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On the Dirac-like equation in 7-component space-time and generalized Clifford-Dirac algebra

Simulik V.M.

The generalized Dirac equation related to 7-component space-time with one time coordinate and six space coordinates has been introduced. Three 8-component Dirac equations have been derived from the same 256-dimensional Clifford-Dirac matrix algebra. Corresponding Clifford-Dirac algebra is considered in the Pauli-Dirac representation of 8×8 gamma matrices. It is proved that this matrix algebra over the field of real numbers has 256-dimensional basis and it is isomorphic to geometric $C\ell^{\mathbb{R}}(1,7)$ algebra. The corresponding gamma matrix representation of 45-dimensional $\mathrm{SO}(1,9)$ algebra is derived and the way of its generalization to the $\mathrm{SO}(m,n)$ algebra is demonstrated. The Klein-Gordon equation in 7-component space-time is considered as well. The way of corresponding consideration of the Maxwell equations and of equations for an arbitrary spin is indicated.

Key words and phrases: Clifford-Dirac algebra, $\mathrm{SO}(1,9)$ algebra, 8-component Dirac equation, Fermi-Bose duality, generalized Minkowski space-time.

Institute of Electron Physics, National Academy of Sciences of Ukraine, 21 Universytetska Str., 88017 Uzhgorod, Ukraine
E-mail: vsimulik@gmail.com

Introduction

The main result of this article is as follows. Here we consider three different Dirac equations on the basis of the same 256-dimensional gamma matrix Clifford-Dirac algebra. Two of them are defined in ordinary Minkowski space-time and are isomorphic to each other. The third one is related to 7-component space-time with one time and six space coordinates and is suggested for the first time, see Section 3 below. This new equation may have new applications in contemporary theoretical physics. Probably, for the consideration of dark matter and dark energy as well.

Contemporary theoretical physics is now one step from the centenary of quantum mechanics, Dirac equation [6] and corresponding formalism. We can observe the wide-range applications of the gamma matrix Clifford-Dirac algebra [1, 16], which was introduced in [6] as well. Thirteen times the "International Conference on Clifford Algebras and Their Applications" has been held. The important step is the relationship between the gamma matrix and some from the set of the Lie algebras $\mathrm{SO}(m,n)$ representations [13, 17, 23]. The role of the Lie algebra $\mathrm{SO}(m,n)$ in quantum field theory is also well-known (see, e.g. [5, 7, 10, 12, 28]).

Nevertheless, even slight generalization of the gamma matrix algebra enabled us to prove the existence of bosonic symmetries, solutions and conservation laws for the standard

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Dirac equation (see, e.g., [21–23, 25, 26]). We call this the Fermi-Bose duality property of the Dirac equation. The found algebra is not a pure matrix, it contains additional operators of complex conjugation. In the papers [21–23, 25, 26], it is proved that this matrix algebra over the field of real numbers has 64-dimensional basis and it is isomorphic to $C\ell^{R}(0,6)$ algebra.

The usefulness of the Clifford algebraic approach to the relativistic quantum mechanics recently was demonstrated in [18], where the spin of electron was derived on the basis of start from the Fock space and Clifford algebra, generated by creation-annihilation operators.

Here and in [21–23, 25, 26] we use the following idea. We work not only with matrices but with matrices and operators acting on matrices, and use these operators together with the matrices to generate the higher dimensional gamma matrix algebras being isomorphic to Clifford algebras. As soon as it is the operator of complex conjugation, we deal with algebras over the field of real numbers. An example of how fruitful this idea might be was provided not only in [21–23,25,26], but in the article [30] as well.

In our investigations we essentially used the ideas, methods and main notations of [4,8,15, 29,31,34]. In the proof of relationship between the Clifford and $SO(1,9)$ algebra the results and methods of [8, 13, 17, 21–26] were applied. Our local goal is to find for the 8-component Dirac equation much more interesting algebras as were introduced in [21–23, 25, 26] for the ordinary 4-component Dirac equation.

The ordinary 8-component Dirac equations are considered here in the rigged Hilbert space $S^{3,8} \subset H^{3,8} \subset S^{*3,8}$, where the solutions of such Dirac equations are well-defined. Here the Schwartz test function space $S^{3,8}$ is dense in the Schwartz generalized function space $S^{*3,8}$ and $H^{3,8}$ is the quantum-mechanical Hilbert space of 8-component functions over $R^3 \subset M(1,3)$. The space of coordinates R^3 is the subset of the Minkowski space-time $M(1,3)$.

It is well known that the Dirac equation possesses the solutions from the class of generalized functions, which do not belong to the quantum mechanical Hilbert space. Including these solutions into the formalism leads to the application of the rigged Hilbert space $S^{3,8} \subset H^{3,8} \subset S^{*3,8}$. For the square-root operator equation in the proof of Theorem 2 the situation is similar and similar rigged Hilbert space $S^{6,8} \subset H^{6,8} \subset S^{*6,8}$ is applied. The details on the rigged Hilbert space and its application in field theory were presented in many monographs (see, e.g., [3,32]).

The new Dirac equation from Section 3 is considered in the rigged Hilbert space $S^{6,8} \subset H^{6,8} \subset S^{*6,8}$. This equation is determined in 7-component space-time $M(1,6)$, where $R^6 = R^3 \otimes R^{*3}$. The possible interpretations of the space R^{*3} are briefly considered in the Section 4.

Here for the first time the Pauli-Dirac representation of the 256-dimensional gamma matrix algebra (constructed with the help of additional operators) is suggested (see Section 2 below). It is the Pauli-Dirac representation, which allows us to observe the three Dirac equations on the basis of the same Clifford-Dirac algebra. Such matrix algebra is proved to be 256-dimensional and is shown to be isomorphic to the geometric Clifford algebra $C\ell^{R}(1,7)$ over the field of real numbers. Corresponding representation of $SO(1,9)$ algebra over the field of real numbers is considered and the way of its generalization to the $SO(m,n)$ form is suggested.

1 Matrix representation of the 256-dimensional Clifford algebra $C\ell^{\mathrm{R}}(1,7)$ in the Pauli-Dirac representation

The system of units $\hbar = c = 1$ is chosen, the metric tensor in Minkowski space-time $\mathrm{M}(1,3)$ is given by

$$g^{\mu\nu} = g_{\mu\nu} = g^{\mu}_{\nu}, \quad \left(g^{\mu}_{\nu}\right) = \mathrm{diag}\left(1,-1,-1,-1\right), \quad x_{\mu} = g_{\mu\nu}x^{\mu},$$

and summation over the twice repeated indices is implied.

The ordinary Dirac equation (see, e.g., [4, 22, 29]) for a free non-interacting spinor field is given by

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0,$$

where $x \in \mathrm{M}(1,3)$, $\partial_{\mu} \equiv \partial/\partial x^{\mu}$, $\mu = \overline{0,3}$, $j = 1,2,3$, and

$$\mathrm{M}(1,3) = \left\{x \equiv \left(x^{\mu}\right) = \left(x^{0} = t,\ \overrightarrow{x} \equiv \left(x^{j}\right)\right)\right\}$$

is the Minkowski space-time.

However, ordinary Dirac formalism is not the subject of our consideration. Here we consider the 8-component representations of the Dirac equation. For our purposes the 8×8 gamma matrices should be involved. Note that in the Pauli-Dirac representation the set of gamma matrices below is suggested for the first time.

Consider a set of the following nine 8×8 gamma matrices

$$\Gamma^{0} = \begin{vmatrix} \mathrm{I}_4 & 0 \\ 0 & -\mathrm{I}_4 \end{vmatrix}, \quad \Gamma^{1} = i\Sigma^{1}_{8} = i\begin{vmatrix} 0 & \mathrm{I}_4 \\ \mathrm{I}_4 & 0 \end{vmatrix}, \quad \Gamma^{2} = \begin{vmatrix} 0 & \Sigma^{2}_{4} \\ -\Sigma^{2}_{4} & 0 \end{vmatrix}, \quad \Gamma^{3} = \begin{vmatrix} 0 & \Sigma^{3}_{4} \\ -\Sigma^{3}_{4} & 0 \end{vmatrix},$$
$$\Gamma^{4} = \begin{vmatrix} 0 & 0 & 0 & \sigma^{1} \\ 0 & 0 & \sigma^{1} & 0 \\ 0 & -\sigma^{1} & 0 & 0 \\ -\sigma^{1} & 0 & 0 & 0 \end{vmatrix}, \quad \Gamma^{5} = \begin{vmatrix} 0 & 0 & 0 & \sigma^{2} \\ 0 & 0 & \sigma^{2} & 0 \\ 0 & -\sigma^{2} & 0 & 0 \\ -\sigma^{2} & 0 & 0 & 0 \end{vmatrix}, \quad \Gamma^{6} = \begin{vmatrix} 0 & 0 & 0 & \sigma^{3} \\ 0 & 0 & \sigma^{3} & 0 \\ 0 & -\sigma^{3} & 0 & 0 \\ -\sigma^{3} & 0 & 0 & 0 \end{vmatrix}, \tag{1}$$
$$\Gamma^{7} = \begin{vmatrix} 0 & 0 & \sigma^{2}\widehat{C} & 0 \\ 0 & 0 & 0 & -\sigma^{2}\widehat{C} \\ \sigma^{2}\widehat{C} & 0 & 0 & 0 \\ 0 & -\sigma^{2}\widehat{C} & 0 & 0 \end{vmatrix}, \quad \Gamma^{8} = \begin{vmatrix} 0 & 0 & -i\sigma^{2}\widehat{C} & 0 \\ 0 & 0 & 0 & i\sigma^{2}\widehat{C} \\ -i\sigma^{2}\widehat{C} & 0 & 0 & 0 \\ 0 & i\sigma^{2}\widehat{C} & 0 & 0 \end{vmatrix},$$

where the 4×4 Pauli matrices are given by

$$\Sigma^{2}_{4} = i\begin{vmatrix} 0 & -\mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{vmatrix}, \quad \Sigma^{3}_{4} = \begin{vmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{vmatrix},$$

and the explicit form of Σ^{1}_{8} is evident from the first row in (1). The 2×2 Pauli matrices σ^{j} are taken in the standard form. Further, here $\widehat{C}$ is the operator of complex conjugation. Symbol I_4 denotes 4×4 unit matrix. Note that all Pauli matrices, which for different dimensions are denoted here as Σ_8, Σ_4 and σ, satisfy the corresponding anti-commutation relations of the Pauli algebra generators.

Lemma 1. *The operators* (1) *satisfy the anti-commutation relations*

$$\Gamma^{\mathrm{A}}\Gamma^{\mathrm{B}} + \Gamma^{\mathrm{B}}\Gamma^{\mathrm{A}} = 2g^{\mathrm{AB}}, \quad g = (+--------), \quad \mathrm{A},\mathrm{B} = \overline{0,8}, \tag{2}$$

of matrix representation of the Clifford algebra generators.

Proof. The above asserion is proved by the direct calculation of the relationship (2). Note that here $\widehat{C}$ is the operator of complex conjugation, $\widehat{C}\psi = \psi^*$ (the operator of involution in the Hilbert spaces $\mathrm{H}^{3,2}$ and $\mathrm{H}^{6,2}$). Thus, in the space of real numbers the assertion is proved. □

Theorem 1. *The matrix algebra* (1)*,* (2) *is 256-dimensional* $\left(2^8 = 256\right)$ *and it is isomorphic to Clifford algebra $C\ell^R$(1,7) over the field of real numbers.*

Proof. Lemma 1 is the first necessary step. Further, it is useful to observe that all matrices from (1) are traceless. Furthermore, one can easy calculate that

$$\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\Gamma^6\Gamma^7\Gamma^8 = \mathrm{I}_8,$$

but the verification of relation

$$\Gamma^6 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5 \tag{3}$$

is necessary for further consideration. Note that formula (3) is the analogy of the relation $\gamma^4 = \gamma^0\gamma^1\gamma^2\gamma^3$, which is well-known from the gamma matrix representation of the Clifford algebra $C\ell^{\mathrm{C}}$(1,3). Therefore, our steps below can be similar to the corresponding proof for $C\ell^{\mathrm{C}}$(1,3).

The next step is to prove that all matrices in the set (1) are linearly independent. Indeed, the equation

$$\sum x_{\mathrm{A}}\Gamma^{\mathrm{A}} = 0$$

yields if and only if the condition $x_0 = x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8$ is fulfilled. It may seem that the dimension of this gamma matrix algebra is equal to $2^9 = 512$. Nevertheless, the dimension is $2^8 = 256$. Here the relation (3) (not a matter of fact that Γ^6 is a linearly independent element of the set (1)) plays the decisive role.

In order to prove that the dimension is $2^8 = 256$, let us once more recall the similar situation with ordinary Clifford-Dirac algebra, which is well-known from [4, 11].

For the basic step it is enough to consider the subalgebra, which is determined by the first seven generators from the matrix algebra (1). Let us recalculate the basis of this subalgebra. The solution follows after taking into account the equality (3). Indeed, the candidates for the additional elements of the basis like Γ^6, $\Gamma^0\Gamma^6 = i\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5$, $\Gamma^0\Gamma^1\Gamma^6 = -i\Gamma^0\Gamma^2\Gamma^3\Gamma^4\Gamma^5$, etc., are already among the 64-dimensional set under consideration and do not contribute to the basis of the algebra. Therefore, we deal here with the 64-dimensional subalgebra.

It means that we have only 8 generators for the matrix algebra determined by formulas (1) and (2). After that, it is evident that the dimension of the matrix algebra determined by the formulas (1), (2) is 256. The anti-commutation relations for such algebra should be given as

$$\Gamma^{\bar{\mathrm{A}}}\Gamma^{\bar{\mathrm{B}}} + \Gamma^{\bar{\mathrm{B}}}\Gamma^{\bar{\mathrm{A}}} = 2g^{\bar{\mathrm{A}},\bar{\mathrm{B}}}, \quad g = \mathrm{diag}(+1-1-1-1-1-1-1-1), \tag{4}$$

where $\bar{\mathrm{A}}, \bar{\mathrm{B}} = \overline{0,7}$.

It is easy to understand that this algebra, which in the set (1) includes two additional matrix operators Γ^7 and Γ^8, is defined only over the field of real numbers.

Finally, according to the formalism from papers [2, 15], this matrix algebra is isomorphic to the geometric Clifford algebra $C\ell^{\mathrm{R}}$(1,7). The designations 1 and 7 are related to the number of $+1$ and -1 on the diagonal in (4). Thus, the theorem is proved. □

Similarly, the 64-dimensional subalgebra, which is determined by the first seven generators from the matrix algebra (1), is isomorphic to geometric algebra $C\ell^{\mathrm{C}}$(1,5).

2 Three Dirac equations from one and the same Clifford-Dirac algebra

After the above given consideration it is evident that in (1) we have two linearly independent sets of the gamma matrices $\Gamma^1, \Gamma^2, \Gamma^3$. The first one is given by $\Gamma^1, \Gamma^2, \Gamma^3$ and the second has the form $\Gamma^4 \rightarrow \Gamma^1, \Gamma^5 \rightarrow \Gamma^2, \Gamma^6 \rightarrow \Gamma^3$. Therefore, two equivalent formulations of the 8-component Dirac equation are evident. Who knows if each one will be more useful in future?

The first possible Dirac equation is given by

$$\left(i\Gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0, \tag{5}$$

where $\psi(x)$ is the 8-component wave function and the matrices Γ^{μ} are given as the first four from the set (1).

The second possible Dirac equation is given by

$$\left(i\Gamma^{\alpha}\partial_{\alpha} - m\right)\psi(x) = 0, \tag{6}$$

where the matrices Γ^{α} follows from the set (1) after the substitution $\Gamma^4 \rightarrow \Gamma^1, \Gamma^5 \rightarrow \Gamma^2, \Gamma^6 \rightarrow \Gamma^3$. Therefore, here we have the same Γ^0 but another Γ^j. Of course, the equations (5) and (6) are equivalent.

Nevertheless, now we can introduce the third Dirac-like equation, which follows from the formulas (1), (2) and visually is another one

$$\left(i\Gamma^{\tilde{\mathrm{A}}}\partial_{\tilde{\mathrm{A}}} - m\right)\psi(X) = 0, \quad \tilde{\mathrm{A}} = \overline{0,6}. \tag{7}$$

Here we have first seven gamma matrices from (1), $X \in \mathrm{M}(1,6)$, where

$$\mathrm{M}(1,6) = \left\{X \equiv \left(X^{\tilde{\mathrm{A}}}\right) = \left(X^0 = t,\ \overrightarrow{X} \equiv (X^{\tilde{j}}) \in \mathrm{R}^6\right)\right\}, \quad \tilde{j} = \overline{1,6},$$

is the corresponding Minkowski type space-time. Therefore, here the 8-component function $\psi(X)$ in equation (7) belongs to the rigged Hilbert space $\mathrm{S}^{6,8} \subset \mathrm{H}^{6,8} \subset \mathrm{S}^{6,8*}$. Equation (7) essentially differs from the ordinary 8-component Dirac equations (5), (6). Nevertheless, the Hamiltonian $H = \Gamma^0 \overrightarrow{\Gamma}\overrightarrow{P} + \Gamma^0 m$ in equation (7) has all mathematical properties of the Dirac Hamiltonian.

Consider the general solutions of the equations (5), (6), (7). The general solution of the equation (5) is given by

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \left[e^{-ikx}c^{\mathrm{r}}(\overrightarrow{k})\mathrm{v}_{\mathrm{r}}^{-}(\overrightarrow{k}) + e^{ikx}c^{*\tilde{\mathrm{r}}}(\overrightarrow{k})\mathrm{v}_{\tilde{\mathrm{r}}}^{+}(\overrightarrow{k})\right], \tag{8}$$

where

$$kx = \widetilde{\omega}t - \overrightarrow{k}\overrightarrow{x}, \quad \widetilde{\omega} = \sqrt{\overrightarrow{k}^2 + m^2}, \quad \mathrm{r} = \overline{1,4}, \quad \tilde{\mathrm{r}} = \overline{5,8}, \tag{9}$$

$c^{\mathrm{r}}(\overrightarrow{k})$, $c^{*\tilde{\mathrm{r}}}(\overrightarrow{k})$ are the amplitudes of the particle and antiparticle, respectively, which relation to the quantum-mechanical momentum-spin amplitudes is considered in [18] in details, and the 8-component spinors $\left(\mathrm{v}_{\mathrm{r}}^{-}(\overrightarrow{k}),\ \mathrm{v}_{\tilde{\mathrm{r}}}^{+}(\overrightarrow{k})\right)$ are given by

$$\mathrm{v}_1^-(\overrightarrow{k}) = N\begin{vmatrix} \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ -ik^1+k^3 \\ 0 \\ ik^2 \\ 0 \end{vmatrix}, \quad \mathrm{v}_2^-(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ -ik^1+k^3 \\ 0 \\ ik^2 \end{vmatrix}, \quad \mathrm{v}_3^-(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ -ik^2 \\ 0 \\ -ik^1-k^3 \\ 0 \end{vmatrix},$$

$$\mathrm{v}_4^-(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ -ik^2 \\ 0 \\ -ik^1-k^3 \end{vmatrix}, \quad \mathrm{v}_5^+(\overrightarrow{k}) = N\begin{vmatrix} ik^1+k^3 \\ 0 \\ ik^2 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \mathrm{v}_6^+(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ ik^1+k^3 \\ 0 \\ ik^2 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \end{vmatrix}, \qquad (10)$$

$$\mathrm{v}_7^+(\overrightarrow{k}) = N\begin{vmatrix} -ik^2 \\ 0 \\ ik^1-k^3 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \end{vmatrix}, \quad \mathrm{v}_8^+(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ -ik^2 \\ 0 \\ ik^1-k^3 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \end{vmatrix},$$

where

$$N \equiv \frac{1}{\sqrt{2\widetilde{\omega}(\widetilde{\omega}+m)}}, \quad \widetilde{\omega} \equiv \sqrt{\overrightarrow{k}^{\,2}+m^2}. \qquad (11)$$

The spinors (10) satisfy the relations of the orthonormalization and completeness similar to the corresponding relations for the standard 4-component Dirac spinors.

The general solution of the equation (6) has the form

$$\psi(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k \left[e^{-ikx}c^{\mathrm{r}}(\overrightarrow{k})\bar{\mathrm{v}}_{\mathrm{r}}^-(\overrightarrow{k}) + e^{ikx}c^{*\tilde{\mathrm{r}}}(\overrightarrow{k})\bar{\mathrm{v}}_{\tilde{\mathrm{r}}}^+(\overrightarrow{k})\right]. \qquad (12)$$

The principal difference between (8) and (12) is only in the explicit form of 8-component Dirac spinors, which here are given by

$$\bar{\mathrm{v}}_1^-(\overrightarrow{k}) = N\begin{vmatrix} \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k^3 \\ k^1+ik^2 \end{vmatrix}, \quad \bar{\mathrm{v}}_2^-(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \\ 0 \\ k^1-ik^2 \\ -k^3 \end{vmatrix}, \quad \bar{\mathrm{v}}_3^-(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ k^3 \\ k^1+ik^2 \\ 0 \\ 0 \end{vmatrix},$$

$$\bar{\mathrm{v}}_4^-(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ k^1-ik^2 \\ -k^3 \\ 0 \\ 0 \end{vmatrix}, \quad \bar{\mathrm{v}}_5^+(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ 0 \\ k^3 \\ k^1+ik^2 \\ \widetilde{\omega}+m \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \bar{\mathrm{v}}_6^+(\overrightarrow{k}) = N\begin{vmatrix} 0 \\ 0 \\ k^1-ik^2 \\ -k^3 \\ 0 \\ \widetilde{\omega}+m \\ 0 \\ 0 \end{vmatrix},$$

$$\bar{\mathrm{v}}_7^+(\overrightarrow{k}) = N\begin{vmatrix} k^3 \\ k^1+ik^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \\ 0 \end{vmatrix}, \quad \bar{\mathrm{v}}_8^+(\overrightarrow{k}) = N\begin{vmatrix} k^1-ik^2 \\ -k^3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{\omega}+m \end{vmatrix}.$$

In the solution (12) the interpretation of amplitudes is the same as in the solution (8), other notations are already explained in formulas (9) and (11).

Theorem 2. *The general solution of the equation* (7) *is given by*

$$\psi(X) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^6K\left[e^{-iKX}c^{\mathrm{r}}(\overrightarrow{K})\widetilde{\mathrm{v}}_{\mathrm{r}}^-(\overrightarrow{K}) + e^{iKX}c^{*\tilde{\mathrm{r}}}(\overrightarrow{K})\widetilde{\mathrm{v}}_{\tilde{\mathrm{r}}}^+(\overrightarrow{K})\right],$$

where $K \equiv (\overrightarrow{K}) = \left(k^1, k^2, k^3, k^4, k^5, k^6\right)$, $d^6K \equiv d^1kd^2kd^3kd^4kd^5kd^6k$,

$$KX = \widehat{\omega}t - \overrightarrow{K}\overrightarrow{X}, \quad \widehat{\omega} = \sqrt{\overrightarrow{K}^2 + m^2}, \quad \mathrm{r} = \overline{1,4}, \quad \tilde{\mathrm{r}} = \overline{5,8},$$

$c^{\mathrm{r}}(\overrightarrow{K})$, $c^{*\tilde{\mathrm{r}}}(\overrightarrow{K})$ *are the amplitudes of the particle and antiparticle, respectively, which relation to the quantum-mechanical momentum-spin amplitudes can be considered in complete analogy with the consideration in [22], and the 8-component spinors* $\left(\widetilde{\mathrm{v}}_{\mathrm{r}}^-(\overrightarrow{K}), \widetilde{\mathrm{v}}_{\tilde{\mathrm{r}}}^+(\overrightarrow{K})\right)$ *are given just below after the Corollary 2.*

Proof. The original very simple and very useful procedure of solving of the equation (7) is completely similar to the presented in the monograph [22]. Here below we apply it to the equation (7). Indeed, in [22, Chapter 5] the corresponding solutions are, without any doubt, evidently being postulated on the level of relativistic canonical quantum mechanics and the any-dimensional set of equations of the following type

$$\partial_t f(X) = i\sqrt{m^2 - \Delta_6} f(X), \quad f \equiv \text{column}\left(f^1, f^2, \ldots, f^8\right), \tag{13}$$

where Δ_6 is given in R^6. The solution of this equation have the form

$$f(X) = \left|\begin{array}{c} f_{\text{particle}} \\ f_{\text{antiparticle}} \end{array}\right| = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^6 K e^{-iKX} b^{\mathrm{A}}(\overrightarrow{K}) \mathrm{d}_{\mathrm{A}}, \quad \mathrm{A} = \overline{1,8},$$

where $b^{\mathrm{A}}(\overrightarrow{K})$ are the corresponding quantum-mechanical momentum-spin amplitudes (first four of a particle, next of an antiparticle) and d_{A} are the vectors of 8-component Cartesian basis $\mathrm{d}_{\mathrm{A}} = \{\delta_{\mathrm{A}\dot{\mathrm{B}}}\}$, $\mathrm{A}\dot{\mathrm{B}} = \overline{1,8}$, $\mathrm{d}_1 = \text{column}(1,0,0,0,0,0,0,0)$, etc.

For reasons already mentioned in the Introduction we consider equation (13) in the rigged Hilbert space $\mathrm{S}^{6,8} \subset \mathrm{H}^{6,8} \subset \mathrm{S}^{*6,8}$.

The essence of the second step in the Chapter 6 was given by the help of very simple but very useful operator known from [20], which in the space of 8-component functions has the form

$$v_8 = \left|\begin{array}{cc} \mathrm{I}_4 & 0 \\ 0 & \mathrm{CI}_4 \end{array}\right|, \quad v_8^{-1} = v_8^{\dagger} = v_8, \quad v_8 v_8 = \mathrm{I}_8.$$

Here CI_4 is the 4×4 operator of complex conjugation, which already is determined in the text above exactly. The canonical Foldy-Wouthuysen type classical field formalism was obtained on the basis of this operator. Indeed, operator v_8 translates the quantum-mechanical Hamiltonian into the Foldy-Wouthuysen Hamiltonian and the quantum-mechanical solution into the solution of canonical field theory.

Thus, after the application of the operator v_8, we have

$$\left(\partial_0 + i\Gamma_8^0\sqrt{m^2 - \Delta_6}\right)\phi(X) = 0, \quad \phi \equiv \text{column}\left(\phi^1, \phi^2, \ldots, \phi^8\right),$$

$$\phi(X) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^6 K \left[e^{-ikx} b^{\mathrm{A}}(\overrightarrow{K}) \mathrm{d}_{\mathrm{A}} + e^{ikx} b^{*\mathrm{B}}(\overrightarrow{K}) \mathrm{d}_{\mathrm{B}}\right],$$

where $\mathrm{A} = \overline{1,4}$, $\mathrm{B} = \overline{5,8}$, and the vectors of the 8-component Cartesian basis are already given above.

Note that this transition with the help of operator v_8 is valid only for the anti-Hermitian form of the Hamiltonians, spin operators, etc. (see the explanations in [8,20,22,34]).

The final step in [22, Chapter 7] applies the well-known inverse Foldy-Wouthuysen type operator (here being considered in R^6), which leads to the equation (7) and its solution. In this step the result is independent of the choice of Hermitian or anti-Hermitian form of corresponding operators of field theory.

This formalism in [22] is presented in details.

After the obtaining of the solution presented in the formulation of the theorem everybody can easy verified its validity by the direct substitution into the equation (7). Thus, the theorem is proved. □

Corollary 1. *All transitions presented here have their inverse operators. The inverse procedure is valid as well. Therefore, the quantum-mechanical equation* $i\partial_t f(X) = \sqrt{m^2 - \Delta_6} f(X)$ *can be derived from the Dirac-like equation* (7) *as well.*

Corollary 2. *The Theorem 2 and its proof have the direct relation to the solutions* (8), (12) *of the equations* (5), (6) *given above, where the situation is completely similar. The equation* (7) *contrary to the equations* (5) *and* (6) *is new, while the equations* (5), (6), *are ordinary 8-component Dirac equations in the exotic representations of gamma matrices. Therefore, here as the main object the equation* (7) *is chosen.*

The 8-component Dirac-like spinors in the solution presented in the Theorem 2 have the form

$$\widetilde{\mathrm{v}}_1^-(\overrightarrow{K}) = \widehat{N}\left|\begin{array}{c} \widehat{\omega}+m \\ 0 \\ 0 \\ 0 \\ -iK^1+K^3 \\ 0 \\ iK^2+K^6 \\ K^4+iK^5 \end{array}\right|, \quad \widetilde{\mathrm{v}}_2^-(\overrightarrow{K}) = \widehat{N}\left|\begin{array}{c} 0 \\ \widehat{\omega}+m \\ 0 \\ 0 \\ 0 \\ -iK^1+K^3 \\ K^4-iK^5 \\ iK^2-K^6 \end{array}\right|, \quad \widetilde{\mathrm{v}}_3^-(\overrightarrow{K}) = \widehat{N}\left|\begin{array}{c} 0 \\ 0 \\ \widehat{\omega}+m \\ 0 \\ -iK^2+K^6 \\ K^4+iK^5 \\ -iK^1-K^3 \\ 0 \end{array}\right|,$$

$$\widetilde{\mathrm{v}}_4^-(\overrightarrow{K}) = \widehat{N}\left|\begin{array}{c} 0 \\ 0 \\ 0 \\ \widehat{\omega}+m \\ K^4-iK^5 \\ -iK^2-K^6 \\ 0 \\ -iK^1-K^3 \end{array}\right|, \quad \widetilde{\mathrm{v}}_5^+(\overrightarrow{K}) = \widehat{N}\left|\begin{array}{c} iK^1+K^3 \\ 0 \\ iK^2+K^6 \\ K^4+iK^5 \\ \widehat{\omega}+m \\ 0 \\ 0 \\ 0 \end{array}\right|, \quad \widetilde{\mathrm{v}}_6^+(\overrightarrow{K}) = \widehat{N}\left|\begin{array}{c} 0 \\ iK^1+K^3 \\ K^4-iK^5 \\ iK^2-K^6 \\ 0 \\ \widehat{\omega}+m \\ 0 \\ 0 \end{array}\right|,$$

$$\widetilde{\mathrm{v}}_7^+(\overrightarrow{K}) = \widehat{N}\left|\begin{array}{c} -iK^2+K^6 \\ K^4+iK^5 \\ iK^1-K^3 \\ 0 \\ 0 \\ 0 \\ \widehat{\omega}+m \\ 0 \end{array}\right|, \quad \widetilde{\mathrm{v}}_8^+(\overrightarrow{K}) = \widehat{N}\left|\begin{array}{c} K^4-iK^5 \\ -iK^2-K^6 \\ 0 \\ iK^1-K^3 \\ 0 \\ 0 \\ 0 \\ \widehat{\omega}+m \end{array}\right|,$$

where

$$\widehat{N} \equiv \frac{1}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}}, \quad \widehat{\omega} \equiv \sqrt{\overrightarrow{K}^2 + m^2}.$$

Thus, we have three Dirac equations from the same Clifford-Dirac algebra (1), (2). Note that in order to derive three Dirac equations from the same Clifford-Dirac algebra it is sufficient to consider the 64-dimensional subalgebra, which is determined by the first seven generators from the matrix algebra (1).

Remark. *In the proof of the Theorem 2, so-called square-root operators as Hamiltonian* $H = \sqrt{m^2 - \Delta_6}$ *appeared. Note that the mathematical properties (self-adjointness, symmetrical properties) of such nonlocal pseudo-differential operators are under consideration in literature on such subjects as the Salpeter equation, spinless Salpeter equation, Foldy-Wouthuysen equation, equation for Levy flight [9]. A long list of corresponding references was given in [22, Chapter 5] (see, e.g., the papers of R.A. Weder [33] or of I.W. Herbst [14]). All necessary definitions were given in the formulas (5.3)–(5.7) of this Chapter 5. The details are not the subject of this brief communication.*

3 On the Dirac-like equation in 7-component space-time

The most interesting among the equations (5), (6), (7) is the 8-component Dirac-like equation (7) in 7-component space-time. This equation is completely different in comparison with the ordinary Dirac equations (5), (6) and is introduced here for the first time. This special place among the other investigations can be determined on the basis of the review of different approaches to the Dirac formalism in [22, Chapter 2].

It is useful to note that other equations of elementary particle physics may be represented in a similar way. Consider briefly the Klein-Gordon and the Maxwell equations.

The Klein-Gordon equation in 7-component space time is given by

$$\left(\partial_{\mathrm{A}}\partial^{\mathrm{A}} + m^2\right)\psi(X) = 0, \quad \mathrm{A} = \overline{0,6}. \tag{14}$$

It is easy to understand that equation (7) can be derived from the procedure of factorization of the Klein-Gordon equation (14) as follows

$$-\partial_{\mathrm{A}}\partial^{\mathrm{A}} - m^2 = \left(i\Gamma^{\mathrm{A}}\partial_{\mathrm{A}} - m\right)\left(i\Gamma^{\mathrm{A}}\partial_{\mathrm{A}} + m\right), \quad \mathrm{A}, \mathrm{B} = \overline{0,6}.$$

Therefore, the $\psi(X)$ is the scalar field here (real or complex). The detailed formalism and corresponding solutions without our generalization are described in the monograph [4]. The argument X in $\psi(X)$ belongs to the 7-component $\mathrm{M}(1,6)$ Minkowski space-time.

In addition to the possible derivation of equation (7) from the equation (14) the procedure of Lagrange approach suggestion may be constructed here similarly to the consideration in the book [4]. The specification will be minimal. Many variants of derivation of the ordinary Dirac equation reviewed in the monograph [22] may be used for the derivation of the equation (7) as well.

For the application to the higher spin models we may represent the Klein-Gordon operator in the matrix-differential form. In the case of 8-component function $\psi(X)$ it will be the 8×8 matrix with operator $\partial_{\mathrm{A}}\partial^{\mathrm{A}} + m^2$ in diagonal (all other elements are equal to zero) acting on 8-component column $\psi(X)$. The 8-component Dirac-like wave equation without redundant components for the particle having spin $3/2$ was discussed briefly in the papers [20, 27]. The details were presented in the monograph [22]. The 8-component Klein-Gordon equation is related to that case of 8-component Dirac equation.

Relationships between the massless Dirac and the Maxwell equations were discussed immediately after the appearance of P.A.M. Dirac's paper [6]. The author was C.G. Darwin (see [24] and references therein). Today on the basis of papers [19, 20, 23, 24] we were able to suggest the Maxwell equations in the 7-component Minkowski space-time $\mathrm{M}(1,6)$ as well.

Note that Dirac Hamiltonian is a very specific one, which differs essentially from other Hamiltonians in quantum mechanics and quantum field theory. On the other hand, least action variational principles for the Maxwell equation immediately differ between each other in the terms of potentials and in the terms of electromagnetic field strengths. This brief article is not the place of the long discussion which follows.

The formalism suggested briefly in the article [20] and in details in the monograph [22] for the equations in ordinary 4-component Minkowski space-time is the method for the consideration of arbitrary spin equations in Minkowski space time $\mathrm{M}(1,6)$ as well.

Let us return to the equation (7) and its interpretation.

Equation (7) under consideration couples together two different coordinate spaces with the same time in the form $\mathrm{R}^6 = \mathrm{R}^3 \otimes \mathrm{R}^{*3}$. They can be similar to each other as two duplications of $\mathrm{R}^3 \subset \mathrm{M}(1,3)$, but, of course, can be completely different fragments as $\mathrm{R}^3 \subset \mathrm{M}(1,3)$ and $\mathrm{R}^{*3} \subset \mathrm{M}^*(1,3)$ corresponding to the different physical reality.

Not only the equation (7), but suggested here representations of Clifford and $\mathrm{SO}(m,n)$ algebras as well, may be useful for the supersymmetry (SUSY) approaches. Furthermore, the interpretation of $\mathrm{R}^{*3} \subset \mathrm{M}^*(1,3)$ as the space for dark matter – dark energy and of $\mathrm{R}^3 \subset \mathrm{M}(1,3)$ as our well-known living space is not senseless as well. Note that the time axis in $\mathrm{M}^*(1,3)$ and in $\mathrm{M}(1,3)$ is the same. Therefore, here the Dirac-like equation (7) coupled together these two possibly different space-time manifolds. It will be better to consider the consequences not in the framework of this brief communication, which, in general, is about mathematical aspects of the problem.

Indeed, the popular assertions of many authors that such elementary particles as hypothetical even in Standard model "axions" are the carriers of the dark matter cannot inspire any clearly thinking scientist.

Briefly, the simple question "Why is the Dirac equation one of the general equations of the Universe?" appears. Without any hope to the final conclusion our tiny answer is under consideration. The Dirac equation directly follows from the Clifford algebra. The Clifford algebras follow from the geometry of space-time. Recently, in the paper [18] this chain of ideas has been recalled in the author's original approach.

4 Generalization of the algebra $\mathrm{SO}(1,9)$ for an arbitrary dimensions

Another application of the Clifford-Dirac gamma matrix algebra (1), (2) is as follows. The transition according to [13,17,21,23,25,26] from (1), (2) to the corresponding Lie algebra leads to the gamma matrix representation of the $\mathrm{SO}(1,9)$ algebra over the field of real numbers. The corresponding generators have the form

$$s^{\widehat{\mathrm{A}}\widehat{\mathrm{B}}} = \left\{ s^{\mathrm{AB}} = -\frac{1}{4}\left[\Gamma^{\mathrm{A}}, \Gamma^{\mathrm{B}}\right],\, s^{\mathrm{A}9} = -s^{9\mathrm{A}} = \frac{1}{2}\Gamma^{\mathrm{A}} \right\}, \tag{15}$$

where $\widehat{\mathrm{A}}, \widehat{\mathrm{B}} = \overline{0,9}$, $\mathrm{A}, \mathrm{B} = \overline{0,8}$.

The commutation relations for the 45 generators of (15) are given by

$$\left[s^{\widehat{\mathrm{A}}\widehat{\mathrm{B}}}, s^{\widehat{\mathrm{C}}\widehat{\mathrm{D}}}\right] = -g^{\widehat{\mathrm{A}}\widehat{\mathrm{C}}} s^{\widehat{\mathrm{B}}\widehat{\mathrm{D}}} - g^{\widehat{\mathrm{C}}\widehat{\mathrm{B}}} s^{\widehat{\mathrm{D}}\widehat{\mathrm{A}}} - g^{\widehat{\mathrm{B}}\widehat{\mathrm{D}}} s^{\widehat{\mathrm{A}}\widehat{\mathrm{C}}} - g^{\widehat{\mathrm{D}}\widehat{\mathrm{A}}} s^{\widehat{\mathrm{C}}\widehat{\mathrm{B}}},$$

where the metric tensor g is given as

$$g = \mathrm{diag}(+1 -1 -1 -1 -1 -1 -1 -1 -1 -1).$$

Here the anti-Hermitian realization of the SO$(1,9)$ operators is chosen. The necessity of an anti-Hermitian form of generators was explained in [25, 26]. We used the formalism from [8, 34].

Note the formal character of possible generalization of algebra SO$(1,9)$ for arbitrary dimensions. Indeed, above considered Lie algebra of the SO$(1,9)$ group can be generalized to the SO(m,n) algebra of an arbitrary dimension in the form

$$s^{\breve{p}\breve{q}} = \left\{ s^{pq} = \frac{1}{4}\left[\Gamma^{p},\Gamma^{q}\right],\, s^{p\,n+1} = -s^{n+1\,p} = -\frac{1}{2}\Gamma^{p} \right\}, \tag{16}$$

where $\breve{p},\breve{q} = \overline{0,n+1}$, $p,q = \overline{0,n}$.

Nevertheless, after this step the gamma matrices in the formulas (16) are no more of the dimension of 8×8. Indeed, even for the next step SO$(1,10)$ the 12×12 gamma matrices are necessary. Therefore, the gamma matrices in (16) are not identified at the moment and need step by step identification.

Conclusions

The main goal of this brief communication is the generalization of the 8-component Dirac equation for the 7-component Minkowski space-time M$(1,6)$. This specification of the Dirac equation is considered for the first time. The interesting result is that here from the same 256-dimensional gamma matrix Clifford-Dirac algebra three different 8-component Dirac equations have been derived.

Our next goal is the investigation of the corresponding Clifford-Dirac algebra, which is presented in the terms of 8×8 gamma matrices. In the Pauli-Dirac representation this matrix representation of the geometric Clifford algebra is considered for the first time. This matrix algebra over the field of real numbers is proved to have 256 dimensional basis and to be isomorphic to geometric $C\ell^{\mathbb{R}}(1,7)$ algebra.

In our investigations [21–23,25,26], the matrix representations of the Clifford algebras were only the step in the derivation of corresponding Lie algebras SO(8) and SO$(1,7)$. The gamma matrix representations of these Lie algebras were the basis for the proof of Bose properties of the Dirac equation. Here the corresponding gamma matrix representation of 45-dimensional SO$(1,9)$ algebra is derived. Moreover, the way of its generalization to the SO(m,n) algebra is demonstrated.

The first application of the introduced new equation (7) is its free putting into consideration. Possible applications for theoretical physics are given at the end of the Section 4. In this brief article the application to SUSY is not under consideration.

Another application of the new Dirac-like equation is the direct continuation of our investigations [21, 23, 26] for the case of 8-component Dirac equation and special spin 3/2 equations [20, 27], which can be considered in the form of 8-component Dirac equation as well.

Note that here corresponding to the Dirac-like equation (7) quantum-mechanical and canonical field (of Foldy-Wouthuysen type) equations are already introduced. They are presented in the proof of the Theorem 2.

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Симулик В.М. *Про діракоподібне рівняння у 7-компонентному просторі-часі та узагальнену алгебру Кліффорда-Дірака* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 529–542.

Запропоновано узагальнене рівняння Дірака, що відноситься до 7-компонентного простору-часу з однією часовою координатою і шістьма просторовими координатами. З однієї й тієї самої 256-вимірної матричної алгебри Кліффорда-Дірака виведено три 8-компонентні рівняння Дірака. Така розширена алгебра Кліффорда-Дірака вводиться в розгляд у представленні Паулі-Дірака 8×8 гамма-матриць. Доведено, що ця матрична алгебра над полем дійсних чисел має 256-вимірний базис і є ізоморфною геометричній алгебрі $C\ell^{\mathbb{R}}(1,7)$. Виведено відповідне гамма-матричне представлення 45-вимірної алгебри $\mathrm{SO}(1,9)$ та продемонстровано спосіб його узагальнення до $\mathrm{SO}(m,n)$ алгебри. У 7-вимірному просторі-часі введено у розгляд також і рівняння Клейна-Гордона. Вказано шлях аналогічного розгляду рівнянь Максвела та рівнянь для довільного спіну.

Ключові слова і фрази: алгебра Кліффорда-Дірака, алгебра $\mathrm{SO}(1,9)$, 8-компонентне рівняння Дірака, дуалізм Фермі-Бозе, узагальнений простір-час Мінковського.