ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2024, **16** (2), 427–447
doi:10.15330/cmp.16.2.427-447

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2024, Т.16, №2, С.427–447

Dissipativity of dynamical systems on time scales and the relationship between dissipative differential and dynamical systems

Tsan V.B.✉**, Stanzhytskyi O.M., Kapustian O.A.**

This work is devoted to study the dissipativity property of dynamical systems on time scales and relationship between the dissipativity of systems of dynamic equations on time scales and the corresponding systems of ordinary differential equations. It is established that the dissipativity property is preserved when transitioning from equations on time scales $\mathbb{T}_\lambda$ to the corresponding ordinary differential equations and vice versa, provided that the graininess function μ_λ converges to zero as $\lambda \to 0$.

Key words and phrases: dissipation, time scale, graininess function, Lyapunov function, bounded solution.

Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska str., 01601, Kyiv, Ukraine
✉ Corresponding author
E-mail: `viktoriia.tsan@knu.ua` (Tsan V.B.), `stanzhytskyi@knu.ua` (Stanzhytskyi O.M.), `olenakapustian@knu.ua` (Kapustian O.A.)

Introduction

The theory of dynamic equations on time scales has been a rapidly developing area of mathematics in recent years (see, for example, [2,3] and the references therein). Such equations represent a generalization of difference equations (discrete with a constant step – Euler scale), covering equations with a variable step, or take values from fractal sets [18,19]. These equations are investigated in [12], where the concept of a derivative (Δ-derivative) was introduced on any closed subset in real axis. Such an approach unifies discrete and continuous analyses, as the Δ-derivative transitions to the classical derivative, when the time scale $\mathbb{T} = \mathbb{R}$, and in the case of the Euler scale $\mathbb{T} = \{kn : k \in \mathbb{Z}\}$, it transitions to a difference ratio.

Special interest is drawn to the behavior of the solutions of dynamic equations that are defined on a family of time scales $\mathbb{T}_\lambda$, when the graininess function μ_λ goes to zero as $\lambda \to 0$. In this case, the intervals of the time scale $[t_0, t_1] \cap \mathbb{T}_\lambda$ approach $[t_0, t_1]$. The question naturally arises whether solutions of equations on time scales and the corresponding solutions of differential equations have the same properties. The question of preservation of the boundedness property of solutions is investigated in the works [1, 14, 15]. We also note that the existence of bounded solutions of dynamic equations in the case of Euler time scales has been studied, for example, in the following works [10, 17].The issue of the relationship between the oscillations of such solutions is studied in the works [6, 21]. Similar issues for optimal control problems are considered in [4, 8, 16, 20]. The relationship between the existence of periodic solutions of

УДК 517.9
2020 *Mathematics Subject Classification:* 34N05, 34C10.

systems of dynamic equations on time scales and their corresponding systems of differential equations is studied in article [22].

In this work, we investigate the dissipativity of systems of dynamic equations on time scales and the relationship between the dissipativity of dynamic equation systems on the family of time scales $\mathbb{T}_\lambda$ and their corresponding differential equation systems, under the condition that the graininess function $\mu_\lambda \to 0$ as $\lambda \to 0$.

This paper is organized as follows. In Section 2, we briefly introduce the basic concepts of the time scale theory, and in Section 3, we provide the problem statement and formulate the main results of the article. Section 4 is devoted to auxiliary propositions necessary for proving the main theorems. The main results are proven in Section 5. Section 6 provides an illustration of the main results using an example of a Liénard-type equation.

1 Some concepts of time scale theory

(1) Any nonempty closed subset of the real line is called a *time scale* $\mathbb{T}$. For any subset $A \subset \mathbb{R}$, the corresponding subset of the time scale is defined as $A_\mathbb{T} = A \cap \mathbb{T}$.

(2) For each point t of the time scale $\mathbb{T}_\lambda$, three functions characterizing the scale are defined. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is such that $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$; the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined as $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$; and the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is such that $\mu(t) = \sigma(t) - t$.

(3) According to the properties of the scale at points $t \in \mathbb{T}$, the points of the scale are divided into *left-dense* (LD) if $\rho(t) = t$; *left-scattered* (LS) if $\rho(t) < t$; *right-dense* (RD) if $\sigma(t) = t$; and *right-scattered* (RS) if $\sigma(t) > t$. If the scale $\mathbb{T}$ has a right-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} \setminus M$ is defined; otherwise, $\mathbb{T}^k = \mathbb{T}$.

(4) A function $f : \mathbb{T} \to \mathbb{R}^n$ is called Δ-*differentiable* at $t \in \mathbb{T}^k$ if the limit

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^n$. Then the corresponding value $f^\Delta(t)$ is called the Δ-*derivative* at the point t.

(5) A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd*-continuous if it is continuous at right-dense points of the time scale $\mathbb{T}$ and has a finite limit at left-dense points of this scale.

(6) A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t) f(t) \neq 0$ for all $t \in \mathbb{T}^k$.

(8) If p is regressive, then the generalized exponential function $e_p(t, x)$ is defined using the expression

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for } s, t \in \mathbb{T},$$

where $\xi_h(z)$ is a cylinder transformation. The cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ is defined as

$$\xi_h(z) = \frac{1}{h}\mathrm{Log}(1 + zh),$$

where Log is the principal logarithm function.

Theorem 1. *If p, q are regressive, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$*;*

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*.*

Theorem 2. *If p is regresive, $a, b, c \in \mathbb{T}$, then*

$$[e_p(c,\cdot)]^{\Delta} = -p[e_p(c,\cdot)]^{\sigma}$$

and

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

2 Problem statement and main results

Let D be a domain in $\mathbb{R}^n$. We consider the system of ordinary differential equations

$$\frac{dx}{dt} = X(t,x) \tag{1}$$

with $t \in \mathbb{R}$, $x \in D$, and the corresponding system of dynamic equations

$$x_\lambda^{\Delta} = X(t,x_\lambda) \tag{2}$$

on the family of time scales $\mathbb{T}_\lambda$, where $t \in \mathbb{T}_\lambda$, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^n$, and $x_\lambda^{\Delta}(t)$ is Δ-derivative of $x_\lambda(t)$ on $\mathbb{T}_\lambda$. We assume that $\inf \mathbb{T}_\lambda = -\infty$, $\sup \mathbb{T}_\lambda = \infty$, $\lambda \in \Lambda \subset \mathbb{R}$, and $\lambda = 0$ is a limit point of Λ. Here and in the following, we will consider time scales $\mathbb{T}_\lambda$ with $\lambda > 0$ and the point $t = 0$ belongs to $\mathbb{T}_\lambda$ for all $\lambda \in \Lambda$.

We also assume that the function $X(t,x)$ is defined for all $t \geq 0$, $x \in D$, it is continuous with respect to the variables t and x and has bounded partial derivatives with respect to t and x in every bounded domain of $\{t \geq 0\} \times D$, i.e. for every $M > 0$ there exists $L(M)$ such that

$$|X(t,x)| + \left|\frac{\partial X(t,x)}{\partial t}\right| + \left\|\frac{\partial X(t,x)}{\partial x}\right\| \leq L(M), \tag{3}$$

if $t \leq M$ and $\|x\| \leq M$. Here $|\cdot|$ is Euclidean norm in $\mathbb{R}^n$, $\|\cdot\|$ is a matrix norm induced by the vector norm. From inequality (3) it follows that there exist locally integrable functions $M_R(t)$ and $B_R(t)$ such that

$$|X(t,x)| \leq M_R, \tag{4}$$

$$|X(t,x_1) - X(t,x_2)| \leq B_R|x_2 - x_1| \tag{5}$$

for $x, x_i \in U_R$. Here and throughout, U_R denotes the set of points x such that $\|x\| \leq R$.

We set $\mu_\lambda := \sup_{t \in \mathbb{T}_\lambda} \mu_\lambda(t)$, where $\mu_\lambda(t) : \mathbb{T}_\lambda \to [0,\infty)$ is the graininess function. If $\mu_\lambda \to 0$ as $\lambda \to 0$, then $\mathbb{T}_\lambda$ approaches the continuous time scale $\mathbb{T}_0 = \mathbb{R}$, and the system (2) transforms into the system (1). Therefore, it is natural to expect that under certain conditions, the dissipativity of the differential equation system (1) implies the dissipativity of the corresponding dynamic equation system (2) on the time scale $\mathbb{T}_\lambda$.

Dissipativity of system (1) will be understood in the next sense.

Definition 1 ([23]). *System* (1) *is called dissipative with respect to* $t \geq t_0$*, if there exists* $R > 0$ *such that for all* $r > 0$ *there exists* $T = T(r, t_0)$ *for which the solution* $x(t; t_0, x_0)$ *of system* (1) *with initial conditions* (t_0, x_0) *satisfying*

$$|x_0| < r \tag{6}$$

satisfies the inequality

$$\|x(t, t_0, x_0)\| < R$$

for $t \geq t_0 + T$.

Definition 2 ([23]). *System* (1) *is called uniformly dissipative in* t_0 *if, in Definition 1,* T *is independent of* t_0.

We define dissipativity for system (2) analogously.

Definition 3. *System* (2) *is called dissipative with respect to* $t \in [0, +\infty)_{\mathbb{T}_\lambda}$*, if there exists* $R(\lambda) > 0$ *such that for all* $r > 0$ *there exists* $T = T(r, t_0, \lambda)$ *for which the solution* $x_\lambda(t, t_0, x_0)$ *of system* (2) *with initial conditions* (t_0, x_0) *satisfying*

$$|x_0| < r \tag{7}$$

satisfies the inequality

$$\|x_\lambda(t, t_0, x_0)\| < R$$

for $t \geq t_0 + T$, $t \in \mathbb{T}_\lambda$.

Definition 4. *System* (2) *is called uniformly dissipative in* $t_0 \in \mathbb{T}_\lambda$ *and* $\lambda \leq \lambda_0$ *if, in Definition 3,* R *and* T *are independent of* t_0 *and* λ.

We start with the conditions of dissipativity of the system of dynamic equations (2) in terms of the Lyapunov function $V(t, x)$.

Regarding all Lyapunov functions that we will consider further, we assume that $V_\lambda(t, x)$ is Δ-absolutely continuous [9] with respect to t and uniformly continuous with respect to x in a neighborhood of each point. Additionally, it satisfies a local Lipschitz condition with respect to x for each $0 < \lambda \leq \lambda_0$ in the domain $\{t \in [0, T]_{\mathbb{T}_\lambda}\} \times U_R$ with a Lipschitz constant depending on R and T. This fact will be denoted as $V \in \mathbf{C}_0$.

Definition 5. *The Lyapunov operator corresponding to system* (2) *will be denoted as* $d^0/\Delta t$*, defined by the relation*

$$\frac{d^0 V(t, x)}{\Delta t} = \overline{\lim_{t \to t_0 + 0, t \in \mathbb{T}_\lambda}} \frac{1}{t - t_0} \left[V(t, x_\lambda(t, t_0, x_0)) - V(t_0, x_0) \right].$$

From [7] it follows the next comment.

Remark 1. *If* $V(t, x) \in \mathbf{C}_0$ *then for almost all* t *the Lyapunov operator will coincide with the* Δ*-derivative of the function* V *in system* (2)*, which denoted as* $\overset{\Delta}{V}$.

Then the following theorem holds.

Theorem 3. *If system* (2) *on time scale* $\mathbb{T}_\lambda$, $\lambda \geq 0$, *has a nonnegative Lyapunov function* $V(t,x) \in \mathbf{C}_0$, *defined for* $t \geq t_0$, $t \in \mathbb{T}_\lambda$, $x \in D \subset \mathbb{R}^n$, *with the following properties:*

1)
$$\inf_{t\in[t_0,\infty)_{\mathbb{T}_\lambda}, \|x\|\geq\rho} V(t,x) = V_\rho(\lambda) \to \infty, \quad \rho \to \infty, \tag{8}$$

2) for $x \in \bar{U}_{R_0} = \{\|x\| \geq R_0, t \geq t_0\}$ *there exists* $C = C(\lambda) > 0$ *such that*
$$\overset{\Delta}{V}(t,x) \leq -C(\lambda)V(t,x), \quad t \in \mathbb{T}_\lambda, \tag{9}$$
and for $x \in U_{R_0}$ *functions* V *and* $\overset{\Delta}{V}(t,x)$ *are bounded above,*

then system (2) *is dissipative.*

Remark 2. *A similar result to Theorem 3, under different conditions and using a different method, was obtained in [1, Theorem 3.4].*

Remark 3. *If in the conditions of Theorem 3* $V(t,x)$ *and* C *do not depend on* λ *and relation* (8) *holds uniformly over* $\lambda \leq \lambda_0$, *then system* (2) *is uniformly dissipative.*

We obtain an inverse result.

Theorem 4. *If there exists* $\lambda_0 > 0$ *such that system* (2) *is dissipative for every* $\lambda \leq \lambda_0$ *and conditions* (4), (5) *are satisfied, then for each system* (2) *there exists a non-negative function* V *satisfying conditions* (8), (9) *for* $\lambda < \lambda_0$.

Studying the dissipativity of the system of dynamic equations, we also investigate the conditions under which the dissipativity of the system of dynamic equations implies the existence of a similar property in the corresponding system of differential equations, as well as the inverse result.

Theorem 5. *Let* $X(t,x)$ *satisfies condition* (3) *and there exists* λ_0 *such that for all* $\lambda \leq \lambda_0$ *the system of dynamic equations* (2) *is uniformly dissipative with respect to* $t_0 \in \mathbb{T}_\lambda$ *and* λ. *Then system* (1) *is uniformly dissipative with respect to* t_0 *for* $t_0 > 0$.

Theorem 6. *Let* $X(t,x)$ *satisfies condition* (3) *and system* (1) *is uniformly dissipative with respect to* t_0 *for* $t_0 > 0$. *Then there exists* λ_0 *such that the dynamic system* (2) *is uniformly dissipative with respect to* t_0 *and* λ *for all* $\lambda \leq \lambda_0$.

3 Auxiliary results

In this section, we present several auxiliary results necessary for proving the main theorems. The following lemma will be applied to study the dissipativity conditions of system (2).

Lemma 1. *Let* $t_0 \in \mathbb{T}_\lambda$, $y_\lambda : \mathbb{T}_\lambda \to \mathbb{R}$. *If* $y_\lambda(t)$ *is a function defined for* $t \geq t_0$, *whose* Δ*-derivative* y_λ^Δ *satisfies the inequality*
$$y_\lambda^\Delta < A(t)y_\lambda + B(t) \tag{10}$$
for almost all $t \geq t_0$, *where* $A(t), B(t) \in C_{rd}(\mathbb{T})$ *and* $1 + \mu_\lambda(t)A(t) > 0$ *for all* $t \in \mathbb{T}_\lambda$, *then for* $t \geq t_0$, *the inequality*
$$y_\lambda(t) < y_\lambda(t_0)e_A(t,t_0) + \int_{t_0}^{t} e_A(t,\sigma(\tau))B(\tau)\Delta\tau$$
holds.

Proof. Inequality (10) can be equivalently rewritten as

$$y_\lambda(t)^\Delta < A(t)\big(y_\lambda(\sigma(t)) - \mu_\lambda(t)y_\lambda^\Delta(t)\big) + B(t).$$

Thus

$$y^\Delta(t)(1+\mu_\lambda(t)A(t)) < A(t)y_\lambda(\sigma(t)) + B(t).$$

From the statement of the lemma we get $1+\mu_\lambda(t)A(t) \geq 0$, then

$$y_\lambda^\Delta(t) < \frac{A(t)}{1+\mu_\lambda(t)A(t)}y_\lambda(\sigma(t)) + \frac{B(t)}{1+\mu_\lambda(t)A(t)}.$$

Since $\frac{-A(t)}{1+\mu_\lambda(t)A(t)} = (\ominus A)(t)$, we have

$$y_\lambda^\Delta(t) < -(\ominus A)(t)y_\lambda(\sigma(t)) + \frac{B(t)}{1+\mu_\lambda(t)A(t)}.$$

Multiplying by the generalized exponential function both sides of $e_{\ominus A}(t,t_0)$ we obtain

$$e_{\ominus A}(t,t_0)y_\lambda^\Delta(t) < -e_{\ominus A}(t,t_0)(\ominus A)(t)y_\lambda(\sigma(t)) + e_{\ominus A}(t,t_0)\frac{B(t)}{1+\mu_\lambda(t)A(t)}.$$

Then we get

$$(e_{\ominus A}(\cdot,t_0)y_\lambda)^\Delta(t) = e_{\ominus A}(t,t_0)y_\lambda^\Delta(t) + e_{\ominus A}(t,t_0)(\ominus A)(t)y_\lambda(\sigma(t)) < e_{\ominus A}(t,t_0)\frac{B(t)}{1+\mu_\lambda(t)A(t)}.$$

Integrating both sides of this last inequality from t_0 to t, we obtain

$$e_{\ominus A}(t,t_0)y_\lambda(t) - e_{\ominus A}(t_0,t_0)y_\lambda(t_0) < \int_{t_0}^{t} e_{\ominus A}(\tau,t_0)\frac{B(\tau)}{1+\mu_\lambda(\tau)A(\tau)}\Delta\tau.$$

Using Theorem 1 *(i)*, we have $e_{\ominus A}(t_0,t_0) \equiv 1$ and it can be shown that

$$e_{\ominus A}(t,t_0)y_\lambda(t) < y_\lambda(t_0) + \int_{t_0}^{t} e_{\ominus A}(\tau,t_0)\frac{B(\tau)}{1+\mu_\lambda(\tau)A(\tau)}\Delta\tau,$$
$$y_\lambda(t) < \frac{1}{e_{\ominus A}(t,t_0)}y_\lambda(t_0) + \int_{t_0}^{t} \frac{e_{\ominus A}(\tau,t_0)}{e_{\ominus A}(t,t_0)}\frac{B(\tau)}{1+\mu_\lambda(\tau)A(\tau)}\Delta\tau.$$

Therefore, by the properties of exponential function (Theorem 1 *(v)*), we get

$$y_\lambda(t) < \frac{1}{e_{\ominus A}(t,t_0)}y_\lambda(t_0) + \int_{t_0}^{t} e_{\ominus A}(t,\tau)\frac{B(\tau)}{1+\mu_\lambda(\tau)A(\tau)}\Delta\tau.$$

Since $\frac{1}{e_{\ominus A}(t,t_0)} = e_A(t,t_0)$ and $\frac{e_{\ominus A}(t,\tau)}{1+\mu_\lambda(\tau)A(\tau)} = \frac{e_A(t,\tau)}{e_A(\sigma(\tau),\tau)} = e_A(t,\sigma(\tau))$ (Theorem 1 (ii)–(iv)), we get the desired result

$$y_\lambda(t) < e_A(t,t_0)y_\lambda(t_0) + \int_{t_0}^{t} e_A(t,\sigma(\tau))B(\tau)\Delta\tau.$$

□

Lemma 2. *Assume that* $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$*,* $V \in C^1(\mathbb{T} \times \mathbb{R}^n)$ *and* $x : [0, T]_\mathbb{T} \to \mathbb{R}^n$ *is delta differentiable on* $\mathbb{T}^k$*. Let* $z : [0, T]_\mathbb{T} \to \mathbb{R}$*,* $z(\cdot) = V(\cdot, x(\cdot))$*, then* z *is delta differentiable at* t *and*

$$z^\Delta(t_0) = \frac{\partial V}{\Delta t}(t_0, x_0) + F(\sigma(t_0), x) \cdot x^\Delta(t_0),$$

where $F(\sigma(t_0), x) = (F_1(\sigma(t_0), x), \dots, F_n(\sigma(t_0), x))$ *and*

$$F_i(\sigma(t_0), x) = \int_0^1 \frac{\partial V}{\partial x_i}(\sigma(t_0), x_1(\sigma(t_0)), \dots, x_{i-1}(\sigma(t_0)), x_i + h\mu(t_0)x_i^\Delta(t_0), x_{i+1}(t_0), \dots, x_n(t_0))dh.$$

Proof. Fix $t_0 \in [0, T]^k$. First we consider the case, when t_0 is right-dense. In this case $\sigma(t_0) = t_0$, x is delta differentiable and continuous at t_0, and

$$\lim_{(t,x)\to(t_0,x_0)} \alpha_1((t_0, x_0), (t, x)) = \lim_{(t,x)\to(t_0,x_0)} \beta_{ij}((t_0, x_0), (t, x)) = 0.$$

Using the definition of completely delta differentiable at point function [5, Def. 6.97], we have

$$\begin{aligned}
z^\Delta(t_0) &= \lim_{t\to t_0} \frac{V(t, x) - V(t_0, x_0)}{t - t_0} \\
&= \lim_{t\to t_0} \frac{A(t - t_0) + \sum_{i=1}^n B_i(x_i(t) - x_i(t_0)) + \alpha_1(t - t_0) + \sum_{i=1}^n \beta_{i1}((x_i(t) - x_i(t_0)))}{t - t_0} \\
&= \frac{\partial V}{\Delta t}(t_0, x_0) + \sum_{i=1}^n \frac{\partial V}{\partial x_i}(t_0, x_0) \cdot x_i^\Delta(t_0).
\end{aligned}$$

Let $F_i(t, x) = \frac{\partial V}{\partial x_i}(t, x)$ and $F(t, x) = (F_1(t, x), F_2(t, x), \dots, F_n(t, x))$, then

$$z^\Delta(t_0) = \frac{\partial V}{\Delta t}(t_0, x_0) + F(t_0, x) \cdot x^\Delta(t_0).$$

Next consider the other case, when t_0 is right-scattered. Let $x(\sigma(t_0)) = x(\sigma_0)$, then we have $x(\sigma_0) - x(t_0) = \mu(t_0)x^\Delta(t_0)$ and

$$\begin{aligned}
z^\Delta(t_0) &= \frac{V(\sigma_0, x(\sigma_0)) - V(t_0, x(t_0))}{\mu(t_0)} \\
&= \sum_{i=1}^n \frac{V(\sigma_0, x_1(\sigma_0), \dots, x_i(\sigma_0), x_{i+1}(t_0), \dots, x_n(t_0))}{x_i(\sigma_0) - x_i(t_0)} \\
&\quad - \frac{V(\sigma_0, x_1(\sigma_0), \dots, x_{i-1}(\sigma_0), x_i(t_0), \dots, x_n(t_0))}{x_i(\sigma_0) - x_i(t_0)} x_i^\Delta(t_0) + \frac{V(\sigma_0, x(t_0)) - V(t_0, x(t_0))}{\mu(t_0)}.
\end{aligned}$$

Since $V \in C^1([0, T] \times \mathbb{R}^n)$, by applying the mean value theorem, we obtain

$$\begin{aligned}
F_i(\sigma_0, x) &= \frac{V(\sigma_0, x_1(\sigma_0), \dots, x_i(\sigma_0), x_{i+1}(t_0), \dots, x_n(t_0))}{x_i(\sigma_0) - x_i(t_0)} \\
&\quad - \frac{V(\sigma_0, x_1(\sigma_0), \dots, x_{i-1}(\sigma_0), x_i(t_0), \dots, x_n(t_0))}{x_i(\sigma_0) - x_i(t_0)} \\
&= \int_0^1 \frac{\partial V}{\partial x_i}(\sigma_0, x_1(\sigma_0), \dots, x_{i-1}(\sigma_0), x_i + h(x_i(\sigma_0) - x_i(t_0)), x_{i+1}(t_0), \dots, x_n(t_0))dh \\
&= \int_0^1 \frac{\partial V}{\partial x_i}(\sigma_0, x_1(\sigma_0), \dots, x_{i-1}(\sigma_0), x_i + h\mu(t_0)x_i^\Delta(t_0), x_{i+1}(t_0), \dots, x_n(t_0))dh.
\end{aligned}$$

Therefore,

$$z^{\Delta}(t_0) = \frac{\partial V}{\Delta t}(t_0, x_0) + \sum_{i=1}^{n} F_i(\sigma_0, x) x_i^{\Delta}(t_0).$$

We set $F(t, x) := (F_1(t, x), F_2(t, x), \ldots, F_n(t, x))$ to obtain

$$z^{\Delta}(t_0) = \frac{\partial V}{\Delta t}(t_0, x_0) + F(\sigma_0, x) \cdot x^{\Delta}(t_0).$$

□

Remark 4. *Given system* (2)*, we consider the function* $\overset{\Delta}{V}(t, x)$ *defined as*

$$\overset{\Delta}{V}(t, x) = \frac{\partial V}{\Delta t}(t, x) + \sum_{i=1}^{n} F_i(\sigma(t), x_\lambda) X_i(t, x_\lambda) = \frac{\partial V}{\Delta t}(t, x) + F(\sigma(t), x_\lambda) \cdot X(t, x_\lambda).$$

The function $\overset{\Delta}{V}(t, x)$ *represents the* Δ*-derivative of the function* $V(t, x)$ *according to system* (2)*.*

Lemma 3 ([15])**.** *Let* $t_0 \in \mathbb{T}_\lambda$*,* $t_0 + T \in \mathbb{T}_\lambda$*,* x_λ *and* x *are the solutions of* (2) *and* (1) *on* $[t_0, t_0 + T]$ *and on* $[t_0, t_0 + T]_{\mathbb{T}_\lambda}$*, respectively. Then if the initial conditions* $x(t_0) = x_\lambda(t_0) = x_0$*,* $x_0 \in D$*, are satisfied, then*

$$|x(t) - x_\lambda(t)| \le \mu(\lambda) K(T)$$

holds, where $\mu(\lambda) = \sup_{t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}} \mu_\lambda(t)$ *and* $K(T) = e^{C(T+1)} \left(C + \frac{C^2 T}{4} \right) + 3C$ *is constant.*

4 Proofs of main results

Proof of Theorem 3. Indeed, according to the condition (9) we have

$$\overset{\Delta}{V}(t, x) \le -CV(t, x) \quad \text{for } t \ge t_0, \|x\| \ge R_0,$$

and if $\|x\| < R_0$, there exists positive constant $C_1 > 0$ such that

$$\overset{\Delta}{V}(t, x) \le C_1, \quad V(t, x) < C_1. \tag{11}$$

Thus, from (9) and (11) we have the next inequality

$$\overset{\Delta}{V}(t, x) \le -CV(t, x) + C_2 \quad \text{for all } t \ge t_0 \text{ and } x \in \mathbb{R}^n, \tag{12}$$

where $C_2 > 0$ is a certain constant.

Now using Lemma 1 to inequality (12) and Theorem 2, we obtain

$$\begin{aligned} V(t, x) &\le V(t_0, x_0) \cdot e_{-C}(t, t_0) + \int_{t_0}^{t} C_2 e_{-C}(t, \sigma(\tau)) \Delta\tau \\ &= V(t_0, x_0) \cdot e_{-C}(t, t_0) + \frac{C_2}{C} (1 - e_{-C}(t, t_0)) \\ &\le V(t_0, x_0) \cdot e_{-C}(t, t_0) + \frac{C_2}{C} \le e_{-C}(t, t_0) \sup_{|x_0| < r} V(t_0, x_0) + \frac{C_2}{C}. \end{aligned}$$

Hence, there exists $T = T(t_0, r, \lambda)$ such that for $t \ge t_0 + T$ the inquality

$$V(t, x) \le C_3$$

holds. From here according to the condition (8), we get dissipativity of system (2) for all $\lambda > 0$. □

Proof of Theorem 4. From the dissipativity of (2) it follows that for any $r > 0$ there exists $T = T(r, t_0, \lambda) > 0$ such that the solution x_λ of (2) with initial conditions (t_0, x_0), $t_0 \in [0, T_0]_{\mathbb{T}_\lambda}$ and x_0 satisfying $|x_0| < r$ for $t \in [t_0 + T; +\infty)_{\mathbb{T}_\lambda}$ is contained in the sphere of radius R, i.e. $|x_\lambda(t; t_0, x_0)| < R$.

Let us consider the next function

$$G(\zeta) = \begin{cases} \zeta - R, & \text{if } \zeta \geq R; \\ 0, & \text{if } 0 \leq \zeta < R. \end{cases}$$

This function is continuous and takes only non-negative values. Moreover, it is defined for $\zeta \geq 0$, $G(\zeta) \to +\infty$ as $\zeta \to +\infty$ and

$$|G(\zeta) - G(\zeta')| \leq |\zeta - \zeta'|. \tag{13}$$

We define $V(t, x)$ as follows

$$V(t, x) = \sup_{\tau \geq 0} \left\{ G(\|x_\lambda(t_\tau; t, x)\|) \cdot e^\tau \right\},$$

where $t_\tau = \sigma(t + \tau) = \inf\{s \in \mathbb{T}_\lambda : s \geq t + \tau\}$.

Note that if $\tau > T$ then, according to the dissipative nature of system (2), the solution x_λ enters a ball of radius R, implying that $G(\|x\|) = 0$. Hence,

$$V(t, x) = \sup_{\tau \in [0, T]} \left\{ G(\|x_\lambda(t_\tau; t, x)\|) \cdot e^\tau \right\}.$$

From the definition of G, we get $G(\|x\|) \leq V(t, x)$. Therefore, $V(t, x)$ satisfies the condition (8).

Now let us show that that $V(t, x)$ satisfies the local Lipschitz condition with respect to variables t and x. Suppose (t, x) and $(\hat{t}, \hat{x})$ such that $t, \hat{t} \in [0, T_0]_{\mathbb{T}_\lambda}$, $t < \hat{t}$, and x, $\hat{x}$ are included in a ball of radius r.

Due to continuous dependence of solution of system (2) on initial data on a finite interval of time scale [13, Theorem 3.2] we can guarantee that for all $\tau \in [0, T]$ for any initial conditions $t, \hat{t} \in [0, T_0]_{\mathbb{T}_\lambda}$ and $x, \hat{x} \in U_r$ solutions $x_\lambda(t_\tau; t, x)$ and $x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})$ are included in a ball of the fixed radius r. Then there exists N_r such that $\|x_\lambda(t_\tau; t, x)\| \leq N_r$, $\|x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| \leq N_r$. Hence, by (13), we have

$$\begin{aligned} V(t, x) - V(\hat{t}, \hat{x}) &= \sup_{\tau \in [0, T]} \left\{ G(\|x_\lambda(t_\tau; t, x)\|) e^\tau \right\} - \sup_{\tau \in [0, T]} \left\{ G(\|x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\|) e^\tau \right\} \\ &\leq \sup_{\tau \in [0, T]} \left\{ |G(\|x_\lambda(t_\tau; t, x)\|) - G(\|x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\|)| e^\tau \right\} \\ &\leq \sup_{\tau \in [0, T]} \left\{ \|x_\lambda(t_\tau; t, x) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| e^\tau \right\}. \end{aligned} \tag{14}$$

Let us consider in more detail the difference

$$\|x_\lambda(t_\tau; t, x) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\|,$$

where $x_\lambda(t_\tau; t, x)$ is the solution of (2) with initial conditions (t, x). Then

$$\|x_\lambda(t_\tau; t, x) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| \leq \|x_\lambda(t_\tau; t, x) - x_\lambda(\hat{t}_\tau; t, x)\| + \|x_\lambda(\hat{t}_\tau; t, x) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\|. \tag{15}$$

Let us estimate first term in (15). Since the solutions $x_\lambda(t_\tau; t, x)$ and $x_\lambda(\hat{t}_\tau; t, x)$ can be represented as

$$x_\lambda(t_\tau; t, x) = x + \int_t^{t_\tau} X(s, x_\lambda(s; t, x))\Delta s,$$
$$x_\lambda(\hat{t}_\tau; t, x) = x + \int_t^{\hat{t}_\tau} X(s, x_\lambda(s; t, x))\Delta s,$$

we obtain the following estimate

$$\begin{aligned}
&\|x_\lambda(t_\tau; t, x) - x_\lambda(\hat{t}_\tau; t, x)\| \\
&\qquad \leq \left\| x + \int_t^{t_\tau} X(s, x_\lambda(s; t, x))\Delta s - x - \int_t^{\hat{t}_\tau} X(s, x_\lambda(s; t, x))\Delta s \right\| \\
&\qquad \leq \left\| \int_t^{t_\tau} X(s, x_\lambda(s; t, x))\Delta s - \int_t^{t_\tau} X(s, x_\lambda(s; t, x))\Delta s - \int_{t_\tau}^{\hat{t}_\tau} X(s, x_\lambda(s; t, x))\Delta s \right\| \\
&\qquad \leq \int_{t_\tau}^{\hat{t}_\tau} \|X(s, x_\lambda(s; t, x))\|\Delta s, \qquad t \leq t_\tau \leq \hat{t}_\tau.
\end{aligned}$$

Let us denote

$$\max_{t \in [0,T]_{\mathbb{T}_\lambda}, \|x\| \leq N_r} \|X(t, x)\| = M_r, \tag{16}$$

then

$$\|x_\lambda(t_\tau; t, x) - x_\lambda(\hat{t}_\tau; t, x)\| \leq \int_{t_\tau}^{\hat{t}_\tau} M_r \Delta s \leq M_r |\hat{t}_\tau - t_\tau| \leq M_r |\hat{t} - t|. \tag{17}$$

Now estimate the second term in (15), namely $\|x_\lambda(\hat{t}_\tau; t, x) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\|$. Let us denote $x_\lambda(\hat{t}; t, x) := x^*$, then we get

$$\|x_\lambda(\hat{t}_\tau; t, x) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| = \|x_\lambda(\hat{t}_\tau; \hat{t}, x^*) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\|.$$

Since $x_\lambda(\hat{t}_\tau; \hat{t}, x^*)$ and $x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})$ we rewrite as

$$x_\lambda(\hat{t}_\tau; \hat{t}, x^*) = x^* + \int_{\hat{t}}^{\hat{t}_\tau} X(s, x_\lambda(s; \hat{t}, x^*))\Delta s,$$
$$x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x}) = \hat{x} + \int_{\hat{t}}^{\hat{t}_\tau} X(s, x_\lambda(s; \hat{t}, \hat{x}))\Delta s,$$

then

$$\begin{aligned}
&\|x_\lambda(\hat{t}_\tau; \hat{t}, x^*) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| \\
&\qquad \leq \left\| x^* + \int_{\hat{t}}^{\hat{t}_\tau} X(s, x_\lambda(s; \hat{t}, x^*))\Delta s - \hat{x} - \int_{\hat{t}}^{\hat{t}_\tau} X(s, x_\lambda(s; \hat{t}, \hat{x}))\Delta s \right\| \\
&\qquad\qquad \leq \|x^* - \hat{x}\| + \int_{\hat{t}}^{\hat{t}_\tau} \|X(s, x_\lambda(s; \hat{t}, x^*)) - X(s, x_\lambda(s; \hat{t}, \hat{x}))\|\Delta s.
\end{aligned}$$

Since $X(t, x)$ is Lipschitzian with constant K with respect to x, we have

$$\|x_\lambda(\hat{t}_\tau; \hat{t}, x^*) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| \leq \|x^* - \hat{x}\| + K \int_{\hat{t}}^{\hat{t}_\tau} \|x_\lambda(s; \hat{t}, x^*) - x_\lambda(s; \hat{t}, \hat{x})\|\Delta s.$$

Hence, using an analogue of the Gronwall inequality for time scales [2, Theorem 6.4] for $s \in [\hat{t}; \hat{t}_\tau]_{\mathbb{T}_\lambda}$ and Lemma 1 [11], we obtain

$$\begin{aligned} \|x_\lambda(\hat{t}_\tau; \hat{t}, x^*) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| &\le e_K(\hat{t}_\tau, \hat{t}) \|x^* - \hat{x}\| \\ &\le e^{K(\hat{t}_\tau - \hat{t})} \|x^* - \hat{x}\| \le e^{K(\hat{t}_\tau - \hat{t})} \left(\|x^* - x\| + \|x - \hat{x}\| \right). \end{aligned} \tag{18}$$

Considering $x^* := x_\lambda(\hat{t}; t, x)$, we have $\|x_\lambda(\hat{t}; t, x) - x\| \le \int_t^{\hat{t}} \|X(s, x_\lambda(s; t, x))\| \Delta s$. Therefore, by (16), for $t_\tau \in [0, \hat{t}_\tau]$ we obtain

$$\|x^* - x\| = \|x_\lambda(\hat{t}; t, x) - x\| \le \int_t^{\hat{t}} M_r \Delta s \le M_r |\hat{t} - t|.$$

Substituting the last equality into (18), we get

$$\|x_\lambda(\hat{t}_\tau; \hat{t}, x^*) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| \le e^{K(\hat{t}_\tau - \hat{t})} \left(M_r |\hat{t} - t| + \|x - \hat{x}\| \right). \tag{19}$$

Thus, from (15), (17) and (19), we obtain

$$\begin{aligned} \|x_\lambda(t_\tau; t, x) - x_\lambda(\hat{t}_\tau; \hat{t}, \hat{x})\| &\le M_r |\hat{t} - t| + e^{K(\hat{t}_\tau - \hat{t})} \left(M_r |\hat{t} - t| + \|x - \hat{x}\| \right) \\ &\le M_r (e^{K(\hat{t}_\tau - t_\tau)} + 1) |\hat{t} - t| + e^{K(\hat{t}_\tau - \hat{t})} \|\hat{x} - x\|. \end{aligned}$$

For sufficiently small $\mu_\lambda > 0$ the point $\hat{t}_\tau$ is on the interval $[\hat{t} + \tau; \hat{t} + \tau + 1]$, so we have $\hat{t}_\tau - t_\tau \le \hat{t} - t + 1$. Hence, from the condition (14), taking into account the dissipativity of system (2) and inequality (4), we have

$$\begin{aligned} V(t, x) - V(\hat{t}, \hat{x}) &\le \sup_{\tau \in [0, T]} \left\{ \left(M_R (e^{K(\hat{t}_\tau - \hat{t})} + 1) |\hat{t} - t| + e^{K(\hat{t}_\tau - \hat{t})} \|\hat{x} - x\| \right) e^\tau \right\} \\ &\le M_R (e^{K(T+1)+T} + 1) |\hat{t} - t| + e^{K(T+1)+T} \|\hat{x} - x\|. \end{aligned} \tag{20}$$

Similarly, performing transformations as in (14)–(20), we can obtain the estimate

$$\begin{aligned} V(\hat{t}, \hat{x}) - V(t, x) &\le \sup_{\tau \in [0, T]} \left\{ \left(M_R (e^{K(\hat{t}_\tau - \hat{t})} + 1) |t - \hat{t}| + e^{K(\hat{t}_\tau - \hat{t})} \|x - \hat{x}\| \right) e^\tau \right\} \\ &\le M_R (e^{K(T+1)+T} + 1) |t - \hat{t}| + e^{K(T+1)+T} \|x - \hat{x}\|. \end{aligned}$$

Therefore, we have

$$V(t, x) - V(\hat{t}, \hat{x}) \ge -M_R (e^{K(T+1)+T} + 1) |t - \hat{t}| + e^{K(T+1)+T} \|x - \hat{x}\|. \tag{21}$$

Thus, from (20) and (21), we obtain

$$|V(t, x) - V(\hat{t}, \hat{x})| \ge M_R (e^{K(T+1)+T} + 1) |t - \hat{t}| + e^{K(T+1)+T} \|x - \hat{x}\|.$$

So, we conclude that the function $V(t, x)$ is Lipschitzian with respect to t and x.

Next we will show that condition (9) holds. For each point $t \in \mathbb{T}_\lambda$ there are two possible cases: when the point t is right-scattered, i.e. $\mu_\lambda(t) > 0$, and when the point t is right-dense, i.e. $\mu_\lambda(t) = 0$. Let us denote

$$V_\tau(t, x) = G(\|x_\lambda(\sigma(t + \tau); t, x)\|) \cdot e^\tau.$$

Then for $\mu_\lambda(t) \geq 0$ we obtain

$$\begin{aligned} &V_\tau(t+\mu_\lambda(t), x(t+\mu_\lambda(t); t, x)) \\ &\qquad = G(\|x_\lambda(\sigma(t+\mu_\lambda(t)+\tau); t+\mu_\lambda(t), x(t+\mu_\lambda(t); t, x))\|)e^{\tau} \\ &\qquad = G(\|x_\lambda(\sigma(t+\mu_\lambda(t)+\tau); t+\mu_\lambda(t), x(t+\mu_\lambda(t); t, x))\|)e^{\tau+\mu_\lambda(t)}e^{-\mu_\lambda(t)}. \end{aligned} \tag{22}$$

From the uniqueness of solutions of (2) on time scales [7, Proposition 4], we get

$$x_\lambda(\sigma(t+\mu_\lambda(t)+\tau); t+\mu_\lambda(t), x(t+\mu_\lambda(t); t, x)) = x_\lambda(\sigma(t+\mu_\lambda(t)+\tau); t, x). \tag{23}$$

Substituting (23) into (22) and denoting $\tau' = \tau + \mu_\lambda(t)$, we obtain

$$V_\tau(t+\mu_\lambda(t), x(t+\mu_\lambda(t); t, x)) = G(\|x_\lambda(\sigma(t+\tau'); t, x))\|)e^{\tau'}e^{-\mu_\lambda(t)} = V_{\tau'}(t, x)e^{-\mu_\lambda(t)}.$$

Note that if $\tau \in [0, T]$ then $\tau' \in [\mu_\lambda(t), T+\mu_\lambda(t)]$, so $t+\tau' \in [t+\mu_\lambda(t), t+T+\mu_\lambda(t)]$. Since from the dissipativity $\|x_\lambda(t+\tau'; t, x)\| < R$ when $t+\tau' \geq t+T$, then $G(\|x_\lambda(t+\tau'; t, x)\|) = 0$ when $t+\tau' \geq t+T$. Hence,

$$\begin{aligned} V(t+\mu_\lambda(t), x(t+\mu_\lambda(t); t, x)) &= \sup_{\tau\in[0,T]} V_\tau(t+\mu_\lambda(t), x(t+\mu_\lambda(t); t, x)) \\ &= \sup_{\tau'\in[\mu_\lambda(t), T+\mu_\lambda(t)]} V_{\tau'}(t, x)e^{-\mu_\lambda(t)} \\ &= \sup_{\tau'\in[\mu_\lambda(t), T]} V_{\tau'}(t, x)e^{-\mu_\lambda(t)} \\ &\leq \sup_{\tau'\in[0,T]} V_{\tau'}(t, x)e^{-\mu_\lambda(t)}, \end{aligned}$$

that is,

$$V(t+\mu_\lambda(t), x(t+\mu_\lambda(t); t, x)) \leq V(t, x)e^{-\mu_\lambda}. \tag{24}$$

Let us consider the Lyapunov operator corresponding to system (2). Since, $V(t, x)$ satisfies a Lipschitz condition with respect to t and x, it is absolutely continuous with respect to t and x, and therefore, for almost all t and x, it has derivatives (Δ-derivative with respect to t and ordinary derivative with respect to x). By (24), for a right-scattered point t, we obtain

$$\begin{aligned} \frac{d^0V(t,x)}{\Delta t} &= \frac{1}{\mu_\lambda(t)}\left[V(t+\mu_\lambda(t), x(t+\mu_\lambda(t,t,x)) - V(t,x)\right] \\ &\leq \frac{1}{\mu_\lambda(t)}\left[V(t,x)e^{-\mu_\lambda(t)} - V(t,x)\right] = V(t,x)\cdot\frac{e^{-\mu_\lambda(t)}-1}{\mu_\lambda(t)}. \end{aligned}$$

Since $\lim_{\mu_\lambda(t)\to 0}\frac{e^{-\mu_\lambda(t)}-1}{\mu_\lambda(t)} = -1$, from Remark 1 it follows that

$$\overset{\Delta}{V}(t, x) \leq -C(\mu_\lambda)V(t, x),$$

where $C(\mu_\lambda) = \frac{1-e^{-\mu_\lambda(t)}}{\mu_\lambda(t)} > 0$.

If t is right-dense, i.e. $\mu_\lambda(t) = 0$, then there exists a sequence $\{h_n\}$, $h_n \in \mathbb{T}_\lambda$, such that $h_n \to t+0$. Let us consider

$$\begin{aligned} V_\tau(t+h_n, x(t+h_n; t, x)) &= G(\|x(t+h_n+\tau; t+h_n, x(t+h_n; t, x)\|)e^{\tau} \\ &= G(\|x(t+h_n+\tau; t+h_n, x(t+h_n; t, x)\|)e^{\tau+h_n}e^{-h_n}. \end{aligned}$$

By the uniqueness of the solution of (2), $x(t+h_n+\tau; t+h_n, x(t+h_n; t, x)) = x(t+h_n+\tau; t, x)$. Then replacing $\tau + h_n = \tau_n$, we obtain

$$\begin{aligned} V(t+h_n, x(t+h_n; t, x)) &= \sup_{\tau\in[0,T]} V_\tau(t+h_n, x(t+h_n; t, x)) \\ &= \sup_{\tau_n\in[h_n, T+h_n]} V_{\tau_n} e^{-h_n} \le \sup_{\tau_n\in[0,T]} V_{\tau_n}(t,x) e^{-h_n} \le V(t,x)e^{-h_n}. \end{aligned}$$

Subtracting $V(t,x)$ from both sides and dividing by $h_n > 0$, we get

$$\frac{V(t+h_n, x(t+h_n; t, x)) - V(t,x)}{h_n} \le V(t,x)\frac{e^{-h_n}-1}{h_n}.$$

Hence, by the definition of the Δ-derivative at dense points as $h_n \to 0$ we have

$$V^\Delta(t,x) \le -V(t,x).$$

Therefore, we have established that condition (9) holds. □

Proof of Theorem 5. Let choose an arbitrary $r > 0$. Without loss of generality, it can be assumed that $t_0 = 0 \in \mathbb{T}_\lambda$. From the conditions of Theorem 5, if $x_\lambda(t; 0, x_0)$ is a solution of (2) with initial conditions $(0, x_0)$, where $|x_0| < r$, then for all $\lambda \le \lambda_0$ there exists $\tilde{T} = \tilde{T}(r)$ such that the following inequality

$$|x_\lambda(t, 0, x_0)| < R$$

holds for $t \in [\tilde{T}, +\infty)_{\mathbb{T}_\lambda}$.

By the continuous dependence of solution of the system (2) on initial data on a finite interval of time scale [13, Theorem 3.2] it follows that there exists $M > 0$ such that $|x_\lambda(t, 0, x_0)| < M$ for all $t \in \mathbb{T}_\lambda$, $t \le \tilde{T}$ and $|x_0| \le r$. Thus, condition (3) holds for $t \in [0, \tilde{T}]_{\mathbb{T}_\lambda}$ and $\|x\| \le M$.

Let x be a solution of system (1) such that $x(0) = x_\lambda(0) = x_0$ at the initial point $t_0 = 0$. Let us choose $T = \inf\{s \in \mathbb{T}_\lambda | s \ge \tilde{T}\}$. Since $T \ge \tilde{T}$, then the inequality $|x_\lambda(T, 0, x_0)| < R$ holds for it. Because of Lemma 3 concerning the proximity estimation of solutions of the differential equation system and the corresponding system of dynamic equations on time scales with the same initial conditions, for $t \in [0, T]$ the inequality

$$|x(t, 0, x_0) - x_\lambda(t, 0, x_0)| \le \mu_\lambda K(T) \to 0$$

holds as $\lambda \to 0$.

Note that, since $\mu_\lambda \to 0$ as $\lambda \to 0$, then always we can choose $\lambda_1 \le \lambda_0$ such that $\mu_\lambda K(T) < 1$ for all $\lambda \le \lambda_1$. Altogether we have

$$|x(T, 0, x_0)| \le |x(T, 0, x_0) - x_\lambda(T, 0, x_0)| + |x_\lambda(T, 0, x_0)| \le 1 + R. \tag{25}$$

Let $y_\lambda(t)$ be a solution of (2) such that $x(T, 0, x_0) = y_\lambda(T)$. Since system (2) is uniformly dissipative with respect to t_0 and λ, then by putting $r = R + 1$ in inequality (7), we obtain that there exists $T_1 = T_1(R+1)$ such that from the inequality $|y_\lambda(t_1)| < R + 1$ it follows that

$$|y_\lambda(t, T, x(T, 0, x_0))| < R \;\text{ for }\; t \in [T + T_1; +\infty)_{\mathbb{T}_\lambda}.$$

Let $t_1 = \inf\{s \in \mathbb{T}_\lambda : s \ge T + T_1\}$. Then $|y_\lambda(t_1, T, x(T, 0, x_0))| < R$. As earlier, we choose $\lambda_2 \le \lambda_1$ such that for all $\lambda \le \lambda_2$ the point t_1 is in the interval $[T + T_1, T + T_1 + 1]_{\mathbb{T}_\lambda}$ and the inequality $|x(t, T, x(T, 0, x_0)) - y_\lambda(t, T, x(T, 0, x_0))| \le \mu_\lambda K(t_1) < 1$ holds for $t \in [T, t_1]$.

Similarly to (25), we obtain $|x(t_1, T, x(T, 0, x_0))| < R + 1$. By the uniqueness of the solution of system (1), we have $x(t, T, x(T, 0, x_0)) = x(t, 0, x_0)$, from which and from the uniform dissipativity of system (2) in t_0, we obtain

$$|x(t_1, 0, x_0)| < R + 1,$$

where $t_1 \in [T + T_1, T + T_1 + 1]$.

Continuing this process, for any $k \in \mathbb{N}$ we have

$$|x(t_k, 0, x_0)| < R + 1,$$

where t_k is the smallest point in the interval $[T + kT_1, T + kT_1 + 1]_{\mathbb{T}_\lambda}$.

Since $x(t, t_0, x_0)$ continuously depends on the initial data, the solution of (1) also is in a fixed ball of radius $R_1 \geq R$ on the interval (t_k, t_{k+1}). Therefore, according to Definitions 1, 2 for $R = R_1$, system (1) is uniformly dissipative with respect to $t_0 > 0$. □

Proof of Theorem 6. Let us choose an arbitrary $r > 0$. Without loss of generality, it can be assumed that $t_0 = 0$. From the conditions of Theorem 6, if $x(t, 0, x_0)$ is a solution of (1) with initial conditions $(0, x_0)$, where $|x_0| < r$, then there exists $\tilde{T} = \tilde{T}(r)$ such that the following inequality

$$|x(t, 0, x_0)| < R$$

holds for $t \in [\tilde{T}, +\infty)$.

By the continuous dependence of the solution of system (1) on initial data on a finite interval it follows that there exists $M > 0$ such that $|x(t, 0, x_0)| < M$ for all $t \in \mathbb{R}$, $t \leq \tilde{T}$ and $|x_0| \leq r$. Thus, the condition (3) holds for $t \in [0, \tilde{T}]$ and $\|x\| \leq M$.

Let x_λ be a solution of system (2) such that $x_\lambda(0) = x(0) = x_0$ at the initial point $t_0 = 0 \in \mathbb{T}_\lambda$.

Let us choose $T = \inf\{s \in \mathbb{T}_\lambda : s \geq \tilde{T}\}$. Since $T \geq \tilde{T}$, then the inequality $|x_\lambda(T, 0, x_0)| < R$ holds for it. Because of Lemma 3 concerning the proximity estimation of solutions of the differential equation system and the corresponding system of dynamic equations on time scales with the same initial conditions, for $t \in [0, T]_{\mathbb{T}_\lambda}$ the inequality

$$|x_\lambda(t, 0, x_0) - x(t, 0, x_0)| \leq \mu_\lambda K(T) \to 0 \text{ as } \lambda \to 0$$

holds.

Note that, since $\mu_\lambda \to 0$ as $\lambda \to 0$, then always we can choose $\lambda_1 \leq \lambda_0$ such that $\mu_\lambda K(T) < 1$ for all $\lambda \leq \lambda_1$. At the same time, we have

$$|x_\lambda(T, 0, x_0)| \leq |x_\lambda(T, 0, x_0) - x(T, 0, x_0)| + |x(T, 0, x_0)| \leq 1 + R. \tag{26}$$

Let $y(t)$ be a solution of (1) such that $x_\lambda(T, 0, x_0) = y(T)$. Since system (1) is uniformly dissipative in t_0 then by putting $r = R + 1$ in inequality (6), we obtain that there exists $T_1 = T_1(R + 1)$ such that from the inequality $|y(T)| < R + 1$ it follows that

$$|y(t, T, x_\lambda(T, 0, x_0))| < R \text{ for } t \geq T + T_1.$$

Let $t_1 = \inf\{s \in \mathbb{T}_\lambda : s \geq T + T_1\}$. Then

$$|y(t_1, T, x_\lambda(T, 0, x_0))| < R.$$

As earlier, we choose $\lambda_2 \leq \lambda_1$ such that for all $\lambda \leq \lambda_2$ the point t_1 is in the interval $[T + T_1, T + T_1 + 1]_{\mathbb{T}_\lambda}$ and the inequality

$$|x_\lambda(t, T, x(T, 0, x_0)) - y(t, T, x(T, 0, x_0))| \leq \mu_\lambda K(t_1) < 1$$

holds for $t \in [T, t_1]_{\mathbb{T}_\lambda}$. Similarly to (26), we have $|x_\lambda(t_1, T, x(T, 0, x_0))| < R + 1$. By the uniqueness of the solution of system (2) on initial data on a finite interval of time scale [13, Theorem 3.2], we obtain $x_\lambda(t, T, x(T, 0, x_0)) = x_\lambda(t, 0, x_0)$. From here, we get

$$|x_\lambda(t_1, 0, x_0)| < R + 1,$$

where $t_1 \in [T + T_1, T + T_1 + 1]$.

Continuing this process, for any natural number k, we have

$$|x_\lambda(t_k, 0, x_0)| < R + 1,$$

where $t_k \in [T + kT_1, T + kT_1 + 1]$.

Since $x_\lambda(t, t_0, x_0)$ continuously depends on initial data on a finite interval of time scale [13, Theorem 3.2], the solution of (2) also is in a fixed ball of radius $R_1 \geq R$ on the interval $[t_k, t_{k+1}]_{\mathbb{T}_\lambda}$. Therefore, according to Definitions 3, 4 for $R = R_1$, (2) is uniformly dissipative with respect to t_0 and λ. □

5 Example

Let us illustrate the results of our theorems to the Liénard-type equation.

We consider the differential equation

$$x'' + (\cos x + 2)x' + x = 0. \tag{27}$$

To establish the dissipativity of this equation, we rewrite it in the form of a system

$$\begin{cases} x' = y, \\ y' = -(\cos x + 2)y - x. \end{cases} \tag{28}$$

Let

$$F(x) = \int_0^x \cos t + 2\, dt = \sin x + 2x, \quad G(x) = \int_0^x t\, dt = \frac{x^2}{2}$$

and

$$\begin{aligned} W(x, y) &= (F(x) - 2x)y + G(x) + \int_0^x (\cos t + 2)(F(t) - 2t)\, dt + 1 + \frac{y^2}{2} \\ &= y \sin x + \frac{x^2}{2} + \int_0^x (\cos t + 2) \sin t\, dt + 1 + \frac{y^2}{2} \\ &= y \sin x + \frac{x^2}{2} + (5 + \cos x) \sin^2 \frac{x}{2} + 1 + \frac{y^2}{2} \\ &= \frac{1}{2} y^2 + y \sin x + (5 + \cos x) \sin^2 \frac{x}{2} + 1, \end{aligned}$$

then we set a Lyapunov function $V(t,x)$ such that

$$V(t,x) = \begin{cases} (W(x,y))^\alpha - C, & \text{for } (W(x,y))^\alpha > C, \\ 0, & \text{for } (W(x,y))^\alpha \le C. \end{cases}$$

We consider W as a quadratic form in terms of y. Given that $0 < 1 \le \cos x + 2 \le 3$, we obtain $W \to \infty$ as $r = \sqrt{x^2+y^2} \to \infty$. We can choose $\alpha > 0$ such that $V(x,y) \in \mathbf{C}_0$. Using the equality

$$\frac{d^0 W}{dt} = -(2y^2 + x\sin(x)),$$

we get that the condition

$$\frac{d^0 V}{dt} \le -CV \tag{29}$$

is satisfied in the region $r > r_0$. Hence, it follows that for an appropriate C the inequality (29) holds for $V(x,y)$ everywhere. Consequently, the dissipativity of system (28) follows from the dissipativity conditions of the system of differential equations in terms of the Lyapunov function [23, Theorem 11].

Let us construct a solution to system (28) with initial conditions $x_0 = x(0) = 2$, $y_0 = y(0) = 0$ and draw it in Figure 1.

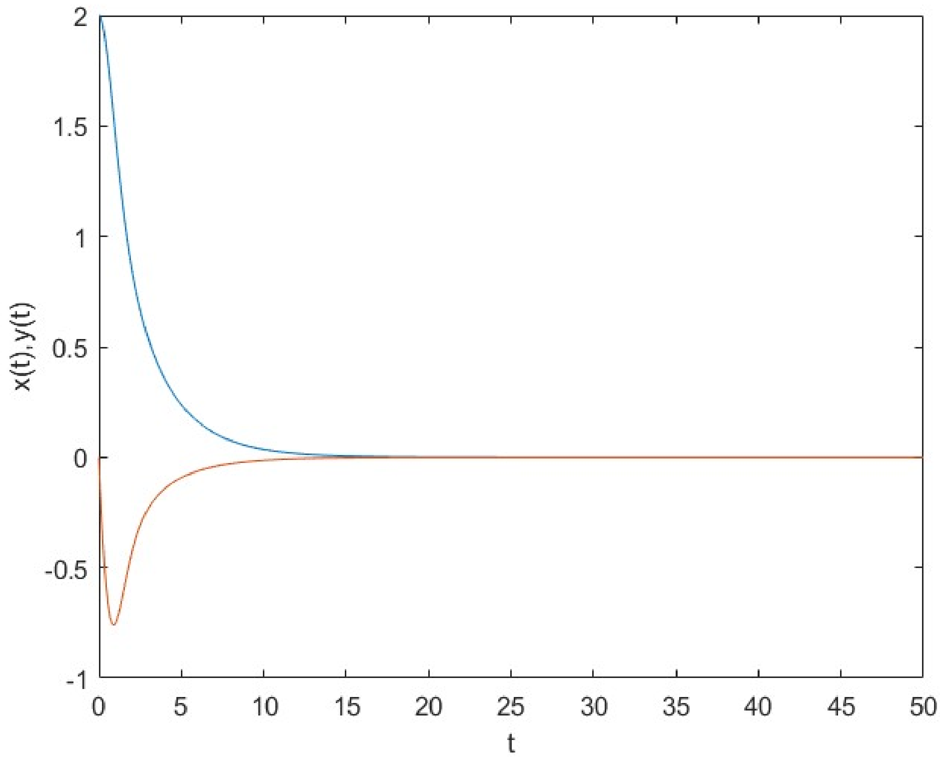


Figure 1. The solution of system (28) on the interval $[0,50]$

Let us consider the corresponding dynamic equation

$$x_\lambda^{\Delta\Delta} + (\cos x_\lambda + 2)x_\lambda^\Delta + x_\lambda = 0 \tag{30}$$

on the set of scales $\mathbb{T}_\lambda$where $\mu_\lambda = \sup_{\mathbb{T}_\lambda} \mu_\lambda(t)$.

The time scale is constructed in such a way that continuous intervals alternate with discrete ones (see Figure 2), and the density of the scale is regulated by the multiplier λ so that as $\lambda \to 0$, $\mu_\lambda \to 0$.

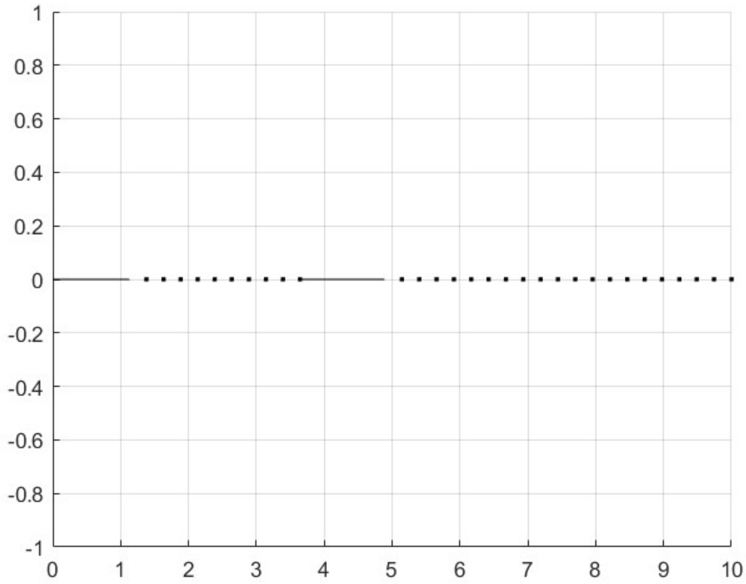
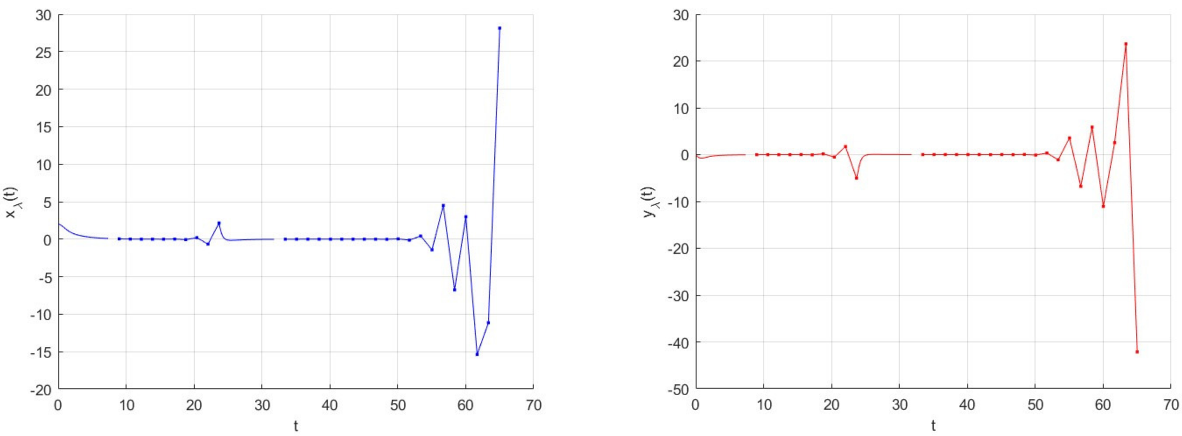


Figure 2. Segment of time scale $[0, 10]_{\mathbb{T}_\lambda}$

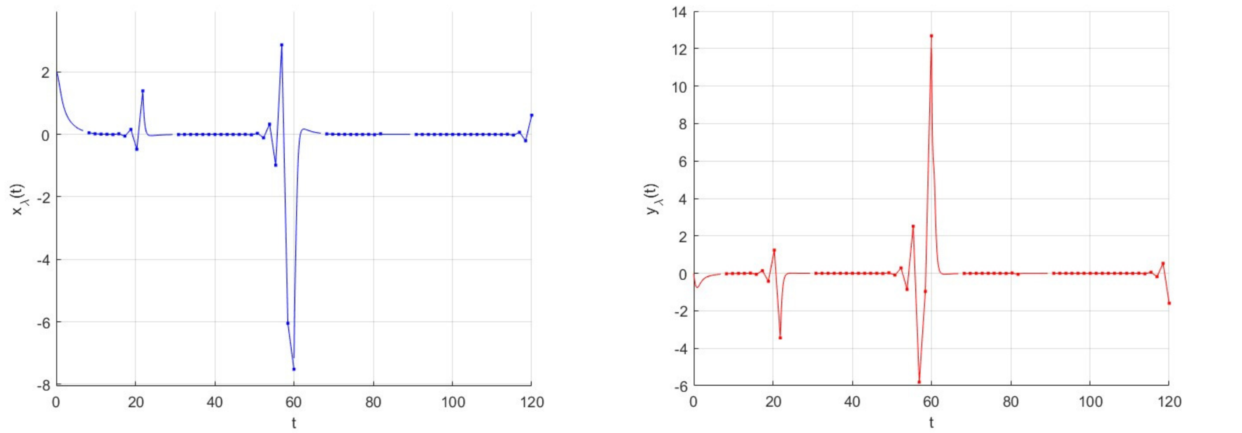
Let us also rewrite dynamic equation (30) in the form of a system

$$\begin{cases} x_\lambda^\Delta = y_\lambda, \\ y_\lambda^\Delta = -(e^{-x_\lambda^2} + 1)y_\lambda - x_\lambda. \end{cases} \tag{31}$$



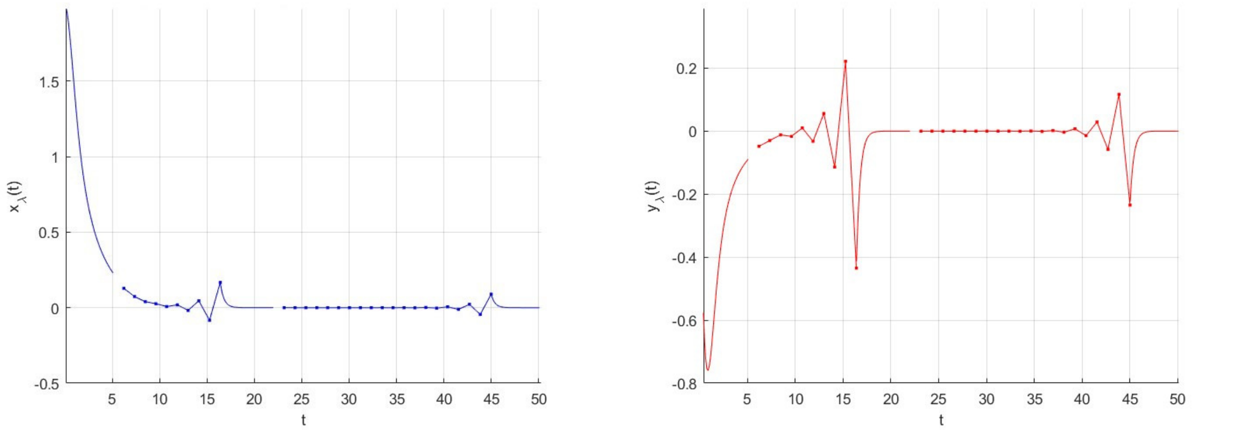
(a) $x_\lambda(t)$ (b) $y_\lambda(t)$

Figure 3. Plot of solutions for $\lambda = 0.65$



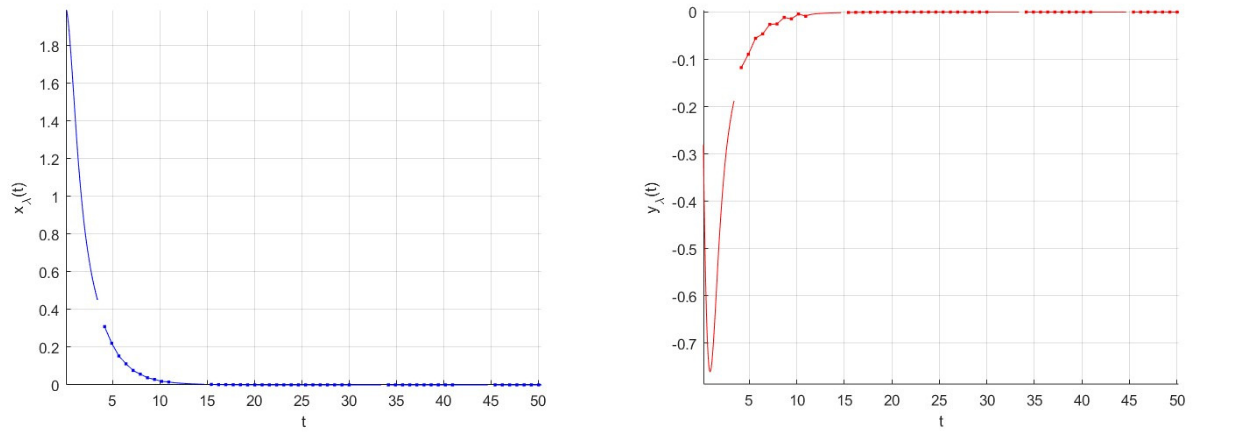
(a) $x_\lambda(t)$ (b) $y_\lambda(t)$

Figure 4. Plot of solutions for $\lambda = 0.6$



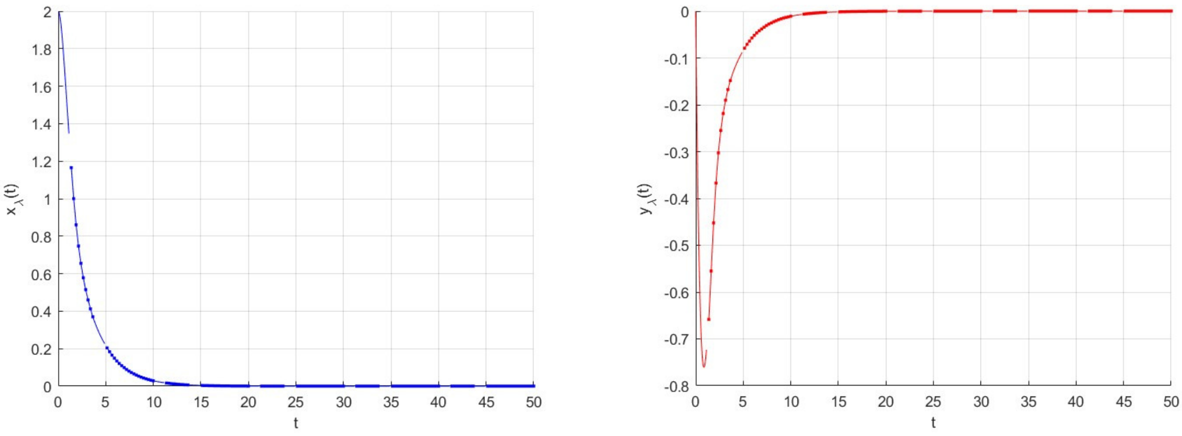
(a) $x_\lambda(t)$ (b) $y_\lambda(t)$

Figure 5. Plot of solutions for $\lambda = 0.45$



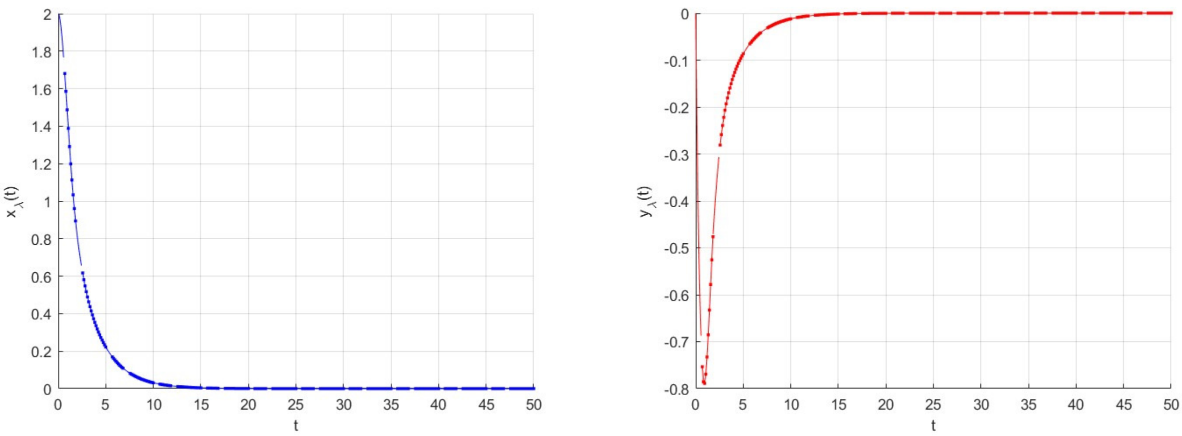
(a) $x_\lambda(t)$ (b) $y_\lambda(t)$

Figure 6. Plot of solutions for $\lambda = 0.3$



(a) $x_\lambda(t)$ (b) $y_\lambda(t)$

Figure 7. Plot of solutions for $\lambda = 0.1$



(a) $x_\lambda(t)$ (b) $y_\lambda(t)$

Figure 8. Plot of solutions for $\lambda = 0.05$

And let us constuct its solution on the interval $[0, 50]_{\mathbb{T}_\lambda}$ for different values of λ. So, for $\lambda = 0.65, 0.6, 0, 45, 0.3, 0.1, 0.05$, we obtain Figures 3, 4, 5, 6, 7 and 8, respectively. We see that, as λ decreases, the solutions of the dynamic system (31) approach solution of the differential equation (27) and exhibit dissipative behavior.

Acknowledgements

The research of Oleksandr Stanzhytskyi and Olena Kapustian was partially supported by Grant No. 0124U001412 and Grant No. 0122U001844, respectively, provided by the Ministry of Education and Science of Ukraine.

References

[1] Akin-Bohner E., Raffoul Y. *Boundedness in functional dynamic equations on time scales*. Adv. Difference Equ. 2006, **2006**, 1–18. doi:10.1155/ADE/2006/79689

[2] Bohner M., Peterson A. Dynamical equations on time scales. An introduction with applications. Birkhauser, Boston, 2001. doi:10.1007/978-1-4612-0201-1

[3] Bohner M., Peterson A. Advances in dynamical equations on time scales. Birkhauser, Boston, 2003. doi:10.1007/978-0-8176-8230-9

[4] Bohner M., Kenzhebaev K., Lavrova O., Stanzhytskyi O. *Pontryagin maximum principle for dynamic systems on time scales*. J. Difference Equ. Appl. 2017, **23** (7), 1161–1189. doi:10.1080/10236198.2017.1284829

[5] Bohner M., Georgiev S.G. Multivariable dynamic calculus on time scales. Springer, Berlin, 2016. doi: 10.1007/978-3-319-47620-9

[6] Bohner M., Karpenko O., Stanzhytskyi O. *Oscillation of solutions of second-order linear differential equations and corresponding difference equations*. J. Difference Equ. Appl. 2014, **20** (7), 1112–1126. doi: 10.1080/10236198.2014.893297

[7] Bourdin L., Trelat E. *General Cauchy-Lipschitz theory for Δ-Cauchy problems with Carathodory dynamics on time scales*. J. Difference Equ. Appl. 2014, **20** (4), 526–547. doi:10.1080/10236198.2013.862358

[8] Bourdin L., Stanzhytskyi O., Trelat E. *Addendum to Pontryagin maximum principle for dynamic systems on time scales*. J. Difference Equ. Appl. 2017, **23** (10), 1760–1763. doi:10.1080/10236198.2017.1363194

[9] Cabada A., Vivero D. *Criterions for absolute continuity on time scales*. J. Difference Equ. Appl. 2005, **11** (11), 1013–1028. doi:10.1080/10236190500272830

[10] Chaikovs'kyi A., Lagoda O. *Bounded solutions of a difference equation with finite number of jumps of operator coefficient*. Carpathian Math. Publ. 2020, **12** (1), 165–172. doi:10.15330/cmp.12.1.165-172

[11] Danilov V.Ya., Lavrova O.E., Stanzhyts'kyi O.M. *Viscous Solutions of the Hamilton-Jacobi-Bellman Equation on Time Scales*. Ukrainian Math. J. 2017, **69** (7), 1085–1106. doi:10.1007/s11253-017-1417-4 (translation of Ukrain. Mat. Zh. 2017, **69** (7), 933–950. (in Ukrainian))

[12] Hilger S. Ein Maskettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. thesis. University of Würzburg, Würzburg, 1988.

[13] Hilscher R., Zeidan V. *Time scale embedding theorem and coercivity of quadratic functionals*. Analysis 2008, **28** (1), 1–28. doi:10.1524/anly.2008.0900

[14] Karpenko O., Stanzhytskyi O. *The relation between the existence of bounded solutions of differential equations and the corresponding difference equations*. J. Difference Equ. Appl. 2013, **19** (12), 1967–1982. doi: 10.1080/10236198.2013.794795

[15] Karpenko O., Stanzhytskyi O., Dobrodzii T. *The relation between the existence of bounded global solutions of the differential equations and equations on time scales*. Turkish J. Math. 2020, **44**, 2099–2112. doi:10.55730/1300-0098.3373

[16] Lavrova O., Mogylova V., Stanzhytskyi O., Misiats O. *Approximation of the Optimal Control Problem on an Interval with a Family of Optimiztion Problems on Time Scales*. Nonlinear Dyn. Syst. Theory 2017, **17** (3), 303–314.

[17] Ogul B., Simsek D. *Dynamical behavior of one rational fifth-orderdifference equation*. Carpathian Math. Publ. 2023, **15** (1), 43–51. doi:10.15330/cmp.15.1.43-51

[18] Pratsiovytyi M., Ratuhniak S. *Properties and distributions of values of fractal functions related to Q_2-representations of real numbers*. Theory Probab. Math. Statist. 2019, **99** (2), 221–228. doi:10.1090/tpms/1091

[19] Pratsiovytyi M., Goncharenko Y., Lysenko I., Ratuhniak S. *Continued A_2-fractions and singular functions*. Mat. Stud. 2022, **58** (1), 3–12. doi:10.30970/ms.58.1.3-12

[20] Stanzhytskyi O., Mogylova V., Lavrova O. *Optimal Control for Systems of Differential Equations on the Infinite Interval of Time Scale*. In: Kelso S. (Ed.) Understanding Complex Systems. Springer, Cham, 2020, 395–405. doi:10.1007/978-3-030-50302-4_18

[21] Stanzhytskyi O., Uteshova R., Tsan V., Khaletska Z. *On the relation between oscillation of solutions of differential equations and corresponding equations on time scales*. Turkish J. Math. 2023, **47** (2), 476–501. doi:10.55730/1300-0098.3373

[22] Tsan V., Stanzhytskyi O., Martynyuk O. *On the correspondence between periodic solutions of differential and dynamic equations on periodic time scales*. Georgian Math. J. 2024, **31** (5), 899–908. doi:10.1515/gmj-2024-2003

[23] Yoshizawa T. Stability Theory by Lyapunovs Second Method. The Mathematical Society of Japan, Tokyo, 1966.

Received 20.12.2023
Revised 01.06.2024

Цань В.Б., Станжицький О.М., Капустян О.А. *Дисипативність динамічних систем на часових шкалах та зв'язок між дисипативними диференціальни та динамічними системами* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 427–447.

Робота присвячена дослідженню властивості дисипативності динамічних систем на часових шкалах та відповідних звичайних диференціальних рівнянь. Встановлено, що властивість дисипативності зберігається при переході від рівнянь на часових шкалах $\mathbb{T}_\lambda$ до відповідних звичайних диференціальних рівнянь та навпаки, коли функція зернистості μ_λ прямує до нуля при $\lambda \to 0$.

Ключові слова і фрази: дисипативність, часова шкала, функція зернистості, функція Ляпунова, обмежений розв'язок.