \le p \le +\infty$. According to [11, Section 8, Proposition 8.6 and Theorem 8.7] and taking into account that every function of bounded type is locally bounded, it is sufficient to show that $J_\lambda(f)$ is a G-holomorphic function of bounded type.

Firstly, we show that the function $J_\lambda(f)$ is G-holomorphic. According to the definition of a G-holomorphic function (see [11, Section 8, Definition 8.1]) we need to prove that for all $a = \{a_n\}_{n=1}^{\infty}, b = \{b_n\}_{n=1}^{\infty} \in \ell_p$ and $\mu \in \mathbb{C}$ the function $h : \mathbb{C} \to \mathbb{C}$ defined as $h(\mu) = J_\lambda(f)(a + \mu b)$, is entire on $\mathbb{C}$. Let $h_{m+1}(\mu)$ be the $(m+1)$-*st* partial sum of the series corresponding to the function h. It is obvious that $h_{m+1} \in H(\mathbb{C})$. According to the above mentioned the following equalities are true:

$$\begin{aligned} h(\mu) &= \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(\lambda I_1^{(p)}(a+\mu b)\right)^{k_1} \cdots \left(\lambda I_n^{(p)}(a+\mu b)\right)^{k_n} \\ &= \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(I_1^{(p)}(y_{a+\mu b})\right)^{k_1} \cdots \left(I_n^{(p)}(y_{a+\mu b})\right)^{k_n} \\ &= f(y_{a+\mu b}) \end{aligned}$$

and

$$\begin{aligned} h_{m+1}(\mu) &= \sum_{n=0}^{m} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(\lambda I_1^{(p)}(a+\mu b)\right)^{k_1} \cdots \left(\lambda I_n^{(p)}(a+\mu b)\right)^{k_n} \\ &= \sum_{n=0}^{m} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(I_1^{(p)}(y_{a+\mu b})\right)^{k_1} \cdots \left(I_n^{(p)}(y_{a+\mu b})\right)^{k_n} \\ &= f_{m+1}(y_{a+\mu b}). \end{aligned}$$

Let us show that the sequence of the partial sums $\{h_{m+1}\}_{m=0}^{\infty}$ of the series corresponding to the function h uniformly converges to the function h. Let $l \in \mathbb{N}$. Since $f \in H_{b\mathbb{I}}(\ell_p)$, we have

$$\|h - h_{m+1}\|_l = \sup_{|\mu|\le l} |h(\mu) - h_{m+1}(\mu)| \le \sup_{\{y\in\ell_p : \|y\| \le \max\{|\lambda|,1\}(\|a\|+l\|b\|)\}} \left|f(y) - f_{m+1}(y)\right| \to 0$$

as $m \to \infty$. Therefore, since the space $H(\mathbb{C})$ is complete and $h_{m+1} \in H(\mathbb{C})$, then $h \in H(\mathbb{C})$. And so the function $J_\lambda(f)$ is G-holomorphic.

Next, let us show that $J_\lambda(f)$ is the function of bounded type. Let S be an arbitrary bounded set in ℓ_p, that is, there exists a ball $B(a,r)$, where $a \in \ell_p, r > 0$ such that $S \subset B(a,r)$. Let us prove that the function $J_\lambda(f)$ is bounded on S, that is, for all $s \in S$ the estimate $\left|J_\lambda(f)(s)\right| < +\infty$ holds. Let s be an arbitrary element from the set S. Then $\|s\| \le r$. Since $f \in H_{b\mathbb{I}}(\ell_p)$, then the following is true

$$\begin{aligned} \left|J_\lambda(f)(s)\right| = \left|f(y_s)\right| &\le \sup_{\left\{z\in\ell_p : \|z\| \le \|y_s\|\right\}} \left|f(z)\right| \\ &\le \sup_{\left\{z\in\ell_p : \|z\| \le r\max\left\{|\lambda|,1\right\}\right\}} \left|f(z)\right| = \|f\|_{r\max\left\{|\lambda|,1\right\}} < +\infty. \end{aligned}$$

Thus, $J_\lambda(f) \in H_{b\mathbb{I}}(\ell_p)$. It is easy to check that the given mapping J_λ preserves the operations of the algebra $H_{b\mathbb{I}}(\ell_p)$.

Prove the second part of the theorem. Analogically, let $f \in H_{b\mathbb{I}}(\ell_p)$, $1 \leq p \leq +\infty$. Let us fix $z \in \ell_p$ and denote by $c = \{c_n\}_{n=1}^{\infty}$ the sequence of complex numbers such that $\{c_n\}_{n=1}^{\infty} = \{\varphi(I_n^{(p)})\}_{n=1}^{\infty}$, where $\varphi = \delta_z \in M_{b\mathbb{I}}(\ell_p)$.

Firstly, let us show that for every $x \in \ell_p$, $1 \leq p \leq +\infty$, and the given sequence $c = \{c_n\}_{n=1}^{\infty} \in \mathbb{C}^{\infty}$ there exists $y_x \in \ell_p$ such that the equality

$$I_n^{(p)}(y_x) = I_n^{(p)}(x) + c_n$$

holds. Consider the sequence

$$y_x = \left(x_1 + c_1, \sqrt{x_2^2 + c_2}, \ldots, \sqrt[n]{x_n^n + c_n}, \ldots\right).$$

It is obvious that for the given sequence y_x the condition $I_n^{(p)}(y_x) = I_n^{(p)}(x) + c_n$ holds. Let us show that $y_x \in \ell_p$, $1 \leq p \leq +\infty$. Firstly, let us consider the case $1 \leq p < +\infty$. It is easy to see that

$$\begin{aligned}
\left|\sqrt[n]{x_n^n + c_n}\right|^p = \left(\sqrt[n]{|x_n^n + c_n|}\right)^p &\leq \left(\sqrt[n]{|x_n|^n + |c_n|}\right)^p = \left(\sqrt[n]{|x_n|^n + |z_n|^n}\right)^p \\
&\leq \left(\sqrt[n]{(|x_n| + |z_n|)^n}\right)^p = (|x_n| + |z_n|)^p = \left(|x_n| + \sqrt[n]{|c_n|}\right)^p \\
&\leq 2^p \left(\max\left\{|x_n|, \sqrt[n]{|c_n|}\right\}\right)^p \leq 2^p \left(|x_n|^p + \sqrt[n]{|c_n|^p}\right).
\end{aligned}$$

Then

$$\sum_{n=1}^{\infty} |y_n|^p = \sum_{n=1}^{\infty} \left|\sqrt[n]{x_n^n + c_n}\right|^p \leq 2^p \sum_{n=1}^{\infty} \left(|x_n|^p + \sqrt[n]{|c_n|^p}\right) = 2^p \left(\sum_{n=1}^{\infty} |x_n|^p + \sum_{n=1}^{\infty} \sqrt[n]{|c_n|^p}\right) < \infty,$$

since $x, z \in \ell_p$. Therefore $y_x \in \ell_p$, $1 \leq p < +\infty$. Besides,

$$\|y_x\| \leq 2\left(\|x\| + \left(\sum_{n=1}^{\infty} \sqrt[n]{|c_n|^p}\right)^{\frac{1}{p}}\right).$$

Let us denote $L = \left(\sum_{n=1}^{\infty} \sqrt[n]{|c_n|^p}\right)^{\frac{1}{p}}$. Then

$$\|y_x\| \leq 2(\|x\| + L).$$

Next, consider the case $p = \infty$. Since $x, z \in \ell_\infty$, the following is true

$$\begin{aligned}
\|y_x\| = \sup_{n\in\mathbb{N}} \left|\sqrt[n]{x_n^n + c_n}\right| = \sup_{n\in\mathbb{N}} \sqrt[n]{|x_n^n + c_n|} &\leq \sup_{n\in\mathbb{N}} \sqrt[n]{|x_n|^n + |c_n|} = \sup_{n\in\mathbb{N}} \sqrt[n]{|x_n|^n + |z_n|^n} \\
&\leq \sup_{n\in\mathbb{N}} \sqrt[n]{(|x_n| + |z_n|)^n} = \sup_{n\in\mathbb{N}} (|x_n| + |z_n|) \\
&\leq \sup_{n\in\mathbb{N}} |x_n| + \sup_{n\in\mathbb{N}} |z_n| \leq \|x\| + \|z\| < \infty.
\end{aligned}$$

Therefore $y_x \in \ell_\infty$. Let $M = \sup_{n\in\mathbb{N}} |z_n| = \sup_{n\in\mathbb{N}} \sqrt[n]{|c_n|}$. Then

$$\|y_x\| \leq \|x\| + M.$$

It is easy to see that for the given sequence c the mapping J_{+c} is well defined. Indeed, taking into account the above mentioned and that the function $f \in H_{b\mathbb{I}}(\ell_p)$, $1 \le p \le +\infty$, is well defined, we have the desired:

$$\begin{aligned} J_{+c}(f)(x) &= \alpha_0 + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(I_1^{(p)}(x)+c_1\right)^{k_1} \cdots \left(I_n^{(p)}(x)+c_n\right)^{k_n} \\ &= \alpha_0 + \sum_{n=1}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(I_1^{(p)}(y_x)\right)^{k_1} \cdots \left(I_n^{(p)}(y_x)\right)^{k_n} \\ &= f(y_x) < \infty. \end{aligned}$$

Now let us prove that $J_{+c}(f) \in H_{b\mathbb{I}}(\ell_p)$, $1 \le p \le +\infty$. Analogically, we need to show that the mapping $J_{+c}(f)$ is G-holomorphic and bounded on bounded sets. We need to show that for all $a = \{a_n\}_{n=1}^{\infty}, b = \{b_n\}_{n=1}^{\infty} \in \ell_p$ and $\mu \in \mathbb{C}$ the function $h : \mathbb{C} \to \mathbb{C}$ defined as $h(\mu) = J_{+c}(f)(a+\mu b)$, is entire on $\mathbb{C}$, that is, $h \in H(\mathbb{C})$. Let $h_{m+1}(\mu)$ be the $(m+1)$-*st* partial sum of the series corresponding to the function h. It is obvious that $h_{m+1} \in H(\mathbb{C})$. According to the above mentioned the following equalities are true:

$$\begin{aligned} h(\mu) &= \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(I_1^{(p)}(a+\mu b)+c_1\right)^{k_1} \cdots \left(I_n^{(p)}(a+\mu b)+c_n\right)^{k_n} \\ &= \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(I_1^{(p)}(y_{a+\mu b})\right)^{k_1} \cdots \left(I_n^{(p)}(y_{a+\mu b})\right)^{k_n} \\ &= f(y_{a+\mu b}) \end{aligned}$$

and

$$\begin{aligned} h_{m+1}(\mu) &= \sum_{n=0}^{m} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(I_1^{(p)}(a+\mu b)+c_1\right)^{k_1} \cdots \left(I_n^{(p)}(a+\mu b)+c_n\right)^{k_n} \\ &= \sum_{n=0}^{m} \sum_{k_1+2k_2+\ldots+nk_n=n} \alpha_{k_1\ldots k_n} \left(I_1^{(p)}(y_{a+\mu b})\right)^{k_1} \cdots \left(I_n^{(p)}(y_{a+\mu b})\right)^{k_n} \\ &= f_{m+1}(y_{a+\mu b}). \end{aligned}$$

Let us show that the sequence of the partial sums $\{h_{m+1}\}_{m=0}^{\infty}$ of the series corresponding to the function h uniformly converges to h.

Let $l \in \mathbb{N}$. Since $f \in H_{b\mathbb{I}}(\ell_p)$, then in the case $1 \le p < +\infty$, we have

$$\begin{aligned} \|h - h_{m+1}\|_l &= \sup_{|\mu| \le l} \left|h(\mu) - h_{m+1}(\mu)\right| \le \sup_{\left\{y \in \ell_p : \|y\| \le 2(\|a\| + l\|b\| + L)\right\}} \left|f(y) - f_{m+1}(y)\right| \\ &= \left\|f - f_{m+1}\right\|_{2(\|a\|+l\|b\|+L)} \to 0 \qquad \text{as} \quad m \to \infty. \end{aligned}$$

And in the case $p = \infty$, since $f \in H_{b\mathbb{I}}(\ell_\infty)$, we have

$$\begin{aligned} \|h - h_{m+1}\|_l &= \sup_{|\mu| \le l} \left|h(\mu) - h_{m+1}(\mu)\right| \le \sup_{\left\{y \in \ell_\infty : \|y\| \le \|a\| + l\|b\| + M\right\}} \left|f(y) - f_{m+1}(y)\right| \\ &= \left\|f - f_{m+1}\right\|_{\|a\|+l\|b\|+M} \to 0 \qquad \text{as} \quad m \to \infty. \end{aligned}$$

Therefore, since the space $H(\mathbb{C})$ is complete and $h_{m+1} \in H(\mathbb{C})$, then $h \in H(\mathbb{C})$ in all these cases. And so, the function $J_{+c}(f)$ is G-holomorphic for every $f \in H_{b\mathbb{I}}(\ell_p)$, $1 \le p \le +\infty$.

Next, let us show that $J_{+c}(f)$ is the function of bounded type. Let S be an arbitrary bounded set in ℓ_p, $1 \leq p \leq +\infty$, that is, there exists a ball $B(a,r)$, where $a \in \ell_p$ and $r > 0$, such that $S \subset B(a,r)$. Let us prove that the function $J_{+c}(f)$ is bounded on S, that is, for all $s \in S$ the estimate $\left|J_{+c}(f)(s)\right| < +\infty$ holds. Let s be an arbitrary element from the set S. Then $\|s\| \leq r$.

Since $f \in H_{b\mathbb{I}}(\ell_p)$, then in the case $1 \leq p < \infty$, the following is true

$$\left|J_{+c}(f)(s)\right| = \left|f(y_s)\right| \leq \sup_{\left\{z\in\ell_p:\|z\|\leq\|y_s\|\right\}} \left|f(z)\right| \leq \sup_{\left\{z\in\ell_p:\|z\|\leq 2(r+L)\right\}} \left|f(z)\right| = \|f\|_{2(r+L)} < +\infty.$$

And in the case $p = \infty$, since $f \in H_{b\mathbb{I}}(\ell_\infty)$, the following is true

$$\left|J_{+c}(f)(s)\right| = \left|f(y_s)\right| \leq \sup_{\left\{z\in\ell_\infty:\|z\|\leq\|y_s\|\right\}} \left|f(z)\right| \leq \sup_{\left\{z\in\ell_\infty:\|z\|\leq r+M\right\}} \left|f(z)\right| = \|f\|_{r+M} < +\infty.$$

Thus $J_{+c}(f) \in H_{b\mathbb{I}}(\ell_p)$, $1 \leq p \leq \infty$.

It is also easy to check that the given mapping J_{+c} preserves the operations of the algebra $H_{b\mathbb{I}}(\ell_p)$, $1 \leq p \leq \infty$. $\square$

Theorem 1 implies the following corollary.

Corollary 3. *For every $\varphi \in M_{b\mathbb{I}}(\ell_p)$, $1 \leq p \leq \infty$, the following equalities hold:*

$$\left(\varphi \circ J_\lambda\big(I_1^{(p)}\big), \varphi \circ J_\lambda\big(I_2^{(p)}\big), \dots\right) = \lambda\left(\varphi\big(I_1^{(p)}\big), \varphi\big(I_2^{(p)}\big), \dots\right), \tag{15}$$

$$\left(\varphi \circ J_{+c}\big(I_1^{(p)}\big), \varphi \circ J_{+c}\big(I_2^{(p)}\big), \dots\right) = \left(\varphi\big(I_1^{(p)}\big) + c_1, \varphi\big(I_2^{(p)}\big) + c_2, \dots\right). \tag{16}$$

Since the equalities (15) and (16) hold, then it is clear that we can perform the operations of scalar multiplication and addition (note that the second addend must be an image of the point-evaluation functional at points of ℓ_p) on the set $\tau\big(M_{b\mathbb{I}}(\ell_p)\big)$, where the mapping τ is defined by (1).

Corollary 4. *The set $\tau\big(M_{b\mathbb{I}}^{(0)}\big)$, where $M_{b\mathbb{I}}^{(0)}$ is defined by* (12)*, has a structure of the linear space in case $X = \ell_p$, $1 \leq p \leq +\infty$.*

Taking into account the continuity of the operator J_{+c} and [4], we have the following assertion.

Corollary 5. *The operator J_{+c} is hypercyclic.*

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Received 13.05.2023

Василишин С.І. *Спектри алгебр аналітичних функцій, породжених послідовностями поліномів на банахових просторах, та операції на спектрах* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 104–119.

Ми розглядаємо підалгебру алгебри Фреше цілих функцій обмеженого типу, породжену зліченною множиною алгебраїчно незалежних однорідних поліномів на комплексному банаховому просторі X. Ми досліджуємо спектр цієї підалгебри у випадку $X = \ell_1$. Ми також розглядаємо деякі операції зсуву, які здійснюються на спектрі цієї підалгебри у випадку $X = \ell_p$ для $p \geq 1$.

Ключові слова і фрази: n-однорідний поліном, аналітична функція, спектр алгебри.